# Constrained Rough Paths

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**Abstract**

We introduce a notion of rough paths on embedded submanifolds and demonstrate that this class of rough paths is natural. On the way we develop a notion of rough integration and an efficient and intrinsic theory of rough differential equations (RDEs) on manifolds. The theory of RDEs is then used to construct parallel translation along manifold valued rough paths. Finally, this framework is used to show there is a one to one correspondence between rough paths on a $d$ – dimensional manifold and rough paths on $d$ – dimensional Euclidean space. This last result is a rough path analogue of Cartan's development map and its stochastic version which was developed by Eells and Elworthy and Malliavin.

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# 1 Introduction

In the series of papers [24–26], Terry Lyons introduced and began the development of the theory of *rough paths* on a Banach space $W$. This theory allow us to model the evolution of interacting systems, driven by highly irregular non-differentiable inputs, modelled as differential equations driven by a rough path $\mathbf{X}$. The theory of rough paths provides existence and uniqueness of solutions to such equations, moreover the solutions depend continuously on the driver $\mathbf{X}$. Among the many applications arising from the interplay of rough paths and stochastic analysis are the study of solutions to stochastic differential equations driven by Gaussian signals see e.g. [4], [3], [5], [16], [7] and the analysis of broad classes of stochastic partial differential equations (SPDEs) [1], [9], [21], [20]. Rough paths also provide us with alternative ways to think about and encode the information presented in a dynamical system.

A rough path of order $p \in [2, 3)$ on $[0, T]$ with values in a Banach space $(W, |\cdot|)$ is a pair of functions

$$\mathbf{X}_{s,t} := (x_{s,t}, \mathbb{X}_{s,t}) \in W \oplus W \otimes W,$$

which may be thought of as the increments of the path itself and a second-order term $\mathbb{X}_{s,t}$. Rough paths are characterised by an algebraic property analogous to the homomorphism property of the Chen series of a path (also known as the multiplicative property) and an analytic $p-$ variation type constraint on $\mathbf{X}$. A great variety of stochastic classical processes may be lifted to rough paths. For example, every $\mathbb{R}^d$ – valued continuous semi-martingale $(x_s)_{0 \le s \le T}$ (e.g. Brownian motion) and large classes of Gaussian processes including fractional Brownian motion (fBM) with Hurst parameter $H > 1/4$ may almost surely be augmented by a process $\mathbb{X}_{s,t}$. For continuous semi-martingales for example one may define

$$\mathbb{X}_{s,t} := \int_{s \le \tau \le t} x_{s,\tau} \otimes dx_\tau$$

where the integral can either be interpreted as an Itô integral or a Fisk-Stratonovich integral. The resulting $\mathbb{X}_{s,t}$ typically *does* depend on which integral is used.

In view of the fact that many (if not most) natural dynamical systems come with geometric constraints it is natural and necessary to develop a theory of constrained rough paths, i.e. rough paths on a manifold, $M$. There is some literature in this direction, (see for example [18, 27, 28]) in which rough paths theory is used to defined certain stochastic evolutions on a manifold. Nevertheless, these papers sidestep the issue of actually *defining* the notion of a rough path in a manifold. The analogue of the approach of these papers in the smooth category would be to proceed as follows. First consider $M^d$ as an embedded submanifold in $\mathbb{R}^N$ for some $N$ and then only consider curves $\sigma : [0, 1] \to M$, that satisfy a differential equation in the ambient space $\left(\mathbb{R}^N\right)$ of the form

$$\dot{\sigma}(t) = \sum_{i=1}^{D} V_i(\sigma(t)) \dot{z}_i(t), \tag{1.1}$$

where $z : [0, 1] \to \mathbb{R}^D$ is a smooth curve, and $\{V_i\}_{i=1}^D$ are smooth vector fields on $\mathbb{R}^N$ such that $V_i(x) \in T_x M$ for all $x \in M$. As a consequence only after one has figured out how to present $\sigma$ in the form of Eq. (1.1) is one allowed to talk about integration along a one form, parallel translation, unrolling, etc. For example, if $\alpha$ is a one form on $M$, then the approach above would define the path integral of a one form $\alpha$ along $\sigma$ by

$$\int_\sigma \alpha := \int_0^1 \sum_{i=1}^{D} \alpha\left(V_i\right)\left(\sigma\left(t\right)\right)\dot{z}_i\left(t\right)dt. \tag{1.2}$$

Clearly this is unsatisfactory. For example, we certainly would like to know the integral is independent of the chosen presentation of $\sigma$ in Eq. (1.1). Moreover, from a practical and computational point of view, carrying around all of this extra structure would at best be very cumbersome and at worst would be an obstruction to developing analysis on paths and loops into a manifold, which are so prevalent in the context of Riemannian geometry and geometric PDEs.

The paper [6], is the first (and until now the only) paper which develops a consistent theory of rough paths on a manifold by viewing them as a sort of non-linear current space. The theory developed in [6] has the advantage of being global and intrinsic, but it does require a rather stringent condition that the manifold is "Lip-$\gamma$." In this paper we will remove this assumption and at the same time provide a concrete realization of the currents appearing [6]. Moreover, it is our goal to carefully develop the tools needed to make the calculus of rough paths on manifolds effective and practical for future applications.

In this paper, we suppose $M^d$ is a $d-$ dimensional smooth manifold which has been embedded into some Euclidean space $E = \mathbb{R}^N$. Our first order of business is to "identify" those weakly geometric rough paths $\mathbf{X} = \{(x_s, \mathbb{X}_{st}) : 0 \le s \le t \le T\} \subset E \oplus E \otimes E$ which are "constrained" to lie in $M$. This point is a bit subtle as the "obvious" definition that $x_s \in M$ and $\mathbb{X}_{st} \in T_{x_s}M \otimes T_{x_s}M$ is not the correct notion. Indeed, the condition that $\mathbb{X}_{st} \in T_{x_s}M \otimes T_{x_s}M$ is too strict and will essentially only hold for constant rough paths. The key starting point of this paper is Definition 3.15 which basically states that a weakly geometric rough path $\mathbf{X} = (x_s, \mathbb{X}_{st}) \in E \oplus E \otimes E$ is constrained to $M$ iff $1-$ forms on $M$ can consistently be integrated along $\mathbf{X}$. It is then shown in Section 3 that this condition is equivalent to $\mathbb{X}_{st}$ being "approximately" in $T_{x_s}M \otimes T_{x_s}M$, see Corollary 3.32. A number of other equivalent characterizations of Definition 3.15 are also given in Section 3.

Our definition of a rough path is natural in that it is the maximal class that permits a consistent definition of rough integration and, though our proofs and definitions will sometimes depend on the embedding, we will however show that the choice of embedding is not important, see Corollaries 3.40, 3.41, and Definition 3.42. In fact, the theory is intrinsic to the manifolds, see Cass, Driver, Litterer [2], where we clarify the relations of intrinsic and embedded definitions of rough paths on manifolds.

In the flat finite dimensional setting it is known that the notions of weakly geometric and geometric rough paths are essentially equivalent. On the other hand in the manifold setting presented here, the analogous result is not a priori known and must be proved which is not done until Theorem 4.17. After this point it would be possible to replace some of our proofs with smooth approximation arguments. However, we choose to avoid doing so as the smooth approximation arguments obscure the interesting second order differential geometric identities which underlie the theory. We believe this important as the smooth approximation arguments will not be available when, in the future, one goes to transfer these results to infinite dimensional settings.

The paper is organized as follows. Section 2 is devoted to introducing fundamental definitions and some preliminary results in Banach space - valued rough path theory. Section 3, as described above is where we define a rough path in an embedded manifold.

Section 4 is devoted to the notion rough differential equations on $M$. Theorem 4.2 shows that one may solve a rough differential equation on $M$ by extending the vector fields defining the differential equation to the ambient space and then applying the Euclidean rough path theory to the resulting dynamical system. The output is a weakly geometric rough path in $M$ which does not depend on any of the choices made in the extensions. Later in Theorem 4.5 we derive an equivalent intrinsic characterisations of these solutions.

Section 5 develops the notion of rough parallel translation along manifold valued rough paths. *Parallel translation* along a rough path $\mathbf{X}$ in $M$ is defined as a rough path $\mathbf{U}$ in the orthogonal frame bundle $O(M)$ over $M$ which solves a prescribed RDE on $O(M)$ driven by $\mathbf{X}$, see Definition

5.13. It is shown in Proposition 5.15 that the RDE defining $\mathbf{U}$ does not explode and so $\mathbf{U}$ exists on the full time interval, $[0,T]$. It is then shown in Theorems 5.16 and 5.17 that two natural classes of RDE's on $O(M)$ give rise to an element $\mathbf{U} \in WG_p(O(M))$ each of which is parallel translation along $\mathbf{X} := \pi_*(\mathbf{U})$ where $\pi : O(M) \to M$ is the natural projection map on $O(M)$. Here $\pi_*(\mathbf{U})$ denotes the pushforward of $\mathbf{U}$ by $\pi$, see Proposition 3.38.

In Section 6 we show in Corollary 6.12 that there is (similar to the smooth theory) a one to one correspondence between rough paths on the orthogonal frame bundle $O(M)$ to $M$ and rough paths on the Euclidean space $\mathbb{R}^d \times so(d)$. Furthermore Theorems 6.18, and Corollary 6.19 show there are one-to-one correspondence between rough paths on $M$, "horizontal" rough paths on $O(M)$ (see Definition 6.14), and rough paths on $\mathbb{R}^d$. These results are rough path versions of the stochastic rolling construction of Brownian motion on a manifold via the orthogonal frame bundle as first appeared in Eells and Elworthy [11] and then further developed in Elworthy [12], [13] and [14] and in Malliavin [31]. We expect that the results of this paper will help lay the foundation of future work that explores the properties of manifold valued solutions to stochastic differential equations driven by Gaussian processes such as fractional Brownian motions.

The paper is completed with two appendices. In Appendix A we gather together some needed results of the Banach space-valued rough path theory while Appendix B explains a few details on how to view $O(M)$ as an embedded submanifold which are needed in Section 6 of the paper.

# 2 Background Rough Path Results

## 2.1 Basic notations

In this section we introduce some basic notations for rough paths on Banach spaces. In addition, we gather some elementary preliminary results that will prove useful in the sequel. Some additional rough path theory results on Banach spaces needed in this paper may also be found in Appendix A. Throughout this section, $V$, $W$ and $U$ will denote real Banach spaces. For simplicity in this paper, we will typically assume that all Banach spaces are finite dimensional. If $(V, |\cdot|)$ is a Banach space we will abuse notation and write $|\cdot|$ for one of the tenor norms on $V \otimes V$. Because $\dim V < \infty$, the choice of tensor norm on $V \otimes V$ is unimportant. For $\mathbb{X} \in V \otimes V$ we denote its symmetric and anti-symmetric part to be $\mathbb{X}^s$ and $\mathbb{X}^a$ respectively. The following definition and (abuse of) notation will frequently be used in the sequel.

**Definition 2.1 (Truncated Tensor Algebra)** *Let $T_2(V) := \mathbb{R} \oplus V \oplus V \otimes V$ which we make into an algebra by using the multiplication in the full tensor algebra and then disregarding any terms that appear in $V^{\otimes 3} \oplus V^{\otimes 4} \dots$. In more detail, if $a, b \in \mathbb{R}$, $x, y \in V$, and $\mathbb{X}, \mathbb{Y} \in V \otimes V$, then*

$$(a, x, \mathbb{X})(b, y, \mathbb{Y}) := (ab, ay + bx, a\mathbb{Y} + x \otimes y + b\mathbb{X}).$$

*In the future we will typically write $a + x + \mathbb{X}$ for $(a, x, \mathbb{X})$.*

**Notation 2.2** *If $B : V \times V \to W$ is a bilinear form with values in a vector space $W$ then by the universal property of the tensor product there is a unique linear map, $\hat{B} : V \otimes V \to W$ such that $B(a,b) = \hat{B}(a \otimes b)$ for all $a, b \in V$. Given $A \in V \otimes V$, it will be useful to abuse notation and abbreviate $\hat{B}(A)$ as $B(a,b)|_{a \otimes b = A}$. For example if $A = \sum_{i=1}^{\ell} a_i \otimes b_i \in V \otimes V$ it follows that*

$$B(a,b)|_{a \otimes b = A} := \sum_{i=1}^{\ell} B(a_i, b_i).$$

Throughout this paper we let $T$ denote a positive finite real number, $p$ be a fixed real number in the interval $[2,3)$ and $\omega$ a *control* whose definition we now recall.

**Definition 2.3** *A **control** $\omega : \Delta_{[0,T]} := \{(s,t) : 0 \le s \le t \le T\} \to \mathbb{R}_+$ is a continuous non-negative function which is superadditive, positive off the diagonal, and zero on the diagonal in $\Delta_{[0,T]}$.*

**Definition 2.4 (Rough Paths)** *For a Banach space $V$, the set of ($\omega$ – controlled $V-$valued) p-rough paths consists of pairs $\mathbf{X} = (x, \mathbb{X})$ of continuous paths*

$$x : [0,T] \to V \quad \text{and } \mathbb{X} : \Delta_{[0,T]} \to V \otimes V,$$

*satisfying the following conditions:*

1. *The Chen identity; i.e.*

$$\mathbb{X}_{s,t} = \mathbb{X}_{s,u} + \mathbb{X}_{u,t} + x_{s,u} \otimes x_{u,t} \ \forall\ 0 \le s \le u \le t \le T, \tag{2.1}$$

   *where here, as throughout, $x_{s,t} := x_t - x_s$ will denote the increment of the path $x$ over $[s,t]$.*

2. *A $p-$variation regularity constraint:*

$$\sup_{0\le s<t\le T} \frac{|x_{s,t}|}{\omega(s,t)^{1/p}} < \infty \text{ and } \sup_{0\le s<t\le T} \frac{|\mathbb{X}_{s,t}|}{\omega(s,t)^{2/p}} < \infty. \tag{2.2}$$

We can identify a rough path as a map taking values in the tensor algebra.

**Remark 2.5** *It is often convenient to identify a rough path, $\mathbf{X} = (x, \mathbb{X})$, with the function $\mathbf{X}$ : $\Delta_{[0,T]} \to T_2(V)$ defined by*

$$\mathbf{X}_{s,t} := 1 + x_{s,t} + \mathbb{X}_{s,t} \text{ for } (s,t) \in \Delta_{[0,T]}.$$

*Using this identification, Chen's identity becomes the following multiplicative property of* $\mathbf{X}$;

$$\mathbf{X}_{s,t} = \mathbf{X}_{s,u}\mathbf{X}_{u,t} \ \ \forall\ 0 \le s \le u \le t \le T, \tag{2.3}$$

*where multiplication is given as in Definition 2.1.*

The collection of $V-$ valued $p-$ rough paths controlled by $\omega$ is denoted by $R_p([0,T], V, \omega)$ (also denoted by $R_p(V)$ where no confusion arises).

**Example 2.6** *Suppose $x : [0,T] \to V$ is a continuous bounded variation path. Then a simple example of a $p$ – rough path is the (truncated) signature $\left(S_2(x)_{s,t} := \mathbf{X}_{s,t} \mid 0 \le s < t \le T\right)$ defined by*

$$\mathbf{X}_{s,t} := 1 + x_{s,t} + \mathbb{X}_{s,t} \in T_2(V) \tag{2.4}$$

*where*

$$\mathbb{X}_{s,t} := \int_{s<t_1<t_2<t} dx_{t_1} \otimes dx_{t_2} = \int_s^t x_{s,u} \otimes dx_u \tag{2.5}$$

*and the latter integral being the Lebesgue-Stieltjes integral. For the control we may take*

$$\omega(s,t) = |x|_{1\text{-}var;[s,t]} := \sup_{D=\{t_i : s=t_0<t_1<\ldots<t_n=t\}} \sum_{i=1}^n \left| x_{t_{i-1},t_i} \right|.$$

*In this case, $\mathbb{X}$ is not an extra piece of information but is in fact determined by the basic path $x$.*

**Remark 2.7** *If $x : [0,T] \to V$ is continuous and of bounded variation and $\mathbb{X}$ is given as in Eq. (2.5), then as a consequence of the fundamental theorem of calculus the symmetric part of $\mathbb{X}_{st}$ satisfies,*

$$\mathbb{X}^s_{st} = \frac{1}{2} x_{s,t} \otimes x_{s,t} \ \forall\ 0 \le s \le t \le T. \tag{2.6}$$

In this paper we are interested in the following two important subsets of $R_p(V)$.

**Definition 2.8** **($WG_p(V)$ and $G_p(V)$)** *Let $\mathbf{X}$ be a $V-$valued p-rough path, i.e. $\mathbf{X} \in R_p(V)$.*

1. *We say that $\mathbf{X}$ is a **geometric** $p$**-rough path**, and write $\mathbf{X} \in G_p(V)$, if $\mathbf{X}$ belongs to the closure of the set:*

$$\{\mathbf{Y} : \mathbf{Y} = S_2(y),\ y \text{ continuous and of finite } 1\text{-variation}\}$$

*with respect to the topology induced by the metric (A.1).*

2. *We say that $\mathbf{X}$ is a **weakly geometric** $p$**-rough path**, and write $\mathbf{X} \in WG_p(V)$, if Eq. (2.6) holds.*

**Remark 2.9** *If $\dim(V) < \infty$ and $p < q$, then we have the strict inclusions $G_p(V) \subset WG_p(V) \subset G_q(V)$ (see Corollary 8.24 of [17]) and so one typically does not have to pay much attention to the difference between geometric and weakly geometric rough paths. However, in infinite dimensions the compactness argument used in the proof that $WG_p(V) \subset G_q(V)$ breaks down.*

## 2.2 Approximate rough paths and integration

The following notation will be used heavily in this paper.

**Notation 2.10** **($\simeq$ and $\simeq_\delta$)** *Let $\omega$ be a control, and assume $g$ and $h$ are continuous functions from $\Delta_{[0,T]}$ into some Banach space $W$. Then we will write*

$$g_{s,t} \simeq h_{s,t}$$

*if there exists $\delta > 0$ and a constant $C(\delta) > 0$ such that for all $s$ and $t$ in $[0,T]$ satisfying $|s-t| \leq \delta$ we have*

$$|g_{s,t} - h_{s,t}| \leq C(\delta)\,\omega(s,t)^{3/p}.$$

*If we wish to emphasize the dependence on $\delta$ then we will write $g_{s,t} \simeq_\delta h_{s,t}$.*

**Remark 2.11** *As a typical application of this notation, let us note that if $g : [0,T] \to V$ is continuous and such that $g_{s,t} \simeq 0$ then, because the increments form an additive function on $\Delta_{[0,T]}$, it must be that $g$ is constant. Indeed, if $D = \{t_i : k = 0, 1..., n\}$ is any partition of $[0,t] \subseteq [0,T]$ with $|D| \leq \delta$ then*

$$|g_{0,t}| = \left|\sum_{i=1}^{n} g_{t_i,t_{i+1}}\right| \leq \sum_{i=1}^{n} \left|g_{t_i,t_{i+1}}\right| \leq C(\delta)\,|D|^{3/p-1}\,\omega(0,t)$$

*which tends to $0$ as $|D| \to 0$.*

This elementary remark may be strengthened to apply to rough paths. The difficulty of course that the second (and higher) order processes are no longer additive with respect to $(s,t)$. The following lemma is due to Lyons [26], and is used to powerful effect in his Extension Theorem.

**Lemma 2.12** *Suppose $(x, \mathbb{X}), (y, \mathbb{Y}) \in R_p(V)$ satisfy $a_{s,t} := x_{s,t} - y_{s,t} \simeq 0$ and $\mathbb{A}_{s,t} := \mathbb{X}_{s,t} - \mathbb{Y}_{s,t} \simeq 0$. Then the two rough paths coincide, i.e. $(x, \mathbb{X}) = (y, \mathbb{Y})$. In particular, this taking $(y, \mathbb{Y})$ to be the zero rough path in $R_p(V)$ we may conclude if $(x, \mathbb{X}) \in R_p(V)$ satisfies $x_{s,t} \simeq 0$ and $\mathbb{X}_{s,t} \simeq 0$, then $x_{st} = 0$ and $\mathbb{X}_{s,t} = 0$ for all $0 \leq s \leq t \leq T$.*

**Proof.** Since $a_{s,t}$ is additive we must have for every partition $D$ of $[s,t]$

$$|a_{s,t}| = \left|\sum_{i:t_i \in D} a_{t_i,t_{i+1}}\right| \to 0 \text{ as } |D| \to 0,$$

and hence $x_{s,t} = y_{s,t}$. It follows from [30], Lemma 3.4 that $\mathbb{A}_{s,t}$ is also additive and repeating the argument with $\mathbb{A}_{s,t}$ in place of $a_{s,t}$ yields the claim. ∎

We say a functional $\mathbf{Z} := (z, \mathbb{Z})$ defined by

$$\mathbf{Z}_{s,t} := 1 + z_{s,t} + \mathbb{Z}_{s,t} \in T_2(V), \ \forall \ (s,t) \in \Delta_{[0,T]}$$

is an **almost rough path** if it satisfies the requirements of Definition 2.4 except identity (2.1), but instead

$$\mathbb{Z}_{s,u} - \mathbb{Z}_{s,t} - \mathbb{Z}_{t,u} - z_{s,t} \otimes z_{t,u} \simeq 0,$$

holds, i.e. it approximately satisfies the multiplicative identity (2.1).The following theorem due to Lyons is a cornerstone for the development of the integration for rough paths. It states that for every almost rough path there exists a unique rough path that is "close." Note that the uniqueness follows from Lemma 2.12 above.

**Theorem 2.13** *Let $\mathbf{Z} := (z, \mathbb{Z})$ be an almost rough path on $V$. Then there exists a unique rough path $\mathbf{X} = (x, \mathbb{X}) \in R_p(V)$ such that $x_{s,t} \simeq z_{s,t}$ and $\mathbb{Z}_{s,t} \simeq \mathbb{X}_{s,t}$.*

The following result due to Terry Lyons [26] allows us to define the integral of a rough path against a sufficiently regular one form.

**Theorem 2.14** *Suppose that $\mathbf{Z} \in WG_p(V)$ and $\alpha \in C^2(V, \mathrm{End}(V,W))$ is a one form on $V$ with values in $W$. Then there is a unique $\mathbf{X} \in WG_p(W)$ such that $x_0 = 0$,*

$$X^1_{s,t} \simeq \alpha(z_s) Z^1_{s,t} + \alpha'(z_s) \mathbb{Z}_{s,t}, \tag{2.7}$$

*and*

$$\mathbb{X}_{s,t} \simeq \alpha(z_s) \otimes \alpha(z_s) \mathbb{Z}_{s,t}. \tag{2.8}$$

In the future we will denote this $\mathbf{X}$ by $\int \alpha(d\mathbf{Z})$ and use it as the definition for the rough integral. The proof is a consequence of Theorem 2.13. The rough path integral has a number of important properties, in particular the map taking

$$\mathbf{Z} \to \int \alpha(d\mathbf{Z})$$

is continuous in the rough path metric $(A.1)$.

## 2.3 Rough differential equations

The following definition of a rough differential equation (RDE) is in the spirit of Davie [8] and may be found for example in Friz, Hairer [15, Proposition 8.4].

**Definition 2.15 (RDE)** *Let $\mathbf{Z} \in WG_p(W)$ and $Y : V \to \mathrm{Hom}(W,V)$ be a $C^1$ – map. Then $\mathbf{X} \in WG_p(E)$ solves the RDE,*

$$d\mathbf{X} = Y(x)\, d\mathbf{Z} \tag{2.9}$$

*if and only if*

$$\begin{aligned} x_{s,t} &\simeq Y(x_s) z_{s,t} + Y'(x_s) Y(x_s) \mathbb{Z}_{s,t} \\ \mathbb{X}_{s,t} &\simeq [Y(x_s) \otimes Y(x_s)] \mathbb{Z}_{s,t} \end{aligned}$$

*where*

$$Y'(x_s) Y(x_s) [a \otimes b] := (Y'(x_s) Y(x_s) a) b = \left(\partial_{Y(x_s)a} Y\right)(x_s) b.$$

*Alternatively if we let $Y_b(x) := Y(x) b$, then*

$$Y'(x_s) Y(x_s) [a \otimes b] := Y'_b(x_s) Y_a(x_s) = \left(\partial_{Y_a(x_s)} Y_b\right)(x_s).$$

Existence and uniqueness of solutions for RDEs defined by sufficiently regular vector fields is due to Lyons [26]. The following theorem is an easy consequence of Theorem 10.14 of [17, Theorem 10.14].

**Theorem 2.16 (RDE existence and uniqueness)** *Let* $p \in [2,3)$, $\mathbf{Z} \in WG_p(W,[0,T])$, $Y : V \to \mathrm{Hom}(W,V)$ *be a smooth map and for* $k \in \{0,1,2,\dots\}$, *let*

$$\left\|Y^{(k)}\right\|_\infty := \sup\left\{\|(\partial_{v_1}\dots\partial_{v_k}Y)(x)\|_{\mathrm{Hom}(W,V)} : x \in V \text{ and } v_i \in V \text{ with } \|v_i\|_V = 1\right\}.$$

*If*

$$M_Y := \max\{\|Y\|_\infty, \|Y'\|_\infty, \|Y''\|_\infty\} < \infty, \tag{2.10}$$

*then there exists a unique* $\mathbf{X} \in WG_p(V,[0,T])$ *that solves the RDE* (2.9) *over* $[0,T]$ *in the sense of Definition 2.15. In addition, there exists a constant* $C_p$ *(depending only on* $p$*) such that*

$$\|x\|_{p-var;[u,v]} \le C_p \max\left(M\|\mathbf{Z}\|_{p-var;[u,v]}, M^p\|\mathbf{Z}\|^p_{p-var;[u,v]}\right) \ \forall\ 0 \le u < v \le T. \tag{2.11}$$

The following corollary is a locallization of Theorem 2.16 which will prove useful later.

**Corollary 2.17 (Local RDE existence)** *Let* $U \subset V$ *be an open neighborhood ,* $U_1$ *be a precompact open neighborhood with closure in* $U$, *and* $Y : U \to \mathrm{Hom}(W,V)$ *be a smooth map. Then there exists* $\delta > 0$ *such that for all* $(x,t_0) \in U_1 \times [0,T]$,

$$d\mathbf{X} = Y(x)\,d\mathbf{Z},\ x_{t_0} = x \tag{2.12}$$

*has a unique solution* $\mathbf{X} \in WG_p(V,[t_0,t_0+\delta\wedge T])$ *in the sense of Definition 2.15 (naturally with trace* $x_t \in U$ *for all* $t \in [t_0,t_0+\delta\wedge T]$*).*

**Proof.** Choose another open precompact subset, $U_2$, of $V$ so that $\bar{U}_1 \subset U_2 \subset \bar{U}_2 \subset U$ and choose $\varphi \in C_c^\infty(U)$ such that $\varphi = 1$ on $\bar{U}_2$. Let $\tilde{Y} = \varphi Y$ which we then extend to be zero outside of $U$. Clearly, $M_{\tilde{Y}} < \infty$ where $M_{\tilde{Y}}$ is as in Eq. (2.10) with $Y$ replaced by $\tilde{Y}$.

Recall that if $u(s,t) := \|\mathbf{Z}\|_{p-var;[s,t]}$ for $(s,t) \in \Delta_{[0,T]}$, then $u^p(s,t)$ is a control and in particular, $u(s,t)$ is continuous on $\Delta_{[0,T]}$ and vanishes on the diagonal. Therefore if $\varepsilon := \mathrm{dist}(U_1, U_2^c) > 0$, then there exists (by the uniform continuity of $u$) a $\delta > 0$ such that

$$C_p \max\left(M_{\tilde{Y}}\|\mathbf{Z}\|_{p-var;[t_0,t_0+\delta\wedge T]}, M^p_{\tilde{Y}}\|\mathbf{Z}\|^p_{p-var;[t_0,t_0+\delta\wedge T]}\right) < \varepsilon\ \forall\ t_0 \in [0,T].$$

By Theorem 2.16, given any $(x,t_0) \in U_1 \times [0,T]$ there exists a unique $\mathbf{X} \in WG_p(V,[t_0,T])$ that solves

$$d\mathbf{X} = \tilde{Y}(x)\,d\mathbf{Z},\ x_{t_0} = x. \tag{2.13}$$

By the choice of $\delta$, the bound in Eq. (2.11), and the triangle inequality, it follows that $x_t \in U_2 \subseteq U$ for all $t \in [t_0,t_0+\delta\wedge T]$. As $Y = \tilde{Y}$ on $U_2$ it follows that $\mathbf{X}$ also solves (2.12) on $[t_0,t_0+\delta\wedge T]$. ■

The solutions of rough differential satisfy a universal limit theorem which states that the map taking $\mathbf{X}$ to the solution $\mathbf{Z}$ is continuous in the $p-$ variation metric on rough paths (see [26]). We also remark that the original definition of the solution of a rough differentiable equations (see Lyons [26]) is given in terms of a fixed point of a rough integral on $V \oplus W$.

The next lemma implies that for sufficiently regular vector fields an RDE solution blows up if and only if both the trace *and* the second-order process of the solution explode. In other words, it is not possible for the explosion of a solution of an RDE to be caused only by the explosion of the second-order process of the solution.

**Lemma 2.18 (Augmentations for free)** *Let* $\mathbf{Z} \in WG_p(W)$ *and* $Y : V \to \mathrm{Hom}(W,V)$ *be a smooth map and consider the RDE*

$$d\mathbf{X} = Y(x)\,d\mathbf{Z} \text{ with } x(0) = x_0 \tag{2.14}$$

*where* $x_0$ *is given. Suppose that we can solve this equation for the trace part, i.e. we can find a path* $x$ *such that*

$$x_{s,t} \simeq Y(x_s)\,z_{s,t} + Y'(x_s)\,Y(x_s)\,\mathbb{Z}_{s,t} \tag{2.15}$$

*holds for* $0 \le s,t \le T$. *Then there exists a lift* $\mathbf{X} \in WG_p(V)$ *of* $x$ *that solves* (2.14) *over* $[0,T]$.

**Proof.** We can augment the trace solution $x$ to a full rough path solution $\mathbf{X} := (x, \mathbb{X})$ as follows. Let

$$\mathbb{A}_{s,t} := [Y(x_s) \otimes Y(x_s)] \mathbb{Z}_{s,t} \text{ and}$$
$$\mathbf{A}_{s,t} := 1 + x_{s,t} + \mathbb{A}_{s,t}.$$

Note that $Y$ is bounded on $x$ and therefore $\mathbf{A}$ has finite $p-$variation in the sense of (2.2). It now suffices to check that $\mathbf{A}$ is an almost multiplicative functional in the language of Lyons. For this it will be enough to check that that $\mathbb{A}$ approximately (in the sense of Notation 2.10) satisfies Chen's identity, which we now do. If $0 \leq s \leq t \leq u \leq T$, then

$$\mathbb{A}_{t,u} = [Y(x_t) \otimes Y(x_t)] \mathbb{Z}_{t,u} \simeq [Y(x_s) \otimes Y(x_s)] \mathbb{Z}_{t,u}$$

so that

$$\begin{aligned} \mathbb{A}_{s,u} - \mathbb{A}_{s,t} - \mathbb{A}_{t,u} &\simeq [Y(x_s) \otimes Y(x_s)] [\mathbb{Z}_{s,u} - \mathbb{Z}_{s,t} - \mathbb{Z}_{t,u}] \\ &= [Y(x_s) \otimes Y(x_s)] [z_{s,t} \otimes z_{t,u}]. \end{aligned} \tag{2.16}$$

Similarly we have

$$x_{s,t} \otimes x_{t,u} \simeq Y(x_s) z_{s,t} \otimes Y(x_t) z_{t,u} \simeq Y(x_s) z_{s,t} \otimes Y(x_s) z_{t,u}$$

which combined with Eq. (2.16) shows

$$\mathbb{A}_{s,u} - \mathbb{A}_{s,t} - \mathbb{A}_{t,u} - x_{s,t} \otimes x_{t,u} \simeq 0$$

which is to say that $\mathbf{A}_{s,t}$ is an almost multiplicative functional. Thus by Theorem 2.13 there exists $\mathbb{X}_{s,t}$ such that $\mathbb{X}_{s,t} \simeq \mathbb{A}_{s,t}$ and $\mathbf{X}_{s,t} = 1 + x_{s,t} + \mathbb{X}_{s,t}$ solves the RDE in Eq. (2.14). ■

# 3 Geometric and weakly geometric rough paths on manifolds

In this section we will introduce the notions of geometric and weakly geometric rough paths on manifolds. The section is split in four parts. Subsection 3.1 introduces the basic geometric notations and facts needed for the rest of the paper. The definitions of constrained rough paths (now called geometric and weakly geometric rough paths) and their path integrals are the introduced in Subsection 3.2, see Definitions 3.15, 3.17, and 3.24. Basic properties of these definitions are then established. The main result of Subsection 3.3, is Proposition 3.35 which gives a more effective criteria for checking that an ambient rough path is in fact a weakly geometric rough path. The final subsection (3.4) explores the behavior of constrained rough paths under change of coordinates and more general smooth transformations, see Proposition 3.38. This result is then used to demonstrate that our constrained rough paths may be formulated to be independent of the choice of embedding, see Corollaries 3.40, 3.41, and Definition 3.42.

## 3.1 Basic geometric definitions

Let $E = \mathbb{R}^N$ and $E' = \mathbb{R}^{N'}$ be Euclidean spaces and let $\langle a, b \rangle = a \cdot b = \sum_{i=1}^N a_i b_i$ for all $a, b \in E$. If $U$ is an open neighborhood in $E$ and $F : U \to E'$ is a smooth map, then for $x \in U$ and $v \in E$ we let $\partial_v F(x) := \frac{d}{dt}|_0 F(x + tv)$ be the directional derivative of $F$ at $x$ along $v$. We will further let $F'(x) : E \to E'$ and $F''(x) : E \otimes E \to V := \mathbb{R}^{N-d}$ be the differential and Hessian of $F$ respectively which are defined by $F'(x) v := (\partial_v F)(x)$ and $F''(x) [v \otimes w] := (\partial_v \partial_w F)(x)$ for all $x \in U$ and $v, w \in E$.

Throughout the rest of this paper, $M^d$ will be a $d-$dimensional embedded submanifold of a Euclidean space $E := \mathbb{R}^N$. The reader may find the necessary geometric background in any number of places including, [10,22,32]. To fix notation let us recall a formulation an embedded submanifold which will be most useful for our purposes.

**Definition 3.1** *A subset $M$ of $E$ is an* ***embedded submanifold*** *of $E$ of dimension $d \in \{1, \ldots, N\}$ provided for each $m \in M$ there is an open neighborhood $U$ in $E$ containing $m$ and smooth* ***local defining function*** *$F : U \to \mathbb{R}^{N-d}$ such that*

$$U \cap M = \{x \in U : F(x) = 0\}$$

*and $F'(x) : E \to \mathbb{R}^{N-d}$ is surjective for $x \in U$.*

Recall that the **tangent plane** to $M$ at $m \in M$ is $\tau_m M := \mathrm{Nul}(F'(m))$. Because of the implicit function theorem, to each $v \in \tau_m M$ there exists a smooth path $\sigma_v : (-\varepsilon, \varepsilon) \to E$ such that $\sigma_v((-\varepsilon, \varepsilon)) \subset M$, $\sigma_v(0) = m$, and $\sigma_v'(0) = v$. From these considerations, one shows $\tau_m M \ni v \to \dot{\sigma}_v(0) \in T_m M$ is a linear isomorphism of vector spaces; we will often use this isomorphism to identify $\tau_m M$ with $T_m M$.

**Remark 3.2** *Around each point $m \in M$ there exists an open set $U$ in $E$ and a smooth map $\pi : U \to M \cap U$ such that $\pi(x) = x$ for all $x \in M \cap U$. As a consequence of this fact, any smooth function $f : M \to W$ defined near $m$ has a smooth extension, $f \circ \pi$, to a neighborhood of $m$ in $E$.*

**Notation 3.3** *Letting $F : U \to \mathbb{R}^{N-d}$ be a local defining function for $M$ as above, we define smooth functions $Q_F, P_F : U \to \mathrm{End}(E)$ by*

$$Q_F(x) := F'(x)^* \left(F'(x)F'(x)^*\right)^{-1} F'(x) \text{ and} \tag{3.1}$$

$$P_F(x) := I_E - Q_F(x) = I - F'(x)^* \left(F'(x)F'(x)^*\right)^{-1} F'(x). \tag{3.2}$$

**Remark 3.4** *We make a number of comments.*

1. *The surjectivity assumption of $F'(x)$ guarantees that $F'(x)F'(x)^*$ is invertible.*
2. *One may easily verify that $Q_F(x)$ is orthogonal projection onto $\mathrm{Ran}(F'(x)^*) = \mathrm{Nul}(F'(x))^\perp$ and $P_F(x)$ is orthogonal projection onto $\mathrm{Nul}(F'(x))$.*
3. *For $m \in U \cap M$ we have that $P_F(m)$ $(Q_F(m))$ is the orthogonal projection onto $\tau_m M$ $\left([\tau_m M]^\perp\right)$ and hence is independent of the choice of local defining function. We will simply write $P(m)$ and $Q(m)$ (or, sometimes, $P_m$ and $Q_m$) for $P_F(m)$ and $Q_F(m)$ when $m \in M$.*

**Remark 3.5** *In the proofs that follow we will often use the following identities*

$$F'(x) = F'(x)\, Q_F(x) \text{ and } Q_F(x) = A_F(x)\, F'(x) \tag{3.3}$$

*which hold for all $x \in M$, where*

$$A_F(x) := F'(x)^*(F'(x)F'(x)^*)^{-1} \in \mathrm{Hom}\left(\mathbb{R}^{N-d}, E\right). \tag{3.4}$$

The last geometric notions we need are vector fields, one forms, and their covariant derivatives.

**Definition 3.6 (Vector Fields)** *A* ***smooth vector field*** *on $M$ is a smooth function $Y : M \to E$ such that $Q(m)Y(m) = 0$ for all $m \in M$, i.e. $Y(m) \in T_m M$ for all $m \in M$. Let $\Gamma(TM)$ denote the collection of smooth vector fields on $M$.*

**Example 3.7** *For $z \in \mathbb{R}^N$ we let $V_z \in \Gamma(TM)$ be defined by $V_z(x) := P_x z$ for all $x \in M$.*

**Definition 3.8** *A* ***smooth one form*** *on $M$ with values in a finite dimensional vector space $W$ is a smooth function $\alpha$ on $M$ with $\alpha_m \in \mathrm{Hom}(T_m M, W)$ for all $m \in M$. Here we can describe the smoothness assumption of $\alpha$ by requiring $M \in m \to \alpha_m P_m \in \mathrm{Hom}(E, W)$ to be a smooth function. Let $\Omega^1(M, W)$ denote the set of smooth one forms on $M$ with values in $W$.*

**Example 3.9** *The function $\alpha_m := P_m$ is in $\Omega^1(M, E)$. If $f : M \to W$ is a smooth function then $\alpha := df \in \Omega^1(M, W)$ where as usual $df(v_m) = v_m f$.*

**Definition 3.10 (Levi-Civita Covariant Derivative)** *Suppose that* $v_m \in T_m M$, $Y \in \Gamma(TM)$, *and* $\alpha \in \Omega^1(M, W)$, *then the covariant derivative at* $v_m$ *of* $Y$ *and* $\alpha$ *are given respectively by*

$$\nabla_{v_m} Y = P_m (\partial_v Y)(m) \in T_m M \text{ and } \nabla_{v_m} \alpha = \partial_{v_m} (\alpha \circ P) \in \mathrm{Hom}(T_m M, W).$$

The next lemma and proposition records some basic well known properties of the Levi-Civita covariant derivative.

**Lemma 3.11** *If* $P$ *and* $Q$ *be the orthogonal projection operators as in Remark 3.4, then* $dP = -dQ$ *and* $PdQ = dQP$.

**Proof.** Differentiate the identities $I = P + Q$ and $0 = PQ$, which hold on $M$ gives the new identities in the statement. ■

**Proposition 3.12** *Let* $Y \in \Gamma(TM)$, $\alpha \in \Omega^1(M, W)$, *and* $\Gamma := dQ \in \Omega^1(M, \mathrm{End}(E))$. *Then;*

1. $\nabla_{v_m} Y = (\partial_v Y)(m) + \Gamma(v_m) Y(m)$,

2. *The product rule holds;*

$$v_m(\alpha(Y)) = (\nabla_{v_m} \alpha)(Y(m)) + \alpha_m(\nabla_{v_m} Y).$$

3. *If* $\alpha_m = \tilde{\alpha}_m|_{T_m M}$ *where* $\tilde{\alpha} : M \to \mathrm{Hom}(E, W)$ *is a smooth function then*

$$\nabla_{v_m} \alpha = (\partial_v \tilde{\alpha})(m) P_m - \tilde{\alpha}_m \Gamma(v_m).$$

4. *If* $\alpha_m = \tilde{\alpha}_m|_{T_m M}$ *as in item 3. and we further assume that* $\tilde{\alpha}_x = \tilde{\alpha}_x \circ P_x$ *for* $x \in M$ *near* $m$, *then*

$$(\nabla_v \alpha)(w) = \tilde{\alpha}'_m [v \otimes w] \text{ for all } v, w \in T_m M.$$

**Proof.** In the proof below recall that $Y = PY$ as $Y \in \Gamma(TM)$.

1. Differentiating the identity, $Y = PY$, shows

$$(\partial_v Y)(m) = dP(v_m) Y(m) + P(m)(\partial_v Y)(m) = -\Gamma(v_m) Y(m) + \nabla_{v_m} Y$$

which proves item 1.

2. Since $\alpha(Y) = \alpha(PY) = (\alpha P) Y$ and $\alpha P : M \to \mathrm{End}(E)$ is a smooth function the ordinary product rule shows,

$$\begin{aligned} v_m(\alpha(Y)) &= (\partial_v(\alpha P)(m)) Y(m) + \alpha_m P_m (\partial_v Y)(m) \\ &= (\nabla_{v_m} \alpha)(Y(m)) + \alpha_m(\nabla_{v_m} Y). \end{aligned}$$

3. If $\alpha_m = \tilde{\alpha}_m|_{T_m M}$ as in item 3., then using the standard product rule again,

$$\nabla_{v_m} \alpha = \partial_v(\alpha P)(m) = \partial_v(\tilde{\alpha} P)(m) = (\partial_v \tilde{\alpha})_m P_m + \tilde{\alpha}_m dP(v_m) = (\partial_v \tilde{\alpha})(m) P_m - \tilde{\alpha}_m \Gamma(v_m).$$

4. From the definitions,

$$(\nabla_v \alpha)(w) = [v_m(\alpha \circ P)] w = [v_m(\tilde{\alpha} \circ P)] w = [v_m \tilde{\alpha}] w = \tilde{\alpha}'_m(v \otimes w).$$

■

## 3.2 (Weakly) Geometric Rough Paths on $M$

In the following let $M$ be a manifold embedded in $E := \mathbb{R}^N$ and $F$ the (local) defining function as introduced in Notation 3.3. In the setting of embedded manifolds there is a natural notion of geometric rough paths that is induced by the rough metric on the ambient Euclidean space $E$. To help prepare the precise definition of a geometric rough path on a manifold we introduce the following set of paths.

Assume that $M \subseteq E$ and let $C_{bv}([0,T], E)$ denote the set of continuous bounded variation paths taking values in $E$. Recall the definition of the truncated signature $S_2$ in (2.4). For any real number $p \in [2,3)$, we define $\bar{G}_p(M)$ to be closure of the lifts of continuous bounded variation paths in $M$; that is, $\bar{G}_p(M)$ is the closure of

$$\{S_2(x) : x \in C_{bv}([0,T], E), x_t \in M \text{ for all } t \in [0,T]\}$$

with respect to the topology induced by the $p-$ variation rough path metric on $E$.

**Remark 3.13** *For $p > 1$ continuous bounded variation paths on $E$ are in the closure of the smooth paths taken in the $p-$variation metric (see e.g. [17, Lemma 5.30]). In addition the truncated signature is a locally Lipschitz continuous map under the (inhomogeneous) rough $p-$ variation metric (see e.g. [17, Theorem 9.10]). Combining these two facts shows we could have replaced $C_{bv}([0,T], E)$ in the definition of $\bar{G}_p(M)$ by the smooth paths $C^\infty([0,T], E)$. This justifies referring to the lifts of $1-$ rough paths as "smooth" rough paths.*

**Lemma 3.14** *Suppose $M$ is a closed subset of $E$, then the trace of any $\mathbf{X}$ in $\bar{G}_p(M)$ lies in $M$.*

**Proof.** By definition $\mathbf{X}$ can be approximated by a sequence of smooth rough paths $\mathbf{X}^n$ (see Remark 3.13) with trace in $M$. The traces of the approximating sequence converges in $p-$variation and therefore also converges pointwise. Since $M$ is assumed to be closed, the proof is complete. ■

**Definition 3.15 (Geometric rough paths)** *We define* ***geometric*** *$p-$***rough paths** *on $M$ to be those elements of $\bar{G}_p(M)$ whose trace $x$ lies inside $M$. The set of geometric $p-$rough paths on $M$ will be denoted by $G_p(M)$. In other words, we have*

$$G_p(M) = \left\{\mathbf{X} = (x, \mathbb{X}) \in \bar{G}_p(M) : x_t \in M \text{ for all } t \in [0,T]\right\}.$$

It follows from Lemma 3.14 that $\bar{G}_p(M) = G_p(M)$ when $M$ is a closed subset of $E$. The next example explains why it is important that we take the closure of paths in $M$, and why it will not be sufficient to *only* assume that the trace of limiting object lies in $M$.

**Example 3.16** *Let $M = \{e_1, e_2\}^\perp \subset \mathbb{R}^N$. Then for any $v, w \in \mathbb{R}^N$ there exists (see [30], [15]) a so-called pure area geometric rough path, $\mathbf{X} = (x, \mathbb{X})$ with the property that $x = 0$, the constant path zero, and*

$$\mathbb{X}_{s,t} = (v \otimes w - w \otimes v)(t-s).$$

*On the other hand if $\mathbf{X} \in WG_p(M)$ we would certainly have $\mathbb{X}_{s,t} \in M \otimes M$ for all $s$ and $t$. Put another way if $\mathbf{X} \in WG_p(M)$ then $[Q \otimes I]\mathbb{X}_{s,t} = 0 = [I \otimes Q]\mathbb{X}_{s,t}$ where $Q$ is orthogonal projection onto $M^\perp = \text{span}\{e_1, e_2\}$. An approximate version of this requirement will appear again in the general manifold setting as well, see Corollary 3.20 below.*

A second set of rough paths on a manifold is, in structure, related to the weakly geometric rough paths in the classical Banach space setting.

**Definition 3.17 (Weakly geometric rough paths)** *We say that $\mathbf{X} = (x, \mathbb{X})$ is a* ***weakly geometric*** *$p-$* ***rough path*** *on the manifold $M$ if: $\mathbf{X}$ is in $WG_p(E)$, its trace $x$ lies in $M$ and for any finite dimensional subspace $W$ and any $\tilde{\alpha} \in \Omega^1(E, W)$ such that $\tilde{\alpha}|_{TM} \equiv 0$ we have $\int \tilde{\alpha}(\mathbf{dX}) \equiv 0$. The set of weakly geometric rough paths will be denoted by $WG_p(M)$.*

In the following we will often make use of the following simple consequence of Taylor's theorem.

**Lemma 3.18** *If $f : E := \mathbb{R}^N \to \mathbb{R}^l$ is a $C^3$ – function which is constant on $M \subset E$, then for $x, y \in M$ we have,*

1. $f'(x)(y-x) = O\left(|y-x|^2\right)$, *and*

2. $f'(x)(y-x) + \frac{1}{2} f''(x)\left[(y-x)\otimes(y-x)\right] = O\left(|y-x|^3\right)$.

**Proof.** By Taylor's theorem,

$$f(y) - f(x) = f'(x)(y-x) + O\left(|y-x|^2\right)$$

and

$$f(y) - f(x) = f'(x)(y-x) + \frac{1}{2} f''(x)\left[(y-x)\otimes(y-x)\right] + O\left(|y-x|^3\right).$$

Since $f$ is constant on $M$ and $x, y \in M$, it follows that $f(y) - f(x) = 0$ and the results follow from the previously displayed equations. ■

An obvious class of one forms having the property that $\alpha|_{TM} \equiv 0$ are those which locally have the form $\alpha = \varphi F'$ , where $\varphi$ is a smooth function and $F$ is a local defining function for the manifold. The following lemma gives simplified description of the level-one component for the integral of any such one form.

**Lemma 3.19** *Let $\mathbf{X} = (x, \mathbb{X}) \in WG_p(E)$ be a weakly geometric $p-$rough path such that the trace $x$ is in $M$. Suppose that $F \in C^\infty(U, V)$, with $V = \mathbb{R}^{N-d}$, is a smooth function which locally defines $M$ as in Definition 3.1 and which has been chosen so that there is a subinterval $[s,t] \subseteq [0,T]$ with*

$$\{x_u : u \in [s,t]\} \subset U.$$

*Assume $W$ is a finite dimensional vector space, and suppose $\varphi \in C^\infty(U, \mathrm{Hom}(V, W))$. Let $\alpha \in \Omega^1(E, W)$ be the one form defined by*

$$\alpha(x)\xi = \varphi(x) F'(x)\xi \in W \text{ for all } \xi \in E.$$

*Then for every $[u,v] \subseteq [s,t]$ we have*

$$\left[\int \alpha(d\mathbf{X})\right]^1_{u,v} \simeq \alpha(x_u) x_{u,v} + \alpha'(x_s)\mathbb{X}_{u,v} \simeq \left[(\varphi' \cdot F')(x_u)\right]\mathbb{X}_{u,v} \tag{3.5}$$

*where $(\varphi' \cdot F')(m)$ denotes the linear map from $E \otimes E \to W$ determined by*

$$\left[(\varphi' \cdot F')(m)\right]\xi_1 \otimes \xi_2 := \left[\varphi'(m)\xi_1\right]\left[F'(m)\xi_2\right] = \left(\partial_{\xi_1}\varphi\right)(m)\left(\partial_{\xi_2}F\right)(m). \tag{3.6}$$

**Proof.** The product rule (written in the notation introduced in Eq. (3.6)) gives

$$\alpha' = \varphi F'' + \varphi' \cdot F'. \tag{3.7}$$

This identity combined with Eq. (2.7) then implies,

$$\begin{aligned}\left[\int \alpha(d\mathbf{X})\right]^1_{u,v} &\simeq \alpha(x_u) x_{u,v} + \alpha'(x_u)\mathbb{X}_{u,v} \\ &= \varphi(x_u) F'(x_u) x_{u,v} + \left[\varphi(x_u) F''(x_u) + (\varphi' \cdot F')(x_u)\right]\mathbb{X}_{u,v} \\ &= \varphi(x_u)\left[F'(x_u) x_{u,v} + F''(x_u)\mathbb{X}_{u,v}\right] + \left[(\varphi' \cdot F')(x_u)\right]\mathbb{X}_{u,v}.\end{aligned} \tag{3.8}$$

Since $F''$ is symmetric and $\mathbf{X} = (x, \mathbb{X})$ is a weakly geometric rough path it follows that

$$F''(x_u)\mathbb{X}_{u,v} = F''(x_u)\mathbb{X}^s_{u,v} = \frac{1}{2} F''(x_u)\left[x_{u,v} \otimes x_{u,v}\right]$$

and therefore by Lemma 3.18,

$$F'(x_u) x_{u,v} + F''(x_u)\mathbb{X}_{u,v} = F'(x_u) x_{u,v} + \frac{1}{2} F''(x_u)\left[x_{u,v} \otimes x_{u,v}\right] \simeq 0.$$

Combining this estimate with Eq. (3.8) gives (3.5). ■

**Corollary 3.20** *Let* $\mathbf{X} = (x, \mathbb{X}) \in WG_p(M) \subset WG_p(E)$ *and* $F : U \to V := \mathbb{R}^{N-d}$ *and* $[s,t] \subseteq [0,T]$ *be as in Lemma 3.19. Then for* $s \le u \le v \le t$,

$$I_E \otimes F'(x_u)\,\mathbb{X}_{u,v} \simeq 0 \simeq F'(x_u) \otimes I_E \mathbb{X}_{u,v} \text{ and} \tag{3.9}$$

$$I_E \otimes Q(x_u)\,\mathbb{X}_{u,v} \simeq 0 \simeq Q(x_u) \otimes I_E \mathbb{X}_{u,v}, \tag{3.10}$$

*where* $Q$ *is defined in Notation 3.3 and Remark 3.4.*

**Proof.** Choose $\varphi \in C_c^\infty(U,E)$ such that $\varphi(x) = x$ for $x$ in a neighborhood $\{x_u : u \in [s,t]\}$ and let $\alpha \in \Omega^1(E, E \otimes V)$ be defined by

$$\alpha(\xi_x) := \varphi(x) \otimes dF(\xi_x) = \varphi(x) \otimes F'(x)\,\xi \text{ for all } \xi_x \in TE \cong E \times E.$$

Then $\alpha|_{TM} = 0$ and therefore $\int \alpha(d\mathbf{X}) \equiv 0$. By Theorem 2.14, Lemma 3.19, and the fact that $\varphi'(x_u) = I_E$, it follows that

$$0 \simeq \alpha(x_u)\,x_{u,v} + \alpha'(x_u)\,\mathbb{X}_{u,v} \simeq [(\varphi' \otimes F')(x_u)]\,\mathbb{X}_{u,v} = I \otimes F'(x_u)\,\mathbb{X}_{u,v}$$

for $s \le u \le v \le t$ and the left member of Eq. (3.9) is proved. This also easily proves the left member of Eq. (3.10) since

$$I_E \otimes Q(x_u)\,\mathbb{X}_{u,v} = [I_E \otimes A_F(x_u)]\, I_E \otimes F'(x_u)\,\mathbb{X}_{u,v}$$

where $A_F(x) := F'(x)^*(F'(x)F'(x)^*)^{-1} \in \operatorname{Hom}\left(\mathbb{R}^{N-d}, E\right)$ as in Remark 3.5. The other approximate identities in Eqs. (3.9) and (3.10) follow similarly, one need only now define $\alpha \in \Omega^1(E, V \otimes E)$ by

$$\alpha(\xi_x) := dF(\xi_x) \otimes \varphi(x) = F'(x)\,\xi \otimes \varphi(x) \text{ for all } \xi_x \in TE \cong E \times E.$$

■

**Remark 3.21** *The conditions in Eqs. (3.9) and (3.10) are equivalent. Indeed the proof of Corollary 3.20 has already shown Eq. (3.9) implies Eq. (3.10). For the converse direction we need only observe that* $F'(x_u) = F'(x_u)\,Q(x_u)$ *so that, for example,*

$$I_E \otimes F'(x_u)\,\mathbb{X}_{u,v} = [I_E \otimes F'(x_u)]\, I_E \otimes Q(x_u)\,\mathbb{X}_{u,v}.$$

**Corollary 3.22** *If* $\mathbf{X} = (x, \mathbb{X}) \in WG_p(M)$, *then* $\mathbb{X}_{s,t} \simeq [P_{x_s} \otimes P_{x_s}]\,\mathbb{X}_{s,t}$.

**Proof.** The result follows by observing that

$$\mathbb{X}_{s,t} = (P_{x_s} + Q_{x_s}) \otimes (P_{x_s} + Q_{x_s})\,\mathbb{X}_{s,t}$$

and then using Eq. (3.10) to conclude $(I_E \otimes Q_{x_s})\,\mathbb{X}_{s,t} \simeq 0$, $(Q_{x_s} \otimes I_E)\,\mathbb{X}_{s,t} \simeq 0$, and

$$(Q_{x_s} \otimes Q_{x_s})\,\mathbb{X}_{s,t} = (I_E \otimes Q_{x_s})\,(Q_{x_s} \otimes I_E)\,\mathbb{X}_{s,t} \simeq 0.$$

■

The following lemma prepares the definition of the integral of a rough path against smooth one forms.

**Lemma 3.23** *Suppose* $\mathbf{X} \in WG_p(M)$, $U$ *is an open neighborhood of* $M$, *and* $\alpha,\ \beta \in \Omega^1(U,W)$. *If* $\alpha|_{TM} = \beta|_{TM}$, *then*

$$\int \alpha(d\mathbf{X}) = \int \beta(d\mathbf{X}). \tag{3.11}$$

**Proof.** The one form, $\psi := \alpha - \beta \in \Omega^1(U,W)$, vanishes on $TM$ and so by Definition 3.17 $\int \psi(d\mathbf{X}) \equiv \mathbf{0}$. As the rough path integral is linear on $\Omega^1(U,W)$ at level one it immediately follows that

$$0 = \left[\int \psi(d\mathbf{X})\right]^1_{s,t} = \left[\int \alpha(d\mathbf{X})\right]^1_{s,t} - \left[\int \beta(d\mathbf{X})\right]^1_{s,t}.$$

Moreover by Corollary 3.22,

$$\begin{aligned}\left[\int \alpha(d\mathbf{X})\right]^2_{s,t} &\simeq \alpha_{x_s}\otimes\alpha_{x_s}\mathbb{X}_{s,t}\simeq \alpha_{x_s}\otimes\alpha_{x_s}\left[P_{x_s}\otimes P_{x_s}\right]\mathbb{X}_{s,t}\\ &=\beta_{x_s}\otimes\beta_{x_s}\left[P_{x_s}\otimes P_{x_s}\right]\mathbb{X}_{s,t}\simeq \beta_{x_s}\otimes\beta_{x_s}\mathbb{X}_{s,t}\\ &\simeq \left[\int \beta(d\mathbf{X})\right]^2_{s,t}.\end{aligned}$$

The last two displayed equations along with Lemma 2.12 now gives Eq. (3.11). ■

Another proof of this lemma could be given along the lines of Proposition 3.29 below. The previous lemma justifies the following definition of integration $\alpha\in\Omega^1(M,W)$ along a weakly geometric rough path $\mathbf{X}\in WG_p(M)$.

**Definition 3.24** *The **rough path integral** of a rough path $\mathbf{X}\in WG_p(M)$ along a smooth one form $\alpha\in\Omega^1(M,W)$ is defined by*

$$\mathbf{Y}=\int\tilde{\alpha}(d\mathbf{X}) \text{ as in Theorem 2.14} \tag{3.12}$$

*where $\tilde{\alpha}\in\Omega^1(U,W)$ is any extension of $\alpha$ to a one form on some open neighborhood $M$ in $E$. [Later, in Proposition 3.29, we will show how to characterize $\int\alpha(d\mathbf{X})$ without using any extension of $\alpha$.]*

As a corollary we immediately see that the rough integrals against smooth one forms are sufficient to characterise a rough path.

**Corollary 3.25** *Suppose that $\mathbf{X}$ and $\mathbf{Y}$ are two elements of $WG_p(M)$ such that $x=y$, and which satisfy*

$$\int\alpha(d\mathbf{X})=\int\alpha(d\mathbf{Y})$$

*for all $\alpha\in\Omega^1(M,V)$. Then $\mathbf{X}=\mathbf{Y}$ as elements of $WG_p(E)$.*

**Proof.** Let $\tilde{\alpha}\in\Omega^1(E,V)$ so that $\tilde{\alpha}$ is a smooth extension of $\alpha=\tilde{\alpha}|_{TM}$. By Lemma 3.23 we have

$$\int_s^t\tilde{\alpha}(d\mathbf{X})=\int_s^t\alpha(d\mathbf{X})=\int_s^t\alpha(d\mathbf{Y})=\int_s^t\tilde{\alpha}(d\mathbf{Y}).$$

Let $\mathbb{A}_{s,t}:=\mathbb{Y}_{s,t}-\mathbb{X}_{s,t}.$ It follows that

$$\tilde{\alpha}_{x_s}(x_{s,t})+\tilde{\alpha}'_{x_s}(\mathbb{X}_{s,t})\simeq\tilde{\alpha}_{x_s}(y_{s,t})+\tilde{\alpha}'_{x_s}(\mathbb{Y}_{s,t})\simeq\tilde{\alpha}_{x_s}(x_{s,t})+\tilde{\alpha}'_{x_s}(\mathbb{Y}_{s,t})$$

which implies $\tilde{\alpha}'_{x_s}(\mathbb{A}_{s,t})\simeq 0$ for every $\tilde{\alpha}\in\Omega^1(E,V)$ and every $s$ and $t$ in $[0,T]$. If we choose $\tilde{\alpha}\in\Omega^1(E,E\otimes E)$ to be defined by $\tilde{\alpha}(\xi_x)=\tilde{\alpha}_x\xi=x\otimes\xi$, then $\tilde{\alpha}'_{x_s}[\eta\otimes\xi]=\eta\otimes\xi$ for all $\eta,\xi\in E$. So for this $\tilde{\alpha}$ it follows that $\mathbb{A}_{s,t}=\tilde{\alpha}'_{x_s}(\mathbb{A}_{s,t})\simeq 0$ and the result follows from Lemma 2.12. ■

Analogous to the Banach space setting every geometric $p$ -rough path on a manifold is a weakly geometric $p$ -rough path.

**Proposition 3.26** *For $p\in[2,3)$ we have*

1. *$G_p(M)\subseteq WG_p(M)$.*
2. *Suppose $\mathbf{X}\in WG_p(E)$ and $\mathbf{X}\in G_{p'}(M)$ for some $3>p'>p$, then $\mathbf{X}\in WG_p(M)$.*

**Proof.** Let $\mathbf{X} \in G_p(M)$. For the first claim, we note by definition there exists a sequence $x_n$ of smooth paths in $M$ such that the lifts $\mathbf{X}_n := S_2(x_n) \in G_p(M) \subset G_p(E)$ converge to $\mathbf{X}$ in the rough $p$ -variation metric on $G_p(E)$. Let $\tilde{\alpha} \in \Omega^1(E, W)$ be such that $\tilde{\alpha}|_{TM} \equiv 0$,then

$$\int \tilde{\alpha}(dx_n) = 0, \text{ hence } 0 = S_2\left(\int \tilde{\alpha}(dx_n)\right) = \int \tilde{\alpha}(d\mathbf{X}_n) \to \int \tilde{\alpha}(d\mathbf{X}) \tag{3.13}$$

as $n \to \infty$. By definition the trace $x$ lies in $M$, and it is immediate that we have $\mathbf{X} \in WG_p(M)$. For the second claim we approximate $\mathbf{X}$ in $p'$ variation to deduce that (3.13) holds provided $\tilde{\alpha}|_{TM} \equiv 0$. ■

In the following we will frequently rely on localisation arguments.

**Remark 3.27 (localisation)** *Suppose $\mathbf{X} = (x, \mathbb{X}) \in WG_p(E)$ has its trace, $x$, lying in $M$. By a simple compactness argument, there exists $k \in \mathbb{N}$, open subsets $U_i$ of $E$, and local defining functions, $F_i : U_i \to \mathbb{R}^{N-d}$, as in Definition 3.1 for $1 \leq i \leq k$ such that $\{U_i\}_{i=1}^k$ is an open cover of $x([0,T])$. Furthermore, since $x$ is uniformly continuous, we can find $\delta = \delta(\mathbf{X}) > 0$, such that for all $s$ and $t$ in the interval $[0,T]$ with $|s-t| < \delta$ the path segment*

$$\{x_u : u \in [s,t]\} \subset U_i \tag{3.14}$$

*for some $i \in \{1, ...., k\}$.*

The next result describes the constraints on $x_{s,t}$ which arise when $\mathbf{X} \in WG_p(M)$ – also see Example 3.30 below.

**Lemma 3.28** *If $\mathbf{X} \in WG_p(M)$ then*

$$Q_{x_s} x_{s,t} \simeq Q_{x_s} \left(\partial_{P_{x_s} a} P b\right)_{a \otimes b = \mathbb{X}_{s,t}}. \tag{3.15}$$

**Proof.** Let $\tilde{\alpha}(\xi_x) = Q_F(x)\xi$ so that $\tilde{\alpha} \in \Omega^1(U, E)$. Then $\tilde{\alpha}|_{T(M \cap U)} \equiv 0$ and therefore by Definition 3.17 and Corollary 3.22,

$$0 = \left[\int \tilde{\alpha}(d\mathbf{X})\right]^1_{s,t} \simeq \tilde{\alpha}_{x_s}(x_{st}) + \tilde{\alpha}'_{x_s}\mathbb{X}_{st} \simeq Q_{x_s} x_{st} + \tilde{\alpha}'_{x_s}\left[P_{x_s} \otimes P_{x_s}\mathbb{X}_{st}\right]. \tag{3.16}$$

Solving Eq. (3.16) for $Q_{x_s} x_{st}$ completes the proof after using the identity,

$$\tilde{\alpha}'_{x_s}\left[P_{x_s} a \otimes P_{x_s} b\right] = dQ\left(P_{x_s} a\right) P_{x_s} b = -Q_{x_s} dP\left(P_{x_s} a\right) P_{x_s} b \;\forall\; a, b \in E,$$

wherein the last inequality made use of Lemma 3.11 and the fact that $P^2 = P$. It is easily seen that this agrees with (3.15). ■

We conclude this section with a theorem that provides a more explicit description of the integral of one forms along $\mathbf{X} \in WG_p([0,T], M)$ which require no extensions of the one form to the ambient space.

**Proposition 3.29 (Integrating one forms without extensions)** *If $\mathbf{X} \in WG_p([0,T], M)$ and $\alpha \in \Omega^1(M, W)$, then*

$$\left[\int \alpha(d\mathbf{X})\right]^1_{s,t} \simeq \alpha_{x_s}\left(P_{x_s} x_{st}\right) + (\nabla\alpha)\left(\left[P_{x_s} \otimes P_{x_s}\right]\mathbb{X}_{s,t}\right) \tag{3.17}$$

*and*

$$\left[\int \alpha(d\mathbf{X})\right]^2_{s,t} \simeq \alpha_{x_s} \otimes \alpha_{x_s}\left[P_{x_s} \otimes P_{x_s}\right]\mathbb{X}_{s,t}, \tag{3.18}$$

*where $\nabla\alpha$ is the Levi-Civita covariant derivative of $\alpha$ as in Definition 3.10.*

**Proof.** By Definition 3.24,

$$\int_s^t \alpha\,(d\mathbf{X}) = \int_s^t \tilde{\alpha}\,(d\mathbf{X}) \simeq \left(\tilde{\alpha}_{x_s}\,(x_{st}) + \tilde{\alpha}'_{x_s}\mathbb{X}_{st}, \left[\tilde{\alpha}_{x_s} \otimes \tilde{\alpha}_{x_s}\right]\mathbb{X}_{s,t}\right) \tag{3.19}$$

where $\tilde{\alpha}$ is any extension of $\alpha$ to an open neighborhood of $M$ in $E = \mathbb{R}^N$. By Corollary 3.22, $\mathbb{X}_{s,t} \simeq [P_{x_s} \otimes P_{x_s}]\,\mathbb{X}_{s,t}$ and hence we may replace Eq. (3.19) by

$$\begin{aligned}\int_s^t \alpha\,(d\mathbf{X}) &\simeq \left(\tilde{\alpha}_{x_s}\,(x_{s,t}) + \tilde{\alpha}'_{x_s}\,[P_{x_s} \otimes P_{x_s}]\,\mathbb{X}_{s,t}, [\tilde{\alpha}_{x_s} \otimes \tilde{\alpha}_{x_s}]\,[P_{x_s} \otimes P_{x_s}]\,\mathbb{X}_{s,t}\right) &(3.20)\\ &\simeq \left(\tilde{\alpha}_{x_s}\,(x_{s,t}) + \tilde{\alpha}'_{x_s}\,[P_{x_s} \otimes P_{x_s}]\,\mathbb{X}_{s,t}, \alpha_{x_s} \otimes \alpha_{x_s}\,[P_{x_s} \otimes P_{x_s}]\,\mathbb{X}_{s,t}\right). &(3.21)\end{aligned}$$

Let us now use Remark 3.2 to locally extend $P$ to a neighborhood of $M$ so that $P = P \circ \pi$. By replacing $\tilde{\alpha}$ by $\tilde{\alpha}P$ if necessary we may assume $\tilde{\alpha} = \tilde{\alpha}P$. Under this assumption, Eq. (3.21) becomes,

$$\int_s^t \alpha\,(d\mathbf{X}) \simeq \left(\alpha_{x_s}\,(P_{x_s}x_{s,t}) + \tilde{\alpha}'_{x_s}\,[P_{x_s} \otimes P_{x_s}]\,\mathbb{X}_{s,t}, \alpha_{x_s} \otimes \alpha_{x_s}\,[P_{x_s} \otimes P_{x_s}]\,\mathbb{X}_{s,t}\right). \tag{3.22}$$

From item 4. of Proposition 3.12,

$$\tilde{\alpha}'_m\,[v \otimes w] = (\nabla_v \alpha)\,(w) \text{ for all } v, w \in T_mM$$

which combined with Eq. (3.22) proves Eqs. (3.17) and (3.18). ∎

## 3.3 Characterising Weakly Geometric Rough Paths on $M$

The goal of this subsection is to show $\mathbf{X} = (x, \mathbb{X}) \in WG_p\,(E)$ is in $WG_p\,(M)$ iff $x_t \in M$ for all $0 \le t \le T$ and that either of the equivalent Eqs. (3.9) or (3.10) holds locally. This will be carried out in Proposition 3.35 below. The next example shows the result in Lemma 3.28 is really about paths in $x_t \in M$ and not so much about its augmentation to a rough path.

**Example 3.30** *Let $x_t$ be any path $M$ with $|x_{s,t}| \le C\omega\,(s,t)^{1/p}$. Then*

$$\begin{aligned}0 = [F\,(x)]_{st} &= F'\,(x_s)\,x_{s,t} + \frac{1}{2}F''\,(x_s)\,x_{s,t} \otimes x_{s,t} + O\left(|x_{s,t}|^3\right)\\ &\simeq F'\,(x_s)\,x_{s,t} + \frac{1}{2}F''\,(x_s)\,[x_{s,t} \otimes x_{s,t}]\,.\end{aligned}$$

*Applying $A_F\,(x_s) \in \mathrm{Hom}\left(\mathbb{R}^{N-d}, E\right)$ (see Remark 3.5) to this equation then shows*

$$Q\,(x_s)\,x_{s,t} \simeq -\frac{1}{2}A_F\,(x_s)\,F''\,(x_s)\,[x_{s,t} \otimes x_{s,t}]\,. \tag{3.23}$$

*From this equation it follows that $x_{s,t} = P\,(x_s)\,x_{s,t} + O\left(|x_{s,t}|^2\right)$ and so we may replace $x_{st} \otimes x_{st}$ in Eq. (3.23) by $P\,(x_s)\,x_{s,t} \otimes P\,(x_s)\,x_{s,t}$ which allows us to rewrite Eq. (3.23) as*

$$Q\,(x_s)\,x_{s,t} \simeq -\frac{1}{2}A_F\,(x_s)\,F''\,(x_s)\,[P\,(x_s)\,x_{s,t} \otimes P\,(x_s)\,x_{s,t}]\,. \tag{3.24}$$

*So if $x_t \in M$ for all $t$, the component of $x_{s,t}$ orthogonal to $\tau_{x_s}M$ is determined modulo terms of order $|x_{s,t}|^3$ by knowing the component of $x_{s,t}$ tangential to $M$ at $x_s$.*

**Lemma 3.31** *Suppose $\mathbf{X} = (x, \mathbb{X}) \in WG_p\,(E)$ such that $x_t \in M$ for all $t \in [0,T]$, then*

$$(I \otimes Q\,(x_s))\,[\mathbb{X}_{s,t}]^s \simeq 0. \tag{3.25}$$

**Proof.** Note that by the definition of $\simeq$ it is sufficient to check (3.25) locally for all $0 < s < t < T$ such that $|t-s| < \delta$ and some $\delta > 0$. Let $\{U_i : i = 1, ...., k\}$ and $F_i$ as in Remark 3.27 be a cover of the trace $x$. By construction of the cover for all $0 \le s < t \le T$ with $|s-t| < \delta$ there exists $U_i$ such that (3.14) holds. By (3.1) we may assume that for $m \in U_i$ we have $Q(m) = A(m) F_i'(m)$, where $A(m) := F_i'(m)^* (F_i'(m) F_i'(m)^*)^{-1}$. From Eq. (2.6) which holds by definition of $\mathbf{X}$ being in $WG_p(E)$, it follows that

$$(I \otimes Q(x_s)) [\mathbb{X}_{s,t}]^s = \frac{1}{2} (I \otimes Q(x_s)) x_{s,t} \otimes x_{s,t} = \frac{1}{2} x_{s,t} \otimes A(x_s) F_i'(x_s) x_{s,t}.$$

Applying item 1. of Lemma 3.18 to the right member of this equations gives the estimate;

$$|(I \otimes Q(x_s)) [\mathbb{X}_{s,t}]^s| \le \frac{1}{2} |A(x_s)| |x_{s,t}| |F_i'(x_s) x_{s,t}| \le C |x_{s,t}|^3 \simeq 0.$$

■

**Corollary 3.32** *If* $\mathbf{X}$ *is an element of* $WG_p(E)$ *such that the trace* $x$ *is in* $M$*, then the following are equivalent:*

1. $(I_E \otimes Q(x_s)) [\mathbb{X}_{s,t}]^a \simeq 0$,
2. $(I_E \otimes Q(x_s)) [\mathbb{X}_{s,t}] \simeq 0$, *and*
3. $(I_E \otimes F'(x_s)) [\mathbb{X}_{s,t}] \simeq 0$ *over the interval* $[u, v]$, *whenever* $F$ *is a local defining function for* $M$ *on* $U$ *in the sense of Definition 3.1, and the path segment of* $x$ *over* $[u, v]$ *satisfies*

$$\{x_r : r \in [u, v]\} \subset U.$$

**Proof.** The equivalence of items 1. and 2. is an immediate corollary of Lemma 3.31. The equivalence of items 2. and 3. is the content of Remark 3.21. ■

**Remark 3.33** *If* $\mathbf{X} \in WG_p(E)$*, then the condition* $(I_E \otimes Q(x_s)) [\mathbb{X}_{s,t}] \simeq 0$ *is equivalent to the condition that* $(Q(x_s) \otimes I_E) [\mathbb{X}_{s,t}] \simeq 0$*. To see this is the case we let* $\mathcal{F} : E \otimes E \to E \otimes E$ *denote the linear flip operator determined by* $\mathcal{F}[a \otimes b] = b \otimes a$ *for all* $a, b \in E$*. Then*

$$\begin{aligned} \mathcal{F}(Q(x_s) \otimes I_E) [\mathbb{X}_{s,t}] &= (I_E \otimes Q(x_s)) [\mathcal{F}\mathbb{X}_{s,t}] \\ &= (I_E \otimes Q(x_s)) \left[\mathcal{F}\mathbb{X}_{s,t}^s + \mathcal{F}\mathbb{X}_{s,t}^a\right] \\ &= (I_E \otimes Q(x_s)) \left[\mathbb{X}_{s,t}^s - \mathbb{X}_{s,t}^a\right] \\ &\simeq (I_E \otimes Q(x_s)) \left[-\mathbb{X}_{s,t}^s - \mathbb{X}_{s,t}^a\right] \\ &= -(I_E \otimes Q(x_s)) [\mathbb{X}_{s,t}] \end{aligned}$$

*where in the second to last line we have used* $I_E \otimes Q(x_s) \mathbb{X}_{s,t}^s = \frac{1}{2} x_{s,t} \otimes Q(x_s) x_{s,t} \simeq 0$.

The next proposition shows Definition 3.17 above and Definition 3.34 below for the notion of a weakly geometric rough path are equivalent.

**Definition 3.34 (Projection Definition of Weakly Geometric Rough Paths)** *We say that* $\mathbf{X} = (x, \mathbb{X})$ *is a* ***weakly geometric*** $p-$ ***rough path*** *on the manifold* $M$ *if:* $\mathbf{X}$ *is in* $WG_p(E)$*, its trace* $x$ *lies in* $M$*, and* $\mathbb{X}$ *satisfies*

$$(I_E \otimes Q(x_s)) [\mathbb{X}_{s,t}]^a \simeq 0 \simeq (I_E \otimes Q(x_s)) \mathbb{X}_{s,t}, \tag{3.26}$$

*wherein* $Q$ *is the orthogonal projection onto the normal bundle as in Notation 3.3 and* $I_E$ *is the identity map on* $E$*.*

**Proposition 3.35 (The Projection Characterization of** $WG_p(M)$**)** *Let* $\mathbf{X} = (x, \mathbb{X}) \in WG_p(E)$ *then* $\mathbf{X} \in WG_p(M)$ *(Definition 3.17) if and only the trace* $x$ *is in* $M$ *and any one of the equivalent conditions in Corollary 3.32 hold.*

**Proof.** ( $\Longrightarrow$ ) This implication has already been demonstrated in Corollary 3.20 and Remarks 3.21 and 3.27.

($\Longleftarrow$) For the converse implication assume $x_t \in M$ for all $t \in [0,T]$ and (again, locally)

$$[I_E \otimes F'(x_s)] \mathbb{X}_{s,t} \simeq 0 \text{ and } [I_E \otimes Q(x_s)][\mathbb{X}_{s,t}] \simeq 0. \tag{3.27}$$

We have to show for any finite dimensional vector space $W$ that

$$\int \alpha(d\mathbf{X}) \equiv 0 \;\forall\; \alpha \in \Omega^1(E,W) \;\;\ni\;\; \alpha|_{TM} \equiv 0. \tag{3.28}$$

The proof will proceed in several stages, considering first one-forms with specific structures, and finally combining those results to deduce the general claim. In what follows we let

$$\mathbf{Y} = (y, \mathbb{Y}) := \int \alpha(d\mathbf{X})$$

and let $Q$, $Q_F$, and $A_F$ be as in Notation 3.3 and Remarks 3.4 and 3.5.

**Case 1.** We begin by supposing that $\alpha = \varphi dF = \varphi F' \in \Omega^1(E,W)$ for some $\varphi \in C_c^\infty(E, \mathrm{Hom}(V,W))$ with $\mathrm{supp}(\varphi) \subset U$ . By Eq. (3.5) of Lemma 3.19 and Eq. (2.7) we learn that

$$y_{s,t} \simeq \alpha(x_s) x_{s,t} + \alpha'(x_s) \mathbb{X}_{s,t} \simeq [\varphi'(x_s) \cdot F'(x_s)] \mathbb{X}_{s,t} \simeq 0 \tag{3.29}$$

where for the last approximation we have used the assumption on Eq. (3.27). Similarly,

$$\mathbb{Y}_{s,t} \simeq \alpha(x_s) \otimes \alpha(x_s) \mathbb{X}_{s,t} = [\varphi(x_s) \otimes \varphi(x_s)][F'(x_s) \otimes I_E][I_E \otimes F'(x_s)] \mathbb{X}_{s,t} \simeq 0. \tag{3.30}$$

Equations (3.29) and (3.30) along with Lemma 2.12 shows $y_{s,t} = 0$ and $\mathbb{Y}_{s,t} \equiv 0$ for all $s$ and $t$.

**Case 2.** Now suppose $\alpha = \beta \circ Q_F$ where $\beta \in \Omega^1(E,W)$ is any one form on $E$. Then locally we have

$$\alpha(x)\xi = \beta(x) Q_F(x)\xi = \beta(x) A_F(x) F'(x)\xi = \varphi(x) F'(x)\xi$$

where $\varphi(x) := \beta(x) A_F(x)$ . We conclude by using case 1 and a suitable application of Remark 3.27.

**Case 3.** Now assume that $\beta \in \Omega^1(E,W)$ is a one-form such that $\beta(m) \equiv 0$ for all $m \in M$. If $\sigma(t)$ is a path in $M$, then $\beta(\sigma(t)) = 0$ and therefore $0 = \frac{d}{dt}\beta(\sigma(t)) = \beta'(\sigma(t))\dot{\sigma}(t)$ . Since $\sigma(t) \in M$ is arbitrary, it follows that $(\partial_{v_m}\beta) = 0$ for all $v_m \in T_m M$. Hence we conclude that

$$(\partial_\xi \beta)_m = (\partial_{P_m\xi}\beta + \partial_{Q_m\xi}\beta)_m = (\partial_{Q_m\xi}\beta)_m \;\;\forall\; m \in M \text{ and } \xi \in E = \mathbb{R}^N$$

or in other words,

$$\beta'(m) = \beta'(m)[Q(m) \otimes I_E] \text{ for all } m \in M.$$

With this in hand, using Lemma 3.18 and Eq. (3.27) again, we find that

$$\left[\int \beta(d\mathbf{X})\right]^1_{s,t} \cong \beta(x_s) x_{s,t} + \beta'(x_s) \mathbb{X}_{s,t} = \beta'(x_s)[Q(x_s) \otimes I_E] \mathbb{X}_{s,t} \simeq 0. \tag{3.31}$$

As usual this together with the additivity of the trace shows $\left[\int \beta(d\mathbf{X})\right]^1_{s,t} = 0$. Then, working as above, the second-order process is given by

$$\left[\int \beta(d\mathbf{X})\right]^2_{s,t} \simeq [\beta(x_s) \otimes \beta(x_s)] \mathbb{X}_{s,t} = [0 \otimes 0] \mathbb{X}_{s,t} = 0.$$

**Case 4.** Finally, if $\alpha$ is any one form on $E$ with the property that $\alpha|_{TM} \equiv 0$, then $\alpha = \alpha \circ Q_F$ on[1] $M$. We now let $\beta := \alpha - \alpha \circ Q_F$ so that $\beta(m) \equiv 0$ for all $m \in M$. Thus we have decomposed $\alpha$ as $\alpha = \beta + \alpha \circ Q_F$ where $\beta \equiv 0$ on $M$ and therefore by cases 2. and 3.,

$$y := \left[\int \alpha(d\mathbf{X})\right]^1 = \left[\int \beta(d\mathbf{X})\right]^1 + \left[\int (\alpha \circ Q_F)(d\mathbf{X})\right]^1 = 0 + 0 = 0.$$

[1] A slightly subtle point here is that $\alpha = \alpha \circ Q$ on $M$ but not necessarily on a neighborhood of $M$. For this reason we can not directly use case 2. here.

We further have, using

$$\alpha_m = \beta_m + \alpha_m Q_m = \alpha_m Q_m \ \forall\ m \in M,$$

and Eq. (3.27) that

$$\begin{aligned}\mathbb{Y}_{s,t} &:= \left[\int \alpha\left(d\mathbf{X}\right)\right]^2_{s,t} \simeq \left[\alpha_{x_s} Q_{x_s} \otimes \alpha_{x_s} Q_{x_s}\right] \mathbb{X}_{s,t} \\ &= \left[\alpha_{x_s} \otimes Q_{x_s}\right]\left[I_E \otimes Q_{x_s}\right]\left[Q_{x_s} \otimes I_E\right] \mathbb{X}_{s,t} \simeq 0.\end{aligned}$$

An application of Lemma 2.12 then shows $\mathbb{Y}_{s,t} \equiv 0$. ■

The defining property in Eq. (3.26) is local and we therefore need a remark analogous to Lemma A.1, which allows us to concatenate rough paths on manifolds.

**Remark 3.36 (gluing)** *Suppose that $D = \{0 = t_0 < t_1 < \cdots < t_n = T\}$ is any partition of $[0, T]$. Let $\delta > 0$, and suppose that the overlapping intervals $J_k$ for $1 \leq k \leq n$ are defined by*

$$J_k = [t_{k-1}, \min(t_k + \delta, T)]$$

*Assume, for each $k$, we are given $\mathbf{X}(k) \in WG_p(J_k, M)$ such that $\mathbf{X}(k)_{s,t} = \mathbf{X}(j)_{s,t}$ for $s, t \in J_k \cap J_j$ and any $i$ and $j$. Then, fixing a starting point $x_0 \in M$, there exists a unique $\mathbf{X} \in WG_p([0, T], M)$ with $x(0) = x_0$ which is consistent with the $\mathbf{X}(k)$s in the sense that for all $1 \leq k \leq n$,*

$$\mathbf{X}(k)_{s,t} = \mathbf{X}_{s,t} \text{ for all } s, t \in J_k.$$

## 3.4 Push forwards and independence of the choice of embedding

Analogous to the Banach space setting (see A.2) we may consider the pushfoward of rough paths on manifolds under sufficiently smooth maps.

**Definition 3.37 (Pushed-forward rough paths )** *Let $M$ and $N$ respectively be smooth embedded submanifolds of the Euclidean spaces $E$ and $E'$. Suppose that $\varphi : M \to N$ is smooth and let $d\varphi \in \Omega^1(M, E')$, i.e. we regard $d\varphi$ as an $E'$ – valued one form. Then if $\mathbf{X}$ is an element of $WG_p(M)$, we define the pushed-forward rough path $\varphi_*(\mathbf{X})$ in $E'$ by setting*

$$\varphi_*(\mathbf{X}) := \int d\varphi\left(d\mathbf{X}\right) = \int \varphi'(x)\, d\mathbf{X},$$

*and taking the starting point to be $\varphi(x_0)$.*

**Proposition 3.38 (Pushing forward rough paths)** *Let $\mathbf{X} \in WG_p(M)$. The rough path $\varphi_*\mathbf{X}$ in Definition 3.37 satisfies;*

1. *$[\varphi_*\mathbf{X}]^1_{s,t} = \varphi(x_t) - \varphi(x_s) \in E'$ for all $s, t \in [0, T]$.*
2. *$\varphi_*\mathbf{X}$ is an element of $WG_p(N)$.*
3. *If $L$ is another smooth submanifold which is embedded in the Euclidean space $E''$ and if $\psi : N \to L$ is a smooth map, then*

$$\psi_*\left[\varphi_*(\mathbf{X})\right] = \left[\psi \circ \varphi\right]_*(\mathbf{X}).$$

4. *If $\beta \in \Omega^1(N, V)$, then*

$$\int \beta\left(d\left[\varphi_*(\mathbf{X})\right]\right) = \int \left(\varphi^*\beta\right)\left(d\mathbf{X}\right).$$

**Proof.** We take each item in turn.

1. If $\varphi : M \to N$ is a smooth map between embedded submanifolds it may be viewed (at least locally) as the restriction of a smooth map from $\Phi : E \to E'$. It then follows that $d\Phi$ is an extension of $d\varphi$ to a neighborhood of $M$ and therefore by Definition 3.24, $\varphi_*(\mathbf{X}) = \Phi_*(\mathbf{X})$, and hence from Lemma A.4 we have that

$$[\varphi_*(\mathbf{X})]^1_{s,t} = [\Phi_*(\mathbf{X})]^1_{s,t} = \Phi(x_t) - \Phi(x_s) = \varphi(x_t) - \varphi(x_s).$$

2. Since $\varphi_*(\mathbf{X}) = \Phi_*(\mathbf{X})$, it follows that $\varphi_*(\mathbf{X}) \in WG_p(E')$. Moreover if $\alpha \in \Omega^1(E', W)$ is such that $\alpha|_{TN} \equiv 0$ then by Theorem A.5

$$\int \alpha(d[\varphi_*(\mathbf{X})]) = \int \alpha(d[\Phi_*(\mathbf{X})]) = \int [\alpha \circ \Phi']\, d\mathbf{X} = 0$$

as $\alpha \circ \Phi' = \Phi^*\alpha \in \Omega^1(E, W)$ which vanishes on $TM$. We deduce from Definition 3.17 and 1. that $\varphi_*(\mathbf{X}) \in WG_p(N)$.

3. Follows by a similar argument to 2. using Corollary A.6.

4. This is a consequence of Theorem A.5 (once again using that $\varphi_*(\mathbf{X}) = \Phi_*(\mathbf{X})$, and the fact that $\Phi^*\beta$ restricts to $\varphi^*\beta$).

■

**Example 3.39** *Suppose that $\varphi : M \to M$ is the identity map, then $\varphi = \Phi|_E$ where $\Phi : E \to E$ is the identity map and therefore,*

$$\varphi_*(\mathbf{X}) = \int d\varphi(d\mathbf{X}) = \int d\Phi(d\mathbf{X}) = \mathbf{X}.$$

The preceding example is a special case of the more general fact that diffeomorphisms give rise to bijections between the respective sets of weakly geometric rough paths on two embedded manifolds. The following corollary is immediate from Proposition 3.38.

**Corollary 3.40** *Let $M, N$ be embedded manifolds and $\varphi : M \to N$ a diffeomorphism. Then the function $\varphi_*$ is a bijection between $WG_p(M)$ and $WG_p(N)$.*

Suppose now $\mathcal{M}$ is an abstract manifold embedded as $M$ and $\tilde{M}$ in two vector spaces $E$ and $\widetilde{E}$. Then there exist smooth maps $f : \mathcal{M} \to E$ and $\tilde{f} : \mathcal{M} \to \tilde{E}$ diffeomorphic onto their image such that $f(\mathcal{M}) = M$ and $\tilde{f}(\mathcal{M}) = \tilde{M}$. The following corollary shows that we have a natural identification between the rough paths on $M$ and $\tilde{M}$. The map we construct is natural in the sense that it respects the integration of one forms (characterizing the rough paths, cf. Corollary 3.25).

**Corollary 3.41** *Let $\mathcal{M}$, $M$, $\tilde{M}$ as above. Then the pushforward $\left(\tilde{f} \circ f^{-1}\right)_*$ is a bijective map from $WG_p(M)$ to $WG_p\left(\tilde{M}\right)$ such that for any finite dimensional vector space valued on form $\alpha \in \Omega^1(\mathcal{M}, W)$ and any $\mathbf{X} \in WG_p(M)$*

$$\int \left(\left(f^{-1}\right)^* \alpha\right)(d\mathbf{X}) = \int \left(\left(\tilde{f}^{-1}\right)^* \alpha\right)\left(d\left(\left[\tilde{f} \circ f^{-1}\right]_* \mathbf{X}\right)\right).$$

**Definition 3.42 (Abstract weakly geometric rough paths)** *Let $\mathcal{M}$ be an abstract manifold and suppose that $f : \mathcal{M} \to M \subset E$ and $\tilde{f} : \mathcal{M} \to \tilde{M} \subset \tilde{E}$ are two embedding of $\mathcal{M}$ into Euclidean spaces $E$ and $\tilde{E}$ respectively. We say that $(f, \mathbf{X})$ and $\left(\tilde{f}, \tilde{\mathbf{X}}\right)$, where $\mathbf{X} \in WG_p(M)$ and $\tilde{\mathbf{X}} \in WG_p\left(\tilde{M}\right)$, are equivalent*[2] *if*

$$\tilde{\mathbf{X}} = \left(\tilde{f} \circ f^{-1}\right)_* (\mathbf{X}).$$

[2] This is an equivalence relation because of item 3. of Proposition 3.38.

*The equivalence class associated to* $(f, \mathbf{X})$ *will be denoted by* $[(f, \mathbf{X})]$. *The* ***weakly geometric rough paths on*** $\mathcal{M}$ *is the collection of these equivalence classes;*

$$WG_p(\mathcal{M}) := \{[(f, \mathbf{X})] : \mathbf{X} \in WG_p(M)\}.$$

If $\alpha \in \Omega^1(\mathcal{M}, W)$ and $[(f, \mathbf{X})] \in WG_p(\mathcal{M})$ then we define

$$Z_{[(f,\mathbf{X})]}(\alpha) := \int \alpha(d[(f, \mathbf{X})]) := \int \left(\left(f^{-1}\right)^* \alpha\right)(d\mathbf{X}) \in WG_p(W).$$

Because of Corollary 3.40, $Z_{[(f,\mathbf{X})]}$ is well defined and because of Corollary 3.25, knowledge of $f$ and $Z_{[(f,\mathbf{X})]}$ uniquely determines $\mathbf{X}$. The functionals $Z_{[(f,\mathbf{X})]}$ are closely related to the notion of manifold valued rough paths as introduced in [6]. An alternative, more explicit, proof of the independence of the embedding for the rough paths will be given in Cass, Driver, Litterer [2] where another intrinsic notion of rough paths will be developed.

# 4 RDEs on manifolds and consequences

In this section we consider rough differential equations constrained to $M$, see Definition 4.1 below. Theorem 4.2 gives the basic existence uniqueness results for constrained RDEs (rough differential equations). The extrinsic Definition 4.1 is shown in Theorem 4.5 to be equivalent to a pair of intrinsic notions of solutions for constrained RDEs. In Example 4.12, we use constrained RDEs to give examples of weakly geometric rough paths on $M$ and then in Theorem 4.18 we show that all $\mathbf{X} \in WG_p(M)$ arise as in Example 4.12. The relationships between $WG_p(M)$ and $G_p(M)$ is spelled out in Theorem 4.17 and a summary of all of our characterizations of $WG_p(M)$ is then given in Theorem 4.18. As an illustration of our results, in subsection 4.3 we study RDEs on a Lie group $G$ whose dynamics are determined by right invariant vector fields on $G$. This added right invariance assumption guarantees that the resulting RDEs have global solutions, see Theorem 4.20.

## 4.1 Rough differential equations on $M$

**Definition 4.1 (Constrained RDE)** *Let* $x_0 \in M$, $Y : \mathbb{R}^n \to \Gamma(TM)$ *be a linear function, and* $\mathbf{Z} \in WG_p(\mathbb{R}^n)$ *be given. We say* $\mathbf{X} \in WG_p(M)$ *solves the RDE*

$$d\mathbf{X}_t = Y_{d\mathbf{Z}_t}(x_t) \text{ with } x(0) = x_0 \in M \tag{4.1}$$

*provided,*

$$x_{s,t} \simeq Y_{z_{s,t}}(x_s) + (\partial_{Y_a} Y_b)(x_s)|_{a \otimes b = \mathbb{Z}_{st}} \text{ and} \tag{4.2}$$

$$\mathbb{X}_{s,t} \simeq Y_a(x_s) \otimes Y_b(x_s)|_{a \otimes b = \mathbb{Z}_{s,t}}. \tag{4.3}$$

Notice that $Y_a(x_s) \in T_{x_s}M$ and therefore there exists a smooth curve $\sigma(t) \in M$ such that $\dot{\sigma}(0) = Y_a(x_s)$ and we then compute $(\partial_{Y_a} Y_b)(x_s)$ using

$$(\partial_{Y_a} Y_b)(x_s) = \frac{d}{dt}|_0 Y_b(\sigma(t)) \in E.$$

This comment shows that the above definition makes sense but it is not yet clear that there is a (local in time) solution to the RDE (4.1). If $\mathbf{X} \in WG_p(M)$ solves Eq. (4.1), $U$ is an open (in $E$) neighborhood of $M$, and $\tilde{Y} : \mathbb{R}^n \to \Gamma(TU)$ is a linear map such that $\tilde{Y}_a = Y_a$ on $M$, then $\mathbf{X}$ solves the standard Euclidean space RDE

$$d\mathbf{X}_t = \tilde{Y}_{d\mathbf{Z}_t}(x_t) \text{ with } x(0) = x_0 \in M. \tag{4.4}$$

From these considerations we see that if there exists $\mathbf{X} \in WG_p(M)$ solving Eq. (4.1) then this solution may be described as the unique solution $\mathbf{X} \in WG_p(E)$ to Eq. (4.4). We will use this remark in our proof of the existence Theorem 4.2 below to Eq. (4.1). Once this is accomplished we develop in Theorem 4.5 alternative intrinsic characterizations of solutions to the RDE in Eq. (4.1).

**Theorem 4.2** *There is a unique solution $\mathbf{X} \in WG_p(M)$ (possibly up to explosion time) of the RDE (4.1). Moreover, either $\mathbf{X}$ exists on all of $[0,T]$ or there exists a $\tau \in [0,T]$ such that $\mathbf{X}$ exists on $[0,\tau)$ and $\overline{\{x(t) : 0 \le t < \tau\}}^M$ is not compact in $M$.*

**Proof.** Let $U$ be a neighborhood of $x_0 \in M$ and $F : U \to \mathbb{R}^k$ be a local defining function of $M$ as in Definition 3.1. We then let $\tilde{Y}_a := P_F[Y_a \circ \pi]$ where $\pi$ is as in Remark 3.2 and $P_F$ is the projection map in Notation 3.3. Let $\mathbf{X} = (x, \mathbb{X}) \in WG_p(E)$ be the RDE solution to Eq. (4.4) defined up to the first exit time $\tau$ from $U$ where we let $\tau = \infty$ if $x_t \in M$ for all $0 \le t \le T$. We are now going to show $x(t) \in M \cap U$ for $0 \le t < \tau$.

Notice by construction that $\tilde{Y}_b = Y_b$ on $M \cap U$ and $F'\tilde{Y}_b = 0$ on $U$. Differentiating this last equation along $\xi = \tilde{Y}_a$ then further implies,

$$F''\tilde{Y}_a \otimes \tilde{Y}_b + F'\partial_{\tilde{Y}_a}\tilde{Y}_b = 0. \tag{4.5}$$

Recall that $\mathbf{X}$ solves Eq. (4.4) iff

$$x_{s,t} \simeq \tilde{Y}_{z_{s,t}}(x_s) + \left(\partial_{\tilde{Y}_a(x_s)}\tilde{Y}_b\right)(x_s)|_{a\otimes b = \mathbb{Z}_{s,t}} \text{ and } \mathbb{X}_{s,t} \simeq \left[\tilde{Y}(x_s) \otimes \tilde{Y}(x_s)\right]\mathbb{Z}_{s,t}. \tag{4.6}$$

Using the first approximate identity in Eq. (4.6) along with $F'\tilde{Y} = 0$ shows

$$\begin{aligned}
[F(x.)]_{s,t} :=& F(x_t) - F(x_s) = F(x_s + x_{s,t}) - F(x_s) \\
\simeq& F'(x_s)x_{s,t} + \frac{1}{2}F''(x_s)x_{s,t} \otimes x_{s,t} \\
\simeq& F'(x_s)\left[\tilde{Y}_{z_{s,t}}(x_s) + \left(\partial_{\tilde{Y}_a(x_s)}\tilde{Y}_b\right)(x_s)|_{a\otimes b=\mathbb{Z}_{s,t}}\right] + \frac{1}{2}F''(x_s)\left[\tilde{Y}_{z_{s,t}}(x_s) \otimes \tilde{Y}_{z_{s,t}}(x_s)\right] \\
=& F'(x_s)\left(\partial_{\tilde{Y}_a(x_s)}\tilde{Y}_b\right)(x_s)|_{a\otimes b=\mathbb{Z}_{s,t}} + \frac{1}{2}F''(x_s)\left[\tilde{Y}_{z_{s,t}}(x_s) \otimes \tilde{Y}_{z_{s,t}}(x_s)\right].
\end{aligned} \tag{4.7}$$

Since $F''(x_s)$ is symmetric and $\mathbf{Z}$ is a geometric rough path it follows that Eq. (4.5) and

$$\frac{1}{2}F''(x_s)\left[\tilde{Y}_{z_{s,t}}(x_s) \otimes \tilde{Y}_{z_{s,t}}(x_s)\right] = F''(x_s)\left[\tilde{Y}_a(x_s) \otimes \tilde{Y}_b(x_s)\right]|_{a\otimes b=\mathbb{Z}_{st}}. \tag{4.8}$$

Combining Eqs. (4.5), (4.7), and (4.8) shows $[F(x.)]_{s,t} \simeq 0$ which implies $F(x_t)$ is constant in $t \in [0,\tau)$. Since $F(x_0) = 0$ it follows that $F(x_t) = 0$ for $t < \tau$, i.e. $x(t) \in M$ for $t < \tau$. Also notice that

$$I \otimes Q(x_s)\mathbb{X}_{s,t} \simeq I \otimes Q(x_s)\left[\tilde{Y}(x_s) \otimes \tilde{Y}(x_s)\right]\mathbb{Z}_{s,t} \simeq 0,$$

and therefore $\mathbf{X} \in WG_p([0,\tau), M)$ and we have proved local existence to Eq. (4.4). This shows local existence to Eq. (4.1).

Suppose that we have found $\mathbf{X} \in WG_p([0,\tau), M)$ solving Eq. (4.1) on $[0,\tau)$ for some $\tau \le T$. If there exists a compact subset $K \subset M$ such that $\{x(t) : t < \tau\} \subset K$, then there exists $t_n \in [0,\tau)$ such that $t_n \uparrow \tau$ and $x_\infty := \lim_{n\to\infty} x(t_n)$ exists in $K \subset M$. We now let $U$ be a precompact neighborhood of $x_\infty \in M$ and $F : U \to \mathbb{R}^k$ be a local defining function of $M$ as in Definition 3.1 and as above let $\tilde{Y}_a := P_F[Y_a \circ \pi]$ on $U$. Moreover we may assume $\tilde{Y}$ is compactly supported. By Corollary 2.17 there exists an $\varepsilon > 0$ and a neighborhood $V \subset U$ of $x_\infty$ such that for any $s \in [\tau - \varepsilon, \tau]$ and $y \in V$ there exists $\hat{\mathbf{X}} \in WG_p([s, \tau+\varepsilon], E)$ with trace in $U$ solving

$$d\hat{\mathbf{X}}_t = \tilde{Y}_{d\mathbf{Z}_t}(x_t) \text{ with } x_s = y \in V.$$

We then choose $n$ sufficiently large so that $t_n \in [\tau - \varepsilon, \tau]$ and let $\hat{\mathbf{X}} \in WG_p([t_n, \tau+\varepsilon], E)$ solve the previous equation with $y = x(t_n)$. We may now apply the concatenation Lemma A.2 to glue $\mathbf{X}$ and $\hat{\mathbf{X}}$ together to show there exists a solution to Eq. (4.1) on $[0, \tau+\varepsilon]$.

Let us now consider the case where Eq. (4.1) does not admit a global solution defined on $[0,T]$. In this case, let

$$\tau = \sup\left\{T_0 \in (0,T) : \exists\, \tilde{\mathbf{X}} \in WG_p([0,T_0], M) \text{ solving } (4.1)\right\} \in (0,T]$$

and for $0 \leq s \leq t < \tau$ let $\mathbf{X}_{s,t} := \tilde{\mathbf{X}}_{s,t}$ where $\tilde{\mathbf{X}} \in WG_p([0,T_0], M)$ solves Eq. (4.1) on $[0,T_0]$ for some $T_0 \in (t,\tau)$. By the uniqueness part of Theorem 2.16, $\{\mathbf{X}_{s,t} : 0 \leq s \leq t < \tau\}$ is well defined and satisfies Eq. (4.1) on $[0,\tau)$. If (for the sake of contradiction) $\overline{\{x(t) : 0 \leq t < \tau\}}^M$ were compact, then the procedure above allows us to produce a solution $\hat{\mathbf{X}}$ to Eq. (4.1) which is valid on $[0, \min(\tau+\epsilon, T)]$ which would violate either the definition of $\tau$ or the assumption that no global solution to Eq. (4.1) exists on $[0,T]$. Hence we must conclude that $\overline{\{x(t) : 0 \leq t < \tau\}}^M$ is not compact. ■

We now prepare an equivalent intrinsic characterizations of an RDE solution. The following proposition is a consequence of the universality property of the full tensor algebra,

$$\mathcal{T}(\mathbb{R}^n) = \mathbb{R} \oplus_{k=1}^{\infty} [\mathbb{R}^n]^{\otimes k},$$

over $\mathbb{R}^n$.

**Proposition 4.3** *Let $\mathcal{L}(M)$ denote the collection of all linear differential operators on $C^\infty(M)$. If $Y : \mathbb{R}^n \to \Gamma(TM)$ is a linear map, then $Y$ extends uniquely to an algebra homomorphism, $\mathcal{Y} : \mathcal{T}(\mathbb{R}^n) \to \mathcal{L}(M)$ such that $\mathcal{Y}_1 := Id \in \mathcal{L}(M)$, where $1 \in \mathcal{T}(\mathbb{R}^n)$.*

**Example 4.4** *If $A \in \mathbb{R}^n \otimes \mathbb{R}^n$, then $\mathcal{Y}_A = Y_a Y_b|_{a\otimes b = A}$ wherein we are using the conventions introduced in Notation 2.2.*

**Theorem 4.5** *Let $Y : \mathbb{R}^n \to \Gamma(TM)$ be a linear map and $\mathbf{X} \in WG_p(M)$. Then the following are equivalent:*

1. *$\mathbf{X}$ solves the RDE in Eq. (4.1).*
2. *For any finite dimensional vector space $W$ and any $\alpha \in \Omega^1(M,W)$,*
$$\left[\int \alpha(d\mathbf{X})\right]^1_{s,t} \simeq \alpha_{x_s}\left(Y_{z_{s,t}}(x_s)\right) + \left[Y_a(x_s)\alpha(Y_b)\right]|_{a\otimes b = \mathbb{Z}_{s,t}}, \tag{4.9}$$
*and*
$$\left[\int \alpha(d\mathbf{X})\right]^2_{s,t} \simeq \left[\alpha_{x_s} Y_a(x_s) \otimes \alpha_{x_s} Y_b(x_s)\right]|_{a\otimes b = \mathbb{Z}_{s,t}}. \tag{4.10}$$
3. *For any finite dimensional vector space $W$ and any $f \in C^\infty(M,W)$,*
$$f(x_t) - f(x_s) \simeq \left(\mathcal{Y}_{\mathbf{Z}_{s,t}} f\right)(x_s) \tag{4.11}$$
*and*
$$(df \otimes df)\left([P_{x_s} \otimes P_{x_s}]\mathbb{X}_{s,t}\right) \simeq Y_a f(x_s) \otimes Y_b f(x_s)|_{a\otimes b = \mathbb{Z}_{s,t}}. \tag{4.12}$$

**Proof.** We will show 1. $\implies$ 2. $\implies$ 3. $\implies$ 1.

(1. $\implies$ 2.) From Eqs. (3.17), (4.2), and (4.3),

$$\begin{aligned}
\left[\int \alpha(d\mathbf{X})\right]^1_{s,t} &\simeq \alpha_{x_s}(P_{x_s} x_{st}) + (\nabla\alpha)\left([P_{x_s} \otimes P_{x_s}]\mathbb{X}_{s,t}\right) \\
&\simeq \alpha_{x_s}\left(P_{x_s}\left[Y_{z_{s,t}}(x_s) + (\partial_{Y_a} Y_b)(x_s)|_{a\otimes b = \mathbb{Z}_{st}}\right]\right) \\
&\qquad + (\nabla\alpha)\left([P_{x_s} \otimes P_{x_s}] Y_a(x_s) \otimes Y_b(x_s)|_{a\otimes b = \mathbb{Z}_{s,t}}\right) \\
&\simeq \alpha_{x_s}\left(Y_{z_{s,t}}(x_s) + P_{x_s}(\partial_{Y_a} Y_b)(x_s)|_{a\otimes b = \mathbb{Z}_{st}}\right) + (\nabla\alpha)\left(Y_a(x_s) \otimes Y_b(x_s)|_{a\otimes b = \mathbb{Z}_{s,t}}\right) \\
&\simeq \alpha_{x_s}\left(Y_{z_{s,t}}(x_s) + \nabla_{Y_a(x_s)} Y_b|_{a\otimes b = \mathbb{Z}_{st}}\right) + \left(\nabla_{Y_a(x_s)}\alpha\right)(Y_b(x_s))|_{a\otimes b = \mathbb{Z}_{s,t}}.
\end{aligned}$$

Combining this approximate identity with the product rule for covariant derivatives in item 2. of Proposition 3.12 gives Eq. (4.9). Equation (4.10) follows easily from Eqs. (3.18) and (4.3);

$$\begin{aligned}
\left[\int \alpha(d\mathbf{X})\right]^2_{s,t} &\simeq \alpha_{x_s} \otimes \alpha_{x_s}[P_{x_s} \otimes P_{x_s}]\mathbb{X}_{st} \\
&\simeq \alpha_{x_s} \otimes \alpha_{x_s}[P_{x_s} \otimes P_{x_s}][Y_a(x_s) \otimes Y_b(x_s)]|_{a\otimes b = \mathbb{Z}_{s,t}} \\
&= \alpha_{x_s} \otimes \alpha_{x_s}[Y_a(x_s) \otimes Y_b(x_s)]|_{a\otimes b = \mathbb{Z}_{s,t}},
\end{aligned}$$

wherein we have used $P_{x_s}Y_{(\cdot)}(x_s) = Y_{(\cdot)a}(x_s)$ in the last equality.

(2. $\implies$ 3.) Applying item 2. with $\alpha = df$ shows

$$\begin{aligned}\left[\int \alpha(d\mathbf{X})\right]^1_{s,t} &\simeq df_{x_s}\left(Y_{z_{s,t}}(x_s)\right) + [Y_a(x_s)\, df_{x_s}(Y_b)]\,|_{a\otimes b=\mathbb{Z}_{s,t}} \\ &\simeq df_{x_s}\left(Y_{z_{s,t}}(x_s)\right) + [Y_bY_bf(x_s)]\,|_{a\otimes b=\mathbb{Z}_{s,t}} \\ &= \left(\mathcal{Y}_{z_{s,t}+\mathbb{Z}_{s,t}}f\right)(x_s)\end{aligned}$$

and

$$\left[\int \alpha(d\mathbf{X})\right]^2_{s,t} \simeq [df_{x_s}Y_a(x_s) \otimes df_{x_s}Y_b(x_s)]\,|_{a\otimes b=\mathbb{Z}_{s,t}}.$$

This shows item 3. holds once we recall that

$$\left[\int df(d\mathbf{X})\right]^1_{s,t} = f_*(\mathbf{X})^1_{s,t} = f(x_t) - f(x_s)$$

and

$$\left[\int df(d\mathbf{X})\right]^2_{s,t} \simeq df_{x_s} \otimes df_{x_s}[P_{x_s} \otimes P_{x_s}]\,\mathbb{X}_{st}.$$

(3. $\implies$ 1.) Let $W = E$ and $f : M \to E$ be the restrictions of the identity map on $E$, i.e. $f(x) = x$ for all $x \in M$. For this $f$, we have $df(v_x) = v$ for all $v_x \in TM$,

$$(Y_bf)(x) = Y_b(x) \text{ and } (Y_aY_bf)(x) = (\partial_{Y_a}Y_b)(x)$$

and so Eq. (4.11) becomes,

$$x_{s,t} = f(x_t) - f(x_s) \simeq \left(\mathcal{Y}_{\mathbf{Z}_{s,t}}f\right)(x_s) = Y_{z_{s,t}}(x_s) + (\partial_{Y_a}Y_b)(x_s)\,|_{a\otimes b=\mathbb{Z}_{s,t}}$$

which is precisely Eq. (4.2). Similarly Eq. (4.12) becomes

$$\begin{aligned}[P_{x_s} \otimes P_{x_s}]\,\mathbb{X}_{s,t} &= (df \otimes df)\left([P_{x_s} \otimes P_{x_s}]\,\mathbb{X}_{s,t}\right) \\ &\simeq Y_af(x_s) \otimes Y_bf(x_s)\,|_{a\otimes b=\mathbb{Z}_{s,t}} = Y_a(x_s) \otimes Y(x_s)\,|_{a\otimes b=\mathbb{Z}_{s,t}}\end{aligned}$$

which is equivalent to Eq. (4.3) by Corollary 3.22. ■

**Remark 4.6** *If we restrict $W$ to be $\mathbb{R}$ in Theorem 4.5 we may still conclude from either of items 2. or 3. of that theorem that $\mathbf{X}$ satisfies Eq. (4.2), i.e. the level one condition for the RDE solution (4.1). Indeed if $f = \ell|_M : M \to \mathbb{R}$ where $\ell \in E^*$ is any linear functional on $E$, then*

$$(Y_af)(x) = \ell Y_a(x) \text{ and } (Y_aY_bf)(x) = (Y_a\ell Y_b)(x) = \ell(\partial_{Y_a}Y_b)(x).$$

*So for $f = \ell|_M$, Eq. (4.11) becomes*

$$\ell(x_{s,t}) = f(x_t) - f(x_s) \simeq \ell Y_{z_{s,t}}(x_s) + \ell(\partial_{Y_a}Y_b)(x_s)\,|_{a\otimes b=\mathbb{Z}_{s,t}}.$$

*As this true for all $\ell \in E^*$ we may conclude Eq. (4.2) holds.*

**Remark 4.7** *We can not get Eq. (4.3) from Eq. (4.10) without allowing for $\dim W > 1$. Indeed, if $\alpha \in \Omega^1(M, \mathbb{R})$, then*

$$\alpha_{x_s} \otimes \alpha_{x_s}\left([P_{x_s} \otimes P_{x_s}]\,\mathbb{X}_{s,t}\right) \simeq \alpha_{x_s} \otimes \alpha_{x_s}[Y_a(x_s) \otimes Y_b(x_s)]\,|_{a\otimes b=\mathbb{Z}_{s,t}}$$

*from which we may only conclude that*

$$[P_{x_s} \otimes P_{x_s}]\,\mathbb{X}^s_{s,t} \simeq [Y_a(x_s) \otimes Y_b(x_s)]\,|_{a\otimes b=\mathbb{Z}^s_{s,t}}. \tag{4.13}$$

*This is because $\alpha \otimes \alpha(\xi \otimes \eta - \eta \otimes \xi) = \alpha(\xi)\alpha(\eta) - \alpha(\eta)\alpha(\xi) = 0$ since scalar multiplication is commutative. Here we have used that $\mathbb{R}\otimes_{\mathbb{R}}\mathbb{R} \cong \mathbb{R}$. The reader should further observe that information contained in Eq. (4.13) is already a consequence of Eq. (4.2) and the assumption that $\mathbf{Z}$ and $\mathbf{X}$ are weakly geometric rough paths.*

**Definition 4.8 (Intrinsic RDEs on Manifolds)** *Given a linear map $Y : \mathbb{R}^n \to \Gamma(TM)$, we say that a geometric rough path $\mathbf{X} \in WG_p(M)$ solves the RDE*

$$d\mathbf{X}_t = Y_{d\mathbf{Z}_t}(x_t) \text{ with } x(0) = x_0 \in M \tag{4.14}$$

*if and only if equations (4.11) and (4.12)hold for all $f \in C^\infty(M, W)$ and every finite dimensional vector space $W$.*

**Notation 4.9 (Intrinsic RDEs)** *To emphasize when we are working with the intrinsic definition of an RDE we sometimes write*

$$d\mathbf{X}_t = \mathcal{Y}_{d\mathbf{Z}_t}(x_t) \text{ with } x(0) = x_0 \in M$$

*in place of (4.14) where now $\mathbf{Z}_{s,t} = z_{s,t} + \mathbb{Z}_{s,t}$ and we interpret*

$$[\mathcal{Y}_{d\mathbf{Z}}(x)]^2_{s,t} \simeq [Y(x_s) \otimes Y(x_s)] \mathbb{Z}_{s,t} \in \mathbb{R}^N \otimes \mathbb{R}^N.$$

We end this subsection with a result describing (in special cases) the push forward of solutions to RDEs.

**Definition 4.10** *Suppose that $\pi : M \to N$ is a smooth map between two smooth manifolds. Also suppose that $Y^M : \mathbb{R}^n \to \Gamma(TM)$ and $Y^N : \mathbb{R}^n \to \Gamma(TN)$ are two linear maps. We say $Y^M$ and $Y^N$ are $\pi$ –* ***related dynamical systems*** *if*

$$\pi_* Y^M_a = Y^N_a \circ \pi \text{ for all } a \in \mathbb{R}^n.$$

**Theorem 4.11** *Suppose $\pi : M \to N$ is a smooth map between manifolds. Let $Y^M : \mathbb{R}^n \to \Gamma(TM)$ and $Y^N : \mathbb{R}^n \to \Gamma(TN)$ be two $\pi$ – related dynamical systems. Further suppose that $\mathbf{Z} \in WG_p(\mathbb{R}^n)$ and $\mathbf{X} = (x, \mathbb{X})$ solves the RDE,*

$$d\mathbf{X}_t = \mathcal{Y}^M_{d\mathbf{Z}_t}(x_t) \text{ with } x_0 \in M \text{ given.}$$

*Then $\mathbf{X}^N := \pi_*(\mathbf{X}) = \left(x^N, \mathbb{X}^N\right)$ solves the RDE,*

$$d\mathbf{X}^N_t = \mathcal{Y}^N_{d\mathbf{Z}_t}\left(x^N_t\right) \text{ with } x^N_0 := \pi(x_0) \in N \text{ given.}$$

**Proof.** Fix a finite dimensional vector space $W$ and let $f \in C^\infty(N, W)$. Applying item 3. of Theorem 4.5 to the function $f \circ \pi \in C^\infty(M, W)$ shows,

$$\begin{aligned} f\left(x^N_t\right) - f\left(x^N_s\right) &= f \circ \pi(x_t) - f \circ \pi(x_s) \\ &\simeq \left(\mathcal{Y}^M_{\mathbf{Z}_{st}}(f \circ \pi)\right)(x_s) = \left(\mathcal{Y}^N_{\mathbf{Z}_{st}} f\right) \circ \pi(x_s) = \left(\mathcal{Y}^N_{\mathbf{Z}_{st}} f\right)\left(x^N_s\right) \end{aligned}$$

and

$$\begin{aligned} \mathbb{X}^N_{s,t} &\simeq \pi_* \otimes \pi_* P(x_s) \otimes P(x_s) \mathbb{X}_{s,t} \\ &\simeq \pi_* \otimes \pi_* P(x_s) \otimes P(x_s) Y^M_{(\cdot)}(x_s) \otimes Y^M_{(\cdot)}(x_s) \mathbb{Z}_{s,t} \\ &= \pi_* Y^M_{(\cdot)}(x_s) \otimes \pi_* Y^M_{(\cdot)}(x_s) \mathbb{Z}_{s,t} \\ &= Y^N_{(\cdot)}\left(x^N_s\right) \otimes Y^N_{(\cdot)}\left(x^N_s\right) \mathbb{Z}_{s,t}. \end{aligned}$$

■

## 4.2 Fundamental properties of rough paths on manifolds

Armed with well-defined notions of integration and RDEs, we now derive some of the fundamental properties of geometric and weakly geometric rough paths on manifolds. We also exhibit some natural examples of elements in $WG_p(M)$ which are constructed by "projecting the increments" of geometric rough paths on $E$ to the tangent space of $M$.

**Example 4.12 (Projection Construction of Geometric Rough Paths)** *Let* $\mathbf{Z}$ *be a weakly geometric* $p-$ *rough path on* $E$ *for some* $p \in [2,3)$, *then there exists a unique rough path solution* $\mathbf{X}$ *(possibly only up to an explosion time) to the RDE*

$$d\mathbf{X}_t = V_{d\mathbf{Z}_t}(x_t) = P_{x_t} d\mathbf{Z}_t \text{ with } x_0 \in M. \tag{4.15}$$

*Moreover it will follow from Theorem 4.2 and Theorem 4.17 that* $\mathbf{X} \in G_{p'}(M) \cap WG_p(M)$ *for all* $p' > p$.

The following proposition shows that, in fact, all weakly geometric rough paths on $M$ may be constructed by this method.

**Proposition 4.13 (Consistency)** *If* $\mathbf{Z} \in WG_p(M) \subset WG_p(E)$, *then the unique solution to Eq. (4.15) with* $x_0 = z_0$ *is* $\mathbf{X} \equiv \mathbf{Z}$. *[So in this setting the solution to Eq. (4.15) exists on all of* $[0,T]$ *.]*

**Proof.** The proof amounts to showing that $\mathbf{X} = \mathbf{Z}$ solves Eq. (4.15), i.e. that

$$z_{s,t} \simeq V_{z_{s,t}}(z_s) + (\partial_{V_a} V_b)(z_s)|_{a\otimes b = \mathbb{Z}_{s,t}} \text{ and} \tag{4.16}$$

$$\mathbb{Z}_{s,t} \simeq V_a(z_s) \otimes V_b(z_s)|_{a\otimes b = \mathbb{Z}_{s,t}} = P(z_s) \otimes P(z_s)\mathbb{Z}_{s,t}. \tag{4.17}$$

Equation (4.17) is a consequence of Corollary 3.20. The right side of Eq. (4.16) is approximated as

$$\begin{aligned} P(z_s) z_{s,t} + dP\left([P(z_s)a]_{z_s}\right) b|_{a\otimes b = \mathbb{Z}_{s,t}} &\simeq P(z_s) z_{s,t} + dP\left([P(z_s)a]_{z_s}\right) P(z_s) b|_{a\otimes b = \mathbb{Z}_{s,t}} \\ &\simeq P(z_s) z_{s,t} + Q(z_s) z_{s,t} = z_{s,t}, \end{aligned}$$

wherein we have used Lemma 3.28 for the second approximate equality above. ■

We now address the relation between geometric and weakly geometric rough paths on manifolds. To do this we first require a couple of elementary lemmas.

**Lemma 4.14** *Suppose* $U$ *is an open neighborhood of* $M$ *and* $\mathbb{R}^n \ni a \to \tilde{Y}_a \in \Gamma(TU)$ *is a linear map such that* $\tilde{Y}_a(m) \in T_m M$ *for all* $m \in M$. *Further suppose that* $z : [0,T] \to \mathbb{R}^n$ *is a smooth function and* $x : [0,T] \to U$ *is a smooth solution to*

$$\dot{x}(t) = \tilde{Y}_{\dot{z}(t)}(x(t)) \text{ with } x(0) = x_0 \in M. \tag{4.18}$$

*If there is an open neighborhood,* $\mathcal{V}$, *in* $E$ *such that* $x([0,T]) \subset \mathcal{V}$ *and* $\overline{\mathcal{V} \cap M}^E \subset M$, *then* $x(t) \in M$ *for all* $t \in [0,T]$ *and* $x(t)$ *satisfies,* $\dot{x}(t) = Y_{\dot{z}(t)}(x(t))$ *with* $x(0) = x_0 \in M$.

**Proof.** By replacing $\mathcal{V}$ by $\mathcal{V} \cap \mathcal{U}$ we may assume that $\mathcal{V} \subset U$. For the sake of contradiction, suppose that $x([0,T])$ is not contained in $M$ and let $\tau = \inf\{t \in [0,T] : x(t) \notin M\}$ be the first exit time of $x_{(\cdot)}$ from $M$. Since $x(0) = x_0 \in M$ and $x(t) \in M$ for all $0 \le t < \tau$ if $\tau > 0$ we may conclude that $x(\tau) \in \overline{\mathcal{V} \cap M}^E \subset M$. As $x([0,T])$ is not contained in $M$ we may now conclude that $\tau < T$.

By the local existence theorem for the ODEs, there exists an $\varepsilon > 0$ and a solution $y : [\tau, \tau + \varepsilon] \to M$ solving

$$\dot{y}(t) = Y_{\dot{z}(t)}(y(t)) \text{ with } y(\tau) = x(\tau).$$

The function $\tilde{x} : [0, \tau + \varepsilon] \to M$ defined by

$$\tilde{x}(t) := \begin{cases} x(t) & \text{if} \quad 0 \le t \le \tau \\ y(t) & \text{if} \quad \tau \le t \le \tau + \varepsilon \end{cases}$$

then solves Eq. (4.18) on $[0, \tau + \varepsilon]$ and hence by uniqueness, $x(t) = \tilde{x}(t)$ for $0 \le t \le \tau + \varepsilon$. This however shows $x(t) \in M$ for $0 \le t \le \tau + \varepsilon$ which contradicts the definition of $\tau$. ■

**Lemma 4.15** *If* $K$ *is a compact subset of* $M$, *there exists an open neighborhood* $\mathcal{V}$ *in* $E$ *containing* $K$ *such that* $\overline{M \cap \mathcal{V}}^E \subset M$.

**Proof.** First suppose $K = \{x\} \subset M$. Let $F : U \to \mathbb{R}^{N-d}$ be a local defining function for $M$ so that $x \in U$ and $U \cap M = \{F = 0\}$. Let $\mathcal{V}_x$ be a precompact open neighborhood of $x$ in $E$ so that $\bar{\mathcal{V}}_x \subset U$. Since $\overline{M \cap \mathcal{V}_x}^E \subset \overline{\mathcal{V}_x}^E \subset U$ and $F \equiv 0$ on $M \cap \mathcal{V}_x$ it follows by continuity that $F = 0$ on $\overline{M \cap \mathcal{V}_x}^E$ from which it follows that $\overline{M \cap \mathcal{V}_x}^E \subset \{F = 0\} \subset M$.

If $K$ is a general compact subset of $M$, to each $x \in K$ there exists a precompact open neighborhood of $\mathcal{V}_x$ in $E$ with $x \in \mathcal{V}_x$ and $\overline{M \cap \mathcal{V}_x}^E \subset M$. Since $K$ is compact there is a finite subset, $\Lambda \subset K$, such that $\mathcal{V} := \cup_{x \in \Lambda} \mathcal{V}_x$ contains $K$. This is the desired open set in $E$ since

$$\overline{M \cap \mathcal{V}}^E = \overline{\cup_{x\in\Lambda} M \cap \mathcal{V}_x}^E = \cup_{x\in\Lambda} \overline{M \cap \mathcal{V}_x}^E \subset M.$$

■

**Lemma 4.16** *Let $\mathbb{R}^n \ni a \to Y_a \in \Gamma(TM)$ be a linear map, $\mathbf{Z} \in G_p(\mathbb{R}^n)$, and suppose $z^k : [0,T] \to \mathbb{R}^n$ are smooth functions such that $S_2(z^k) \to \mathbf{Z}$ in rough $p-$variation metric, see Eq. (A.1). Assume $\mathbf{X} \in WG_p(M)$ satisfies the RDE $d\mathbf{X} = Y_{d\mathbf{Z}}(x)$ with starting point $x_0 = z_0$. Then, for $k$ sufficiently large, there exists smooth functions $x^k : [0,T] \to M$ (note: taking values in $M$) satisfying*

$$\dot{x}^k(t) = Y_{\dot{z}^k(t)}(x(t)) \text{ with } x^k(0) = z_0, \tag{4.19}$$

*and moreover such that $S_2(x^k)$ converges to $\mathbf{X}$ in $WG_p(M)$. Consequently $\mathbf{X} \in G_p(M)$.*

**Proof.** By Remark 3.2 and a partition of unity argument we may find an open neighborhood $U$ of $M$ in $E$ and a linear map $\mathbb{R}^n \ni a \to \hat{Y}_a \in \Gamma(TU)$ such that $\hat{Y}_a = Y_a$ on $M$. By Lemma 4.15 there exists a precompact open neighborhood $\mathcal{V}$ in $E$ containing $K = x([0,T])$ such that $\overline{M \cap \mathcal{V}}^E \subset M$. By replacing $\mathcal{V}$ by $\mathcal{V} \cap \mathcal{U}$ we may assume that $\mathcal{V} \subset U$. We can then find a linear map $\mathbb{R}^n \ni a \to \tilde{Y}_a \in \Gamma(TU)$, such that $\tilde{Y}_a = \hat{Y}_a$ on $\mathcal{V}$ and the vector fields $\tilde{Y}_a$ have compact support. As $x([0,T]) \subset \mathcal{V}$ and $\mathbf{X}$ solves $d\mathbf{X} = Y_{d\mathbf{Z}}(x)$, it follows that $\mathbf{X}$ also solves $d\mathbf{X} = \tilde{Y}_{d\mathbf{Z}}(x)$. By Lemma 4.14 we know the equations $\dot{x}^k(t) = \tilde{Y}_{\dot{z}^k(t)}(x(t))$, $x^k(0) = z_0$ have (global) solutions $x^k(t) \in M$ for all $0 \le t \le T$. In addition, it follows by the universal limit theorem (Theorem 5.3 of [30]) that solutions to the differential equations,

$$d\mathbf{X}^k = \tilde{Y}_{dS_2(z_k)}(x) \text{ with } x^k(0) = z_0$$

satisfy $S_2(x^k) \to \mathbf{X}$ in $p-$ variation as $k \to \infty$ and hence $x^k \to x$ uniformly. Therefore, for sufficiently large $k$, it follows that $x^k(t) \in \mathcal{V}$ for all $0 \le t \le T$ and hence $x^k(t) \in M$ (Lemma 4.14). Since $Y_a = \tilde{Y}_a$ on $\mathcal{V} \cap \mathcal{M}$ we conclude that $x_k$ solve (4.19) as required. ■

**Theorem 4.17** *For all $p' > p \ge 1$ we have $G_p(M) \subseteq WG_p(M) \subseteq G_{p'}(M)$.*

**Proof.** We have already demonstrated the first containment in Corollary 3.32. Suppose now $\mathbf{Z} \in WG_p(M)$, then in particular $\mathbf{Z} \in WG_p(E)$ and hence by classical results (see Corollary 8.24 of [17]) $\mathbf{Z}$ belongs to $G_{p'}(E)$. By Proposition 4.13, $\mathbf{Z}$ solves the RDE,

$$d\mathbf{Z} = V_{d\mathbf{Z}}(z) = P(z)\, d\mathbf{Z}, \text{ with } z_0 \in M. \tag{4.20}$$

Consequently by Lemma 4.16, $\mathbf{Z} \in G_{p'}(M)$. ■

We conclude the section with the following theorem summarizes three equivalent characterizations of weakly geometric rough paths on manifolds. We reemphasize that $WG_p(M)$ are precisely those rough paths in $WG_p(E)$ that consistently integrate finite dimensional vector space valued one forms $\alpha \in \Omega^1(M, W)$.

**Theorem 4.18 (Characterisation of $WG_p(M)$)** *If $\mathbf{Z} \in WG_p(E)$, then the following are equivalent:*

1. $\mathbf{Z} \in WG_p(M)$.

2. *The trace $z$ of $\mathbf{Z} \in WG_p(E)$ is in $M$ and further satisfies, for all finite dimensional vector spaces $W$,*

$$\int \hat{\alpha}(d\mathbf{Z}) = \int \tilde{\alpha}(d\mathbf{Z})$$

*for any $\hat{\alpha}, \tilde{\alpha} \in \Omega^1(E, W)$ such that $\hat{\alpha} = \alpha$ on $TM$.*

3. *The trace $z$ of $\mathbf{Z} \in WG_p(E)$ is in $M$ and $Q_{x_s} \otimes I\mathbb{X}_{st} \simeq 0$ for $0 \le s \le t \le T$.*

4. *The starting point, $z_0$, is in $M$ and $\mathbf{Z}$ solves the projection equation (4.15).*

**Proof.** Lemma 3.23 shows 1. implies 2. and taking $\hat{\alpha} = 0$ in item 2. shows $\mathbf{Z}$ satisfies Definition 3.17 and so items 1. and 2. are equivalent. The equivalence of items 1. and 3. is the content of Proposition 3.35. The equivalence of items 1. and 4. follows from Example 4.12 and Proposition 4.13. ■

## 4.3 Right invariant RDE's on Lie groups

To illustrate some of the results above we are going to consider RDEs on a Lie group $G$ relative to right invariant vector fields. We assume, as is always possible, that $G$ is embedded in some Euclidean space $\mathbb{R}^N$. Although we will be using the results above we will not need to know any information about the embedding other than it exists.

**Definition 4.19** *To each Lie group $G$ with Lie algebra $\mathfrak{g} := \mathrm{Lie}(G)$, let $Y^G : \mathfrak{g} \to \Gamma(TG)$ be the linear map defined by,*

$$\left(Y_\xi^G\right)(g) := -\hat{\xi}(g) := -\frac{d}{dt}|_0 e^{t\xi} g, \tag{4.21}$$

*i.e. $Y_\xi^G$ is the right invariant vector field on $G$ such that $Y_\xi^G(e) = -\xi$.*

**Theorem 4.20 (Global Solutions to Right Invariant RDEs)** *To each $\mathbf{A} = (a, \mathbb{A}) \in WG_p(\mathfrak{g}, \omega)$ there exists a (unique)* ***global*** *solution $\mathbf{G} = (g, \mathbb{G})$ to the RDE,*

$$d\mathbf{G} = \mathcal{Y}_{d\mathbf{A}}^G(g) \text{ with } g_0 = e \in G, \tag{4.22}$$

**Proof.** According to Theorem 4.2, Eq. (4.22) either has a solution on all of $[0, T]$ in which case we are done or there is a $\tau \in (0, T]$ such that the solution $\mathbf{G}$ exists on $[0, \tau)$ while $\overline{\{g_t : 0 \le t < \tau\}}^G$ is not compact. To finish the proof we need only rule out the second case.

By Corollary 2.17, we may find and $\varepsilon > 0$ such that for any $t_0 \in [0, T]$ there is a solution $\mathbf{H} = (h, \mathbb{H})$ on $[t_0, \min(t_0 + \epsilon, T)]$ to the RDE, $d\mathbf{H} = \mathcal{Y}_{d\mathbf{A}}^G(h)$ with $h_{t_0} = e \in G$. For $u \in G$ let $R_u : G \to G$ be the diffeomorphism of $G$ given by $R_u x = xu$ for all $x \in G$. By its very definition, we have $R_{u*} Y_\xi^u = Y_\xi^u \circ R_u$ and so by an application of Theorem 4.11 it follows that $\mathbf{K} = (k, \mathbb{K}) := (R_u)_* (\mathbf{H})$ solves $d\mathbf{K} = \mathcal{Y}_{d\mathbf{A}}^G(k)$ with $k_{t_0} = u$ on $[t_0, \min(t_0 + \epsilon, T)]$.

Choose $t_0 \in (\max\{0, \tau - \varepsilon/2\}, \tau)$ and apply the above result with $u = g_{t_0}$ in order to produce a weakly geometric rough path, $\mathbf{K} = (k, \mathbb{K})$, on $[t_0, \min(\tau + \epsilon/2, T)]$ solving $d\mathbf{K} = \mathcal{Y}_{d\mathbf{A}}^G(k)$ with $k_{t_0} = g_{t_0}$. An application of Lemma A.2 (easily adapted to RDE on manifolds) shows that $\mathbf{G}$ restricted to $[0, t_0]$ and $\mathbf{K}$ on $[t_0, \min(\tau + \epsilon/2, T)]$ may be concatenated into a weakly geometric rough path $\tilde{\mathbf{G}}$ which solves Eq. (4.22) on $[0, \min(\tau + \epsilon/2, T)]$. This then violates the definition of $\tau$ and shows that Eq. (4.22) can not explode. ■

**Theorem 4.21 (Pushing forward solutions by Lie homorphisms)** *Suppose that $\rho : G \to H$ is a Lie group homomorphism and for $\mathbf{A} = (a, \mathbb{A}) \in WG_p(\mathfrak{g}, \omega)$ let*

$$\mathbf{A}^\rho = (d\rho)_* (\mathbf{A}) = ([d\rho]\, a, [d\rho \otimes d\rho]\, \mathbb{A}) \in WG_p(\mathfrak{h}, \omega). \tag{4.23}$$

*If $\mathbf{G} = (g, \mathbb{G}) \in WG_p(G)$ is the unique global solution to the RDE (4.22) and $\mathbf{H} = \rho_*(\mathbf{G})$, then*

$$d\mathbf{H} = \mathcal{Y}_{d\mathbf{A}^\rho}^H(h) \text{ with } h_0 = e_H \in H. \tag{4.24}$$

*Moreover, if* $\mathbb{G}^T_{s,t} := P^G(g_s) \otimes P^G(g_s)\,\mathbb{G}_{s,t}$ *and* $\mathbb{H}^T_{s,t} := P^H(h_s) \otimes P^H(h_s)\,\mathbb{H}_{s,t}$ *denote the tangential components of* $\mathbb{G}$ *and* $\mathbb{H}$ *respectively, then* $\mathbf{H}$ *may also be characterised by;*

$$h_t = \rho(g_t) \text{ and } \mathbb{H}^T_{s,t} \simeq [\rho_* \otimes \rho_*]\,\mathbb{G}^T_{s,t}. \tag{4.25}$$

**Proof.** For $\xi \in \mathfrak{g}$, let $W_\xi \in \Gamma(TH)$ be defined by

$$W_\xi(h) = Y^H_{d\rho(\xi)}(h) = \frac{d}{dt}|_0 h e^{-td\rho(\xi)}.$$

A simple computation then shows $\rho_* Y^G_\xi = W_\xi \circ \rho$ and therefore by Theorem 4.11, $\mathbf{H} \in WG_p(H)$ satisfies the RDE,

$$d\mathbf{H} = \mathcal{W}_{d\mathbf{A}}(h) \text{ with } h_0 = \rho(e_G) = e_H \in H. \tag{4.26}$$

Using $W_{a_{s,t}} = Y^H_{d\rho(a_{s,t})}$ and

$$W_a W_b|_{a\otimes b = \mathbb{A}_{s,t}} = Y^H_{d\rho(a)} Y^H_{d\rho(b)}|_{a\otimes b=\mathbb{A}_{s,t}} = Y_\alpha^H Y_\beta^H|_{\alpha\otimes\beta = d\rho\otimes d\rho[\mathbb{A}_{s,t}]}$$

along with Theorem 4.5 one shows $\mathbf{H}$ also solves Eq. (4.24). From Proposition 3.38 we know $h_t = \rho(g_t)$ and from Equation 4.2 and Corollary 3.22,

$$\mathbb{H}^T_{s,t} \simeq W_{(\cdot)}(h_s) \otimes W_{(\cdot)}(h_s)\,\mathbb{A}_{s,t} = \left[\rho_* Y^G_{(\cdot)}(g_s) \otimes \rho_* Y^G_{(\cdot)}(g_s)\right]\mathbb{A}_{s,t} \simeq [\rho_* \otimes \rho_*]\,\mathbb{G}^T_{st}.$$

■

# 5 Parallel Translation

In subsection 5.1, we recall the definition of parallel translation along smooth curves in $M$ along with some of its basic properties. In order to transfer these results to the rough path setting it is useful to introduce the orthogonal frame bundle $(O(M))$ over $M$ which is done in subsection 5.2. The "lifting" of paths in $M$ to "horizontal" paths in $O(M)$ and the relationship of these horizontal lifts to parallel translation is also reviewed here. After this warm-up, we defined *parallel translation* along $\mathbf{X} \in WG_p(M)$ as an element $\mathbf{U} \in WG_p(O(M))$ solving a prescribed RDE on $O(M)$ driven by $\mathbf{X}$, see Definition 5.13 of subsection 5.3. An alternative characterization of the level one components of $\mathbf{U}$ is then given in Proposition 5.15 which is then used to show that the RDE defining $\mathbf{U}$ exists on the full time interval, $[0,T]$. It is then shown in Theorems 5.16 and 5.17 that two natural classes of RDE's on $O(M)$ give rise to an element $\mathbf{U} \in WG_p(O(M))$ each of which is parallel translation along $\mathbf{X} := \pi_*(\mathbf{U})$ where $\pi : O(M) \to M$ is the natural projection map on $O(M)$.

## 5.1 Smooth Parallel Translation

**Definition 5.1** *Given smooth paths* $x(t) \in M$ *and* $v(t) \in E$ *such that* $v(t)_{x(t)} \in T_{x(t)}M$ *for all* $t$*, the* ***covariant derivative*** *of* $v(\cdot)_{x(\cdot)}$ *is defined as*

$$\frac{\nabla v(t)_{x(t)}}{dt} := [P(x(t))\,\dot{v}(t)]_{x(t)} = [\dot{v}(t) + dQ(\dot{x}(t))\,v(t)]_{x(t)},$$

*wherein the last equality follows by differentiating the identity,* $P(x(t))\,v(t) = v(t)$*, and using* $dQ = -dP$*. A path* $v(t)_{x(t)} \in TM$ *is said to be* ***parallel*** *if* $\frac{\nabla}{dt}\left[v(t)_{x(t)}\right] = 0$ *for all* $t$*, i.e.* $v(t)$ *solves the differential equation,*

$$\dot{v}(t) + dQ(\dot{x}(t))\,v(t) = 0. \tag{5.1}$$

If $v(t)$ solves Eq. (5.1) with $v(0) \in T_{x(0)}M$ then a simple calculation using Eq. (5.1) and Lemma 3.11 shows

$$\frac{d}{dt}[Q(x(t))\,v(t)] = dQ(\dot{x}(t))\,[Q(x(t))\,v(t)] \text{ with } Q(x(0))\,v(0) = 0$$

which forces $Q(x(t))v(t)=0$ by the uniqueness theorem of linear ordinary differential equations. Moreover using $PdQP=0$ (Lemma 3.11),

$$\begin{aligned}\frac{d}{dt}\|v(t)\|_E^2 &= 2\langle v(t), \dot{v}(t)\rangle = -2\langle v(t), dQ(\dot{x}(t))v(t)\rangle \\ &= -2\langle P(x(t))v(t), dQ(\dot{x}(t))P(x(t))v(t)\rangle = 0,\end{aligned}$$

which shows $\|v(t)\|_E = \|v(0)\|_E$.

**Notation 5.2** *Given two inner product spaces, $V$ and $W$, let* $\mathrm{Iso}(V,W)$ *denote the collection of isometries from $V$ to $W$.*

From the previous discussion, if $V$ is an inner product space and $g_0 \in \mathrm{Iso}\left(V, \tau_{x(0)}M\right)$, then the function $g(t) \in \mathrm{Hom}(V,E)$ solving,

$$\dot{g}(t) + dQ(\dot{x}(t))g(t) = 0 \text{ with } g(0) = g_0, \tag{5.2}$$

satisfies $g(t) \in \mathrm{Iso}\left(V, \tau_{x(t)}M\right)$ for $0 \le t \le T$.

**Definition 5.3 (Smooth Parallel Translation)** ***Parallel translation*** *along the smooth path $x(\cdot) \in M$ is the collection of isometries, $//_t(x) : T_{x(0)}M \to T_{x(t)}M$, defined by*

$$//_t(x)\, v_{x(0)} = [g(t)v]_{x(t)} \tag{5.3}$$

*where $g(t)$ solves Eq. (5.2) with $g_0 = Id_{\tau_{x(0)}M} \in \mathrm{Hom}\left(\tau_{x(0)}M, E\right)$.*

## 5.2 The Frame Bundle, $O(M)$

**Definition 5.4** *The **orthogonal frame bundle**, $O(M)$, is the subset of $E \times \mathrm{Hom}\left(\mathbb{R}^d, E\right)$ defined by,*

$$O(M) = \left\{(m,g) : m \in M \text{ and } g \in \mathrm{Iso}\left(\mathbb{R}^d, \tau_m M\right)\right\}. \tag{5.4}$$

*Further, let $\pi : O(M) \to M$ be the restriction to $O(M)$ of projection of $E \times \mathrm{Hom}\left(\mathbb{R}^d, E\right)$ onto its first factor and set*

$$O_m(M) := \pi^{-1}(\{m\}) = \{m\} \times \mathrm{Iso}\left(\mathbb{R}^d, \tau_m M\right). \tag{5.5}$$

**Theorem 5.5 (Embedding the Frame Bundle)** *The orthogonal frame bundle, $O(M)$, is an embedded submanifold of $E \times \mathrm{Hom}\left(\mathbb{R}^d, E\right)$. In fact, if $F : U \to \mathbb{R}^{N-d}$ is a local defining function for $M$, then*

$$G : U \times \mathrm{Hom}\left(\mathbb{R}^d, E\right) \to \mathbb{R}^k \times \mathrm{Hom}\left(\mathbb{R}^d, \mathbb{R}^k\right) \times \mathcal{S}_d$$

*defined by*

$$G(x,g) := \left(F(x), Q(x)g, g^*g - I_d\right), \tag{5.6}$$

*where $\mathcal{S}_d$ denotes the linear subspace of $\mathrm{End}\left(\mathbb{R}^d\right)$ consisting of symmetric $d \times d$ matrices is a local defining function for $O(M)$. Moreover, if $(m,g) \in O(M)$, then*

$$T_{(m,g)}O(M) = \left\{(\xi,h)_{(m,g)} : \xi \in \tau_m M,\ Q(m)h = -dQ(\xi_m)g \text{ and } g^*h \in so(d)\right\}, \tag{5.7}$$

*where $so(d)$ is the vector space of $d \times d$ real skew symmetric matrices.*

The proof of this standard theorem is given in Appendix B for the readers convenience. From Eq. (5.7), if $(\xi,h)_{(m,g)} \in T_{(m,g)}O(M)$, then

$$h = Q(m)h + P(m)h = -dQ(\xi_m)g + P(m)h$$

which leads to the decomposition of $T_{(m,g)}O(M)$ into its *horizontal* and *vertical* components,

$$(\xi,h)_{(m,g)} = \left(\xi, -dQ(\xi_m)g\right)_{(m,g)} + \left(0, P(m)h\right)_{(m,g)}. \tag{5.8}$$

**Definition 5.6** *The* ***vertical sub-bundle,*** $T^vO(M)$, *of* $TO(M)$ *is defined by;*

$$T^v_{(m,g)}O(M) = \mathrm{Nul}\left(\pi_{*(m,g)}\right) = \left\{(0,h)_{(m,g)} : \; Q(m)h = 0 \text{ and } g^*h \in so(d)\right\}. \tag{5.9}$$

*The* ***horizontal sub-bundle,*** $T^\nabla O(M)$, *associated to the Levi-Civita covariant derivative,* $\nabla$, *is defined by*

$$T^\nabla_{(m,g)}O(M) = \left\{(\xi, -dQ(\xi)g)_{(m,g)} : \xi \in \tau_m M\right\}. \tag{5.10}$$

According to Eq. (5.8),

$$T_{(m,g)}O(M) = T^v_{(m,g)}O(M) \oplus T^\nabla_{(m,g)}O(M) \text{ for all } (m,g) \in O(M).$$

**Example 5.7 (Horizontal Lifts)** *A smooth path* $u(t) = (x(t), g(t)) \in O(M)$ *is* ***horizontal*** *if* $\dot{u}(t) \in T^\nabla_{u(t)}O(M)$ *which happens iff* $g(t)$ *solves Eq. (5.2). Given a smooth path,* $x(\cdot)$, *in* $M$ *and* $(x(0), g_0) \in O_{x(0)}(M)$, *there is a unique horizontal path* $u(t) \in O(M)$ *(called the* ***horizontal lift*** *of* $x$*) such that* $u(0) = (x(0), g_0)$. *The relationship of parallel translation to horizontal lifts is given by*

$$//_t(x)\, v_{x(0)} = \left[g(t)g_0^{-1}v\right]_{x(t)} \text{ for all } v \in \tau_{x(0)}M.$$

**Definition 5.8 (Horizontal Lifts of Vector Fields)** *If* $W \in \Gamma(TM)$ *and* $u = (m,g) \in O(M)$, *let*

$$W^\nabla(m,g) = (W(m), -dQ(W(m))g). \tag{5.11}$$

*We may also describe* $W^\nabla$ *by*

$$W^\nabla(u) := \frac{d}{dt}|_0 //_t(\sigma)\, u \text{ where } \dot{\sigma}(0) = W(\pi(u)) \tag{5.12}$$

*or alternatively as the unique horizontal vector field,* $W^\nabla \in \Gamma\left(T^\nabla O(M)\right)$, *such that* $\pi_* W^\nabla = W \circ \pi$.

**Lemma 5.9** *If* $u(t)$ *is the horizontal lift of a smooth path* $x(\cdot)$ *in* $M$ *starting at* $(x(0), g_0)$, *then* $u(t)$ *is the unique solution to the ordinary differential equation,*

$$\dot{u}(t) = V^\nabla_{\dot{x}(t)}(u(t)) \text{ with } u(0) = (x(0), g_0). \tag{5.13}$$

*where* $V_z(m) = P(m)z$ *for all* $z \in E$ *and* $m \in M$ *as in Example 3.7.*

**Proof.** A path $u(t) = (x(t), g(t)) \in O(M)$ solves Eq. (5.13) iff

$$(\dot{x}(t), \dot{g}(t))_{u(t)} = V^\nabla_{\dot{x}(t)}(u(t)) = \left(V_{\dot{x}(t)}(x(t)), -dQ\left(V_{\dot{x}(t)}(x(t))\right)g(t)\right) = (\dot{x}(t), -dQ(\dot{x}(t))g(t)),$$

i.e. iff $g(t)$ solves Eq. (5.2). ■

To end this subsection let us recall that the horizontal/vertical sub-bundle decomposition of $TO(M)$ in Definition 5.6 gives rise to two "canonical" vector fields and one forms on $O(M)$.

**Definition 5.10** *Let* $u = (m,g) \in O(M)$. *The* ***canonical vertical vector field*** *on* $O(M)$ *associated to* $A \in so(d)$ *is defined by*

$$\mathcal{V}_A(u) := \frac{d}{dt}|_0 ue^{tA} = (0, uA)_{(m,g)} \in T^v_u O(M) \tag{5.14}$$

*while the* ***horizontal vector field*** *associated to* $a \in \mathbb{R}^d$ *(determined by* $\nabla$*) is defined by*

$$B_a(u) = B^\nabla_a(u) = (ga, -dQ(ga)g)_{(m,g)} \in T^\nabla_u O(M). \tag{5.15}$$

**Definition 5.11** *Let $u = (m, g) \in O(M)$. The **canonical** $\mathbb{R}^d$ – valued **one-form**, $\theta$, on $O(M)$ is defined by*

$$\theta\left((\xi, h)_{(m,g)}\right) := g^{-1}\xi = g^*\xi \text{ for all } (\xi, h)_{(m,g)} \in T_u O(M). \tag{5.16}$$

*The **connection one-form** on $O(M)$ determined by the covariant derivative $\nabla$ is given by*

$$\omega^\nabla\left((\xi, h)_{(m,g)}\right) = g^{-1}\left[h + dQ(\xi_m) g\right] \in so(d), \tag{5.17}$$

*where $u(t) = (\sigma(t), g(t))$ is any smooth curve in $O(M)$ such that $\dot{u}(0) = (\xi, h)_{(m,g)}$.*

**Remark 5.12** *Since $g^{-1} = g^*$ and*

$$g^* dQ(\xi_m) g = g^* P(m) dQ(\xi_m) P(m) g = 0.$$

*we may express $\omega^\nabla$ more simply as,*

$$\omega^\nabla\left((\xi, h)_{(m,g)}\right) = g^* h. \tag{5.18}$$

*Also, if $u(t) := (x(t), g(t))$ is a smooth path in $O(M)$ then*

$$\frac{\nabla}{dt}\left[(x(t), g(t) a)\right] := \left(x(t), g(t)\omega^\nabla(\dot{u}(t)) a\right) \text{ for all } a \in \mathbb{R}^d$$

*from which it follows that $u(t)$ is horizontal iff $\frac{\nabla}{dt}\left[(x(t), g(t) a)\right] = 0$ for all $a \in \mathbb{R}^d$.*

## 5.3 Rough Parallel Translation on $O(M)$

As in Proposition 3.12 we may choose to write $\Gamma$ for $dQ$. The following definition is motivated by Lemma 5.9 above.

**Definition 5.13 (Parallel Translation on $M$)** *Given $\mathbf{X} \in WG_p(M)$ and $u_0 \in O_{x_0}(M)$, we say $\mathbf{U} \in WG_p(O(M))$ is **parallel translation** along $\mathbf{X}$ starting at $u_0$ if $\mathbf{U}$ solves the RDE,*

$$d\mathbf{U} = V^\nabla_{d\mathbf{X}}(u) \text{ with } u(0) = u_0, \tag{5.19}$$

*where $V_z(x) := P_x z$ as in Example 3.7 and $V_z^\nabla$ is its horizontal lift as in Definition 5.8. [In Proposition 5.15 below it will be shown that Eq. (5.19) has global solutions, i.e. $\mathbf{U}$ exists on $[0, T]$.]*

**Lemma 5.14** *If $\mathbf{U}$ is parallel translation along $\mathbf{X}$ as in Definition 5.13, then $\pi_*(\mathbf{U}) = \mathbf{X}$.*

**Proof.** From Definition 5.8 we know that $V^\nabla$ and $V$ are $\pi$ – related dynamical systems and therefore by Theorem 4.11, $\hat{\mathbf{X}} := \pi_*(\mathbf{U})$ solves the RDE,

$$d\hat{\mathbf{X}} = V_{d\mathbf{X}}(\hat{x}) \text{ with } \hat{x}_0 = \pi(u_0) = x_0.$$

On the other hand by the consistence Proposition 4.13 we know $\mathbf{X}$ satisfies the same RDE and so by uniqueness of solutions to RDEs we conclude that $\mathbf{X} = \hat{\mathbf{X}} = \pi_*(\mathbf{U})$. ■

**Proposition 5.15** *Suppose that $\mathbf{X} \in WG_p(M)$, $\mathbf{A} := \int \Gamma(d\mathbf{X})$, where $\Gamma := dQ$ and $\mathbf{U} = (u = (x_t, g_t), \mathbb{U}) \in WG_p(O(M))$ is parallel translation along $\mathbf{X}$ starting at $u_0 = (x_0, g_0)$. Then $g$ satisfies the level one component of the RDE,*

$$d\mathbf{g} = (-d\mathbf{A}) g = Y^G_{d\mathbf{A}}(g). \tag{5.20}$$

*In particular, the RDE in Eq. (5.19) exists for all time that $\mathbf{X}$ is defined.]*

**Proof.** Using $d\mathbf{X} = V_{d\mathbf{X}}(x)$ along with item 2. of Theorem 4.5 implies

$$a_{s,t} = \left[\int \Gamma(d\mathbf{X})\right]^1_{s,t} \simeq \Gamma\left(V_{x_{s,t}}(x_s)\right) + \left(V_a\Gamma(V_b)\right)(x_s)|_{a\otimes b=\mathbb{X}_{s,t}} \text{ and}$$
$$\mathbb{A}_{st} = \left[\int \Gamma(d\mathbf{X})\right]^2_{s,t} \simeq \Gamma(V_a(x_s)) \otimes \Gamma(V_b(x_s))|_{a\otimes b=\mathbb{X}_{s,t}}$$

Now let $f : O(M) \to \mathrm{End}\left(\mathbb{R}^d, E\right)$ be the projection map, $f(x,g) = g$. From Theorem 4.5,

$$g_{st} = [f(u)]_{s,t} \simeq \left(V^{\nabla}_{x_{s,t}} f\right)(u_s) + \left(\mathcal{V}^{\nabla}_{\mathbb{X}_{st}} f\right)(u_s).$$

Combing this equation with the identities,

$$\left(V^{\nabla}_b f\right)(x,g) = -\Gamma(V_b(x))\, g \text{ and}$$
$$\left(V^{\nabla}_a V^{\nabla}_b f\right)(x,g) = \Gamma(V_b(x))\,\Gamma(V_a(x))\, g - (V_a\Gamma(V_b))(x)\, g,$$

shows

$$g_{st} \simeq -\Gamma\left(V_{x_{s,t}}(x_s)\right) g_s + \left[\Gamma(V_b(x_s))\,\Gamma(V_a(x_s))\, g_s - (V_a\Gamma(V_b))(x_s)\, g_s\right]|_{a\otimes b=\mathbb{X}_{st}}$$
$$\simeq -a_{s,t} g_s + [\mathbb{A}_{s,t}]\, g_s.$$

where $[A \otimes B] := BA$. Similarly if we let $\mathcal{I}(g) = g$, the RDE in Eq. (5.20) is equivalent to

$$g_{s,t} = \left(Y^G_{a_{s,t}}\mathcal{I}\right)(g_s) + \left(Y^G_a Y^G_b \mathcal{I}\right)(g_s)|_{a\otimes b=\mathbb{A}_{s,t}}$$
$$= -a_{s,t} g_s + bag_s|_{|a\otimes b=\mathbb{A}_{s,t}} = -a_{s,t} g_s + [\mathbb{A}_{s,t}]\, g_s.$$

From the theory of linear RDE [30] or by a minor modification of the results in Theorem 4.20 we know that $\mathbf{G}$ solving Eq. (5.20) does not explode. Therefore we may then conclude that $u_t = (x_t, g_t)$ has no explosion. Combining this result with Lemma 2.18 then shows that the RDE of Eq. (5.19) also does not explode. ■

**Theorem 5.16** *Let $\mathbb{R}^n \ni z \to Y_z \in \Gamma(TM)$ be a linear map, $\mathbf{Z} \in WG_p(\mathbb{R}^n)$, and $u_0 \in O(M)$ be given. If $\mathbf{X} \in WG_p(M)$ and $\mathbf{U} \in WG_p(O(M))$ solve the RDEs*

$$d\mathbf{X} = Y_{d\mathbf{Z}}(x) \text{ with } x_0 := \pi(u_0) \in M \text{ and} \tag{5.21}$$
$$d\mathbf{U} = Y^{\nabla}_{d\mathbf{Z}}(u) \text{ with } u(0) = u_0, \tag{5.22}$$

*then $\mathbf{U}$ is a parallel translation along $\mathbf{X}$, i.e. $\mathbf{X} = \pi_*(\mathbf{U})$ and $\mathbf{U}$ satisfies Eq. (5.19).*

**Proof.** Since $Y^{\nabla}$ and $Y$ are $\pi$ related it follows from Theorem 4.11 that $\mathbf{X} = \pi_*(\mathbf{U})$. Using Theorem 4.5 and Remark 4.6, Eq. (5.22) at the first level is equivalent to

$$F(u)_{s,t} \simeq \left(Y^{\nabla}_{z_{s,t}} F\right)(u_s) + \left(\mathcal{Y}^{\nabla}_{\mathbb{Z}_{s,t}} F\right)(u_s) \tag{5.23}$$

while $\mathbf{U}$ solving Eq. (5.19) at the first level is equivalent to

$$F(u)_{s,t} \simeq \left(V^{\nabla}_{x_{s,t}} F\right)(u_s) + \left(\mathcal{V}^{\nabla}_{\mathbb{X}_{s,t}} F\right)(u_s), \tag{5.24}$$

where in each case $F$ is assumed to be an arbitrary smooth function on $O(M)$. Thus to complete the proof we must show Eq. (5.23) implies Eq. (5.24) and show the second-order condition

$$\mathbb{U}_{s,t} \simeq \left[V^{\nabla}_{\cdot}(u_s) \otimes V^{\nabla}_{\cdot}(u_s)\right] \mathbb{X}_{s,t}. \tag{5.25}$$

First recall that

$$
\begin{aligned}
Y_z^{\nabla}(x,g) &= (Y_z(x), -\Gamma(Y_z(x))g) \text{ and} \\
V_\xi^{\nabla}(x,g) &= (V_\xi(x), -\Gamma(V_\xi(x))g) = (P(x)\xi, -\Gamma(P(x)\xi)g)
\end{aligned}
$$

so that $Y_z^\nabla = V_{Y_z}^\nabla$ from which (5.25) can be deduced immediately, and also

$$
\begin{aligned}
\mathcal{Y}_{a\otimes b}^{\nabla}F &= Y_a^\nabla Y_b^\nabla F = V_{Y_a}^\nabla V_{Y_b}^\nabla F \\
&= V_{Y_a}^\nabla V_\beta^\nabla F|_{\beta=Y_b} + V_{(Y_aY_b)}^\nabla F.
\end{aligned}
$$

Putting this together with Eq. (5.23) shows,

$$
F(u)_{s,t} \simeq dF\left(V^\nabla_{Y_{z_{s,t}}(x_s)+\nabla_{Y_a(x_s)}Y_b|_{a\otimes b=\mathbb{Z}_{s,t}}}(u_s)\right) + \left(\mathcal{V}^\nabla_{\alpha\otimes\beta}F\right)(u_s)\,|_{\alpha\otimes\beta=\left[Y_{(\cdot)}(x_s)\otimes Y_{(\cdot)}(x_s)\right]\mathbb{Z}_{s,t}}.
$$

Thus to finish the proof we must show

$$
x_{s,t} \simeq Y_{z_{s,t}}(x_s) + Y_a(x_s)Y_b|_{a\otimes b=\mathbb{Z}_{s,t}}. \tag{5.26}
$$

But we already know that $\mathbf{X}$ solves Eq. (5.21) which applied to the identity function $\mathcal{I}$ on $\mathbb{R}^N$ shows

$$
\begin{aligned}
x_{s,t} = \mathcal{I}(x)_{s,t} &\simeq \left(Y_{z_{s,t}}\mathcal{I}\right)(x_s) + \left(Y_{(\cdot)}Y_{(\cdot)}\mathcal{I}\right)(x_s)\,\mathbb{Z}_{s,t} \\
&= Y_{z_{s,t}}(x_s) + \left(Y_{(\cdot)}Y_{(\cdot)}\mathcal{I}\right)(x_s)\,\mathbb{Z}_{s,t}
\end{aligned}
$$

which is precisely Eq. (5.26). ■

We will actually be more interested in the following variant of Theorem 5.16.

**Theorem 5.17** *Suppose that* $\mathbf{Z} = (z,\mathbb{Z}) \in WG_p\left([0,T], \mathbb{R}^d, 0\right)$ *and* $\mathbf{U} \in WG_p(O(M))$ *solves*

$$
d\mathbf{U} = B^\nabla_{d\mathbf{Z}_t}(u_t) \text{ with } u_0 = u_o \text{ given.} \tag{5.27}
$$

*Then* $\mathbf{U}$ *is a parallel translation along* $\mathbf{X} = \pi_*(\mathbf{U})$, *i.e.* $\mathbf{U}$ *satisfies Eq. (5.19).*

**Proof.** Working as above, Eq. (5.27) is equivalent to

$$
F(u)_{s,t} \simeq \left(B^\nabla_{z_{s,t}}F\right)(u_s) + \left(\mathcal{B}^\nabla_{\mathbb{Z}_{st}}F\right)(u_s) \tag{5.28}
$$

while $\mathbf{U}$ solving Eq. (5.19) is equivalent to Eq. (5.24) where in each case $F$ is assumed to be an arbitrary smooth function on $O(M)$. Thus to complete the proof we must show Eq. (5.28) implies Eq. (5.24) and the correspondence of the second-order pieces by the approximate identity

$$
\mathbb{U}_{s,t} \simeq \left[V_\cdot^\nabla(u_s) \otimes V_\cdot^\nabla(u_s)\right]\mathbb{X}_{s,t} \tag{5.29}
$$

First recall that

$$
\begin{aligned}
B_z^\nabla(x,g) &= \left((gz)_x, -\Gamma\left((gz)_x\right)g\right) \text{ and} \\
V_\xi^\nabla(x,g) &= (V_\xi(x), -\Gamma(V_\xi(x))g) = (P(x)\xi, -\Gamma(P(x)\xi)g)
\end{aligned}
$$

so that $B_z^\nabla(x,g) = V_{gz}^\nabla(x,g)$. (5.29) is then immediate from the calculation

$$
\mathbb{U}_{s,t} \simeq \left[B_\cdot^\nabla(u_s)\otimes B_\cdot^\nabla(u_s)\right]\mathbb{Z}_{s,t} \simeq \left[V_\cdot^\nabla(u_s)\otimes V_\cdot^\nabla(u_s)\right]\mathbb{X}_{s,t}.
$$

Furthermore writing $u = (x,g)$ we have

$$
\begin{aligned}
\left(\mathcal{B}^\nabla_{a\otimes b}F\right)(u) &= \left(B_a^\nabla B_b^\nabla F\right)(u) = B_a^\nabla\left[(x,g)\to\left(V_{gb}^\nabla F\right)(x,g)\right] \\
&= \left(V^\nabla_{-\Gamma\left((ga)_x\right)gb}F\right)(x,g) + \left(\mathcal{V}^\nabla_{ga\otimes gb}F\right)(u)
\end{aligned}
$$

and putting this together with Eq. (5.28) shows,

$$F(u)_{s,t} \simeq dF\left(V^{\nabla}_{g_s z_{s,t} - \Gamma\left((ga)_x\right) gb|_{a\otimes b=\mathbb{Z}_{st}}}(u_s)\right) + \left(\mathcal{V}^{\nabla}_{\alpha\otimes\beta} F\right)(u_s)\,|_{\alpha\otimes\beta=[g_s\otimes g_s]\mathbb{Z}_{s,t}}. \tag{5.30}$$

Applying Eq. (5.28) to $F = \pi$ where $\pi(x,g) = x$ shows

$$x_{s,t} = [\pi(u)]_{s,t} \simeq \left(B_{z_{s,t}}\pi\right)(u_s) + \left(\mathcal{B}^{\nabla}_{\mathbb{Z}_{s,t}}\pi\right)(u_s). \tag{5.31}$$

Using

$$(B_b\pi)(x,g) = gb \text{ and}$$
$$(B_a B_b \pi)(x,g) = -\Gamma\left((ga)_x\right) gb,$$

in Eq. (5.31) gives

$$x_{s,t} \simeq g_s z_{s,t} - \Gamma\left((ga)_x\right) gb|_{a\otimes b=\mathbb{Z}_{s,t}} \tag{5.32}$$

which combined with Eq. (5.30) shows Eq. (5.24) does indeed hold. ■

# 6 Rolling and Unrolling

In this section we develop the rough path analogy of Cartan's rolling map. As a consequence we will see that rough paths on a $d$ -dimensional manifold are in one to one correspondence with rough paths on $d$ - dimensional Euclidean space.

**Definition 6.1** *A manifold $M$ is said to* ***parallelizable*** *if there exists a linear map, $Y : \mathbb{R}^d \to \Gamma(TM)$ such that the map*

$$Y_{\cdot}(m) : a \mapsto Y_a(m) \in T_m M$$

*is a linear isomorphism for all $m \in M$. We refer to any choice of $Y : \mathbb{R}^d \to \Gamma(TM)$ with this property as a* ***parallelism*** *of $M$. Associated to a parallelism $Y$ is an $\mathbb{R}^d$ – valued one form on $M$ given by*

$$\theta^Y(v_m) := Y(m)^{-1} v.$$

It is easy to see that every vector space is parallelizable; we detail some other not so trivial examples which will be useful later.

**Example 6.2** *Every Lie group $G$ is parallelizable. Indeed if we let $d = \dim G$, so that the Lie-algebra $\mathfrak{g} := \mathrm{Lie}(G) \cong \mathbb{R}^d$, then $Y = Y^G$ of Eq. (4.21) defines a parallelism on $G$. In this case, the associated one form $\theta^Y$ is known as the (right) Maurer–Cartan form on $G$.*

**Example 6.3** *Let $M = M^d$ be any Riemannian manifold and $O(M)$ be the associated orthogonal frame bundle. Then $O(M)$ is parallelizable with $T_u O(M) \cong \mathbb{R}^d \times so(d)$. In this case we can define a parallelism by taking*

$$Y^{O(M)}(u)(a,A) := B_a(u) + \mathcal{V}_A(u), \tag{6.1}$$

*where $B_a$ and $\mathcal{V}_A$ were defined in Eqs. (5.14) and (5.15). In this case the associated $\mathbb{R}^d \times so(d)$ – valued one form $\theta^{Y^{O(M)}} := (\theta, \omega)$ on $O(M)$ is determined by*

$$(\theta,\omega)\left(B_a(u) + \mathcal{V}_A(u)\right) := (a,A) \text{ for all } (a,A) \in \mathbb{R}^d \times so(d) \text{ and } u \in O(M) \tag{6.2}$$

*where $\theta$ and $\omega$ are as in Definition 5.11.*

## 6.1 Smooth Rolling and Unrolling

The following "rolling and unrolling" theorems in the smooth category are all relatively easy to prove and therefore most proofs are omitted here. They are included as a warm-up to the more difficult rough path versions which are appear in the next subsection.

**Theorem 6.4 (Rolling and Unrolling I)** *Let $M$ be a parallelizable manifold and $Y : \mathbb{R}^d \to \Gamma(TM)$ be a parallelism and $\theta^Y$ be the associated one form. Fix $o \in M$. Then every $x \in C_o^1([0,T],M)$ determines a path in $z \in C_0^1\left([0,T],\mathbb{R}^d\right)$ by*

$$C_o^1([0,T],M) \ni x \to z := \int_0^{\cdot} \theta^Y(dx_s) = \int_0^{\cdot} Y(x_s)^{-1} \dot{x}_s ds \tag{6.3}$$

*Conversely, given $z \in C_0^1\left([0,T],\mathbb{R}^d\right)$ the solution to the differential equation*

$$\dot{x}_t = Y(x_t)\dot{z}_t \text{ with } x_0 = o \in M, \tag{6.4}$$

*which may explode in finite time $\tau = \tau(z) < T$, is such that $x \in C_o^1([0,\tau],M)$ and over $[0,\tau)$*

$$z = \int_0^{\cdot} \theta^Y(dx_s). \tag{6.5}$$

*The solution to (6.4) determines the inverse of the map (6.3); that is, the solution to (6.4) satisfies (6.5) and any $x \in C_o^1([0,\tau), M)$ agrees with the solution $w$ to the differential equation*

$$\dot{w}_t = Y(w_t)\dot{z}_t \text{ with } x_0 = o \in M,$$

*until the explosion time of this equation.*

**Corollary 6.5** *Fix $o \in M$ and $u_o$ an orthogonal frame at $o$. Then every $u \in C_{u_o}^1([0,T],O(M))$ determines an element of $C_{(0,0)}^1\left([0,T],\mathbb{R}^d \times so(d)\right)$ by the map*

$$C_o^1([0,T],O(M)) \ni u \to \int_0^{\cdot} \theta^{Y^{O(M)}}(du) = \int_0^{\cdot} \theta(du) + \int_0^{\cdot} \omega(du) \in C_{(0,0)}^1\left([0,T],\mathbb{R}^d \times so(d)\right) \tag{6.6}$$

*Suppose that $(a,A) \in C_{(0,0)}^1\left([0,T],\mathbb{R}^d \times so(d)\right)$ and define $u$ to be the solution to the differential equation*

$$\dot{u}_t = B_{\dot{a}_t}(u_t) + \mathcal{V}_{\dot{A}_t}(u_t) \text{ with } u_0 = u_o \text{ given,} \tag{6.7}$$

*which may explode in finite time $\tau := \tau(a,A) < T$. Then $u$ is in $C_{u_o}^1([0,T],O(M))$ and over $[0,\tau)$ we have*

$$\int_0^{\cdot} (\theta,\omega)(du) = (a_\cdot, A_\cdot) \tag{6.8}$$

**Theorem 6.6** *The solution to (6.7) determines the inverse to (6.6) until explosion; that is, the solution to (6.7) satisfies (6.8), and any $u \in C_{u_o}^1([0,T],O(M))$ agrees with $w$, the solution to the differential equation*

$$\dot{w}_t = B_{\theta(\dot{u}_t)}(w_t) + \mathcal{V}_{\omega(\dot{u}_t)}(w_t) \text{ with } w_0 = u_o \in O(M),$$

*until the explosion time of this equation.*

**Definition 6.7** *We say a path $u \in C^1([0,T],O(M))$ is* ***horizontal (or parallel)*** *provided $\omega(\dot{u}_t) = 0$, i.e. provided $\int_0^{\cdot} \omega(du) \equiv 0$. We let $\mathcal{H}C^1([0,T],O(M))$ denote the horizontal path in $C^1([0,T],O(M))$.*

**Theorem 6.8** *Let $u \in C^1([0,T],O(M))$ and $x := \pi(u) \in C^1([0,T],M)$ be its projection to $M$. Then $u$ is horizontal iff $u_t = //_t(x)u_0$ for all $0 \le t \le T$.*

**Proof.** If $u \in C^1([0,T], O(M))$ we have from Eq. (5.17) that $\omega(\dot{u}_t) = u_t^{-1} \frac{\nabla u_t}{dt}$ which is zero iff $\frac{\nabla u_t}{dt} = 0$ iff $u_t$ is parallel iff $u_t = //_t(x) u_0$ for all $0 \leq t \leq T$. ■

**Corollary 6.9** *Let $M$ be a Riemannian manifold with $o \in M$ and $u_o \in O(M)$ given. Then the map,*

$$\mathcal{H}C^1_{u_o}([0,T], O(M)) \ni u \to \pi \circ u \in C^1_o([0,T], M) \tag{6.9}$$

*is a bijection with inverse map given by,*

$$C^1_o([0,T], M) \ni x \to u_t := //_t(x) u_o \in \mathcal{H}C^1_{u_o}([0,T], O(M)). \tag{6.10}$$

**Corollary 6.10** *Let $M$ be a Riemannian manifold with $o \in M$ given. Then there exists a one to one correspondence between $C^1_o([0,T], M)$ and $C^1_0\left([0,T], \mathbb{R}^d\right)$ determined by,*

$$\begin{array}{ccccc} C^1_o([0,T], M) & \to & \mathcal{H}C^1_{u_o}([0,T], O(M)) & \to & C^1_0\left([0,T], \mathbb{R}^d\right) \\ x & \to & //_\cdot(x) u_o & \to & \int_0^\cdot \theta(d[//_s(x) u_o]) = u_o^{-1} \int_0^\cdot //_s(x)^{-1} dx_s \end{array}$$

## 6.2 Rough Rolling and Unrolling

**Theorem 6.11 (Rough Rolling and Unrolling I)** *Let $M$ be a parallelizable manifold and $Y : \mathbb{R}^d \to \Gamma(TM)$ be a parallelism and $\theta^Y$ be the associated one form. Fix $o \in M$. Then every $\mathbf{X} \in WG_p([0,T], M, o)$ determines an element of $WG_p\left([0,T], \mathbb{R}^d, 0\right)$ by the map*

$$WG_p([0,T], M, o) \ni \mathbf{X} \to \mathbf{Z} := \int_0^\cdot \theta^Y(d\mathbf{X}) \in WG_p\left([0,T], \mathbb{R}^d, 0\right). \tag{6.11}$$

*Suppose that $\mathbf{Z} = (z, \mathbb{Z}) \in WG_p\left([0,T], \mathbb{R}^d, 0\right)$ and let $\mathbf{X}$ denote the solution to the RDE*

$$d\mathbf{X} = Y_{d\mathbf{Z}_t}(x_t) \text{ with } x_0 = o \in M, \tag{6.12}$$

*with possible explosion time $\tau := \tau(Y, \mathbf{Z}) < T$. Then $\mathbf{X}$ is in $WG_p([0,\tau), M, o)$ and over $[0,\tau)$ we have*

$$\int_0^\cdot \theta^Y(d\mathbf{X}) = \mathbf{Z}. \tag{6.13}$$

*The solution to (6.12) determines the inverse to (6.11) until explosion; that is, both (6.13) holds, and any $\mathbf{X} \in WG_p(M, o)$ agrees with $\mathbf{W}$ the solution to the RDE*

$$d\mathbf{W}_t = Y(w_t)\, d\left[\int_0^t \theta^Y(d\mathbf{X})\right] \text{ with } w_0 = o \in M \tag{6.14}$$

*until the explosion time of this equation.*

**Proof.** Suppose that $\mathbf{Z} \in WG_p\left(\mathbb{R}^d, 0\right)$ and let $\mathbf{X}$ solve Eq. (6.12). Since $\theta^Y(Y_a) = a$ for all $a \in \mathbb{R}^d$, it follows that $Y_a\left[\theta^Y(Y_b)\right] = Y_a[b] = 0$ and hence from item 2. of Theorem 4.5

$$\left[\int \theta^Y(d\mathbf{X})\right]^1_{s,t} \simeq \theta^Y_{x_s}(Y(x_s) z_{s,t}) = z_{s,t}$$

and

$$\left[\int \theta^Y(d\mathbf{X})\right]^2_{s,t} \simeq [\theta_{x_s} Y(x_s) \otimes \theta_{x_s} Y(x_s)]\, \mathbb{Z}_{s,t} = \mathbb{Z}_{s,t}.$$

Conversely, suppose that $\mathbf{X} \in WG_p(M, o)$ and now *define* $\mathbf{Z} = (z, \mathbb{Z})$ by $\mathbf{Z} = \int_0^\cdot \theta^Y(d\mathbf{X})$. We need to show, making the usual caveat about explosion, that $\mathbf{X}$ is the solution to (6.14). To this end, we first note

$$\mathbb{Z}_{s,t} \simeq [\theta_{x_s} \otimes \theta_{x_s}][P_{x_s} \otimes P_{x_s}]\, \mathbb{X}_{s,t} \text{ and } z_{s,t} \simeq \theta_{x_s} P_{x_s} x_{s,t} + \nabla\theta [P_{x_s} \otimes P_{x_s}]\, \mathbb{X}_{s,t}.$$

Since $Y\theta = Id_{TM}$ it follows from the last two equations that

$$\begin{aligned} Y(x_s) \otimes Y(x_s) \mathbb{Z}_{s,t} &\simeq Y(x_s) \otimes Y(x_s) [\theta_{x_s} \otimes \theta_{x_s}] [P_{x_s} \otimes P_{x_s}] \mathbb{X}_{s,t} \\ &\simeq [P_{x_s} \otimes P_{x_s}] \mathbb{X}_{s,t} \end{aligned}$$

and

$$Y(x_s) z_{s,t} \simeq P_{x_s} x_{s,t} + Y(x_s) \nabla\theta [P_{x_s} \otimes P_{x_s}] \mathbb{X}_{s,t} \simeq P_{x_s} x_{s,t} + Y(x_s) \nabla\theta [Y(x_s) \otimes Y(x_s) \mathbb{Z}_{s,t}]$$

or, equivalently, that

$$\mathbb{X}_{s,t} \simeq [P_{x_s} \otimes P_{x_s}] \mathbb{X}_{st} \simeq Y(x_s) \otimes Y(x_s) \mathbb{Z}_{s,t} \tag{6.15}$$

and

$$P_{x_s} x_{s,t} \simeq Y(x_s) z_{s,t} - Y(x_s) \nabla\theta [Y(x_s) \otimes Y(x_s) \mathbb{Z}_{s,t}]. \tag{6.16}$$

Again using the fact that $Y\theta = Id_{TM}$ we see that

$$0 = \nabla Id_{TM} = \nabla [Y\theta] = (\nabla Y)\theta + Y\nabla\theta,$$

which combined with Eq. (6.16) and the fact that $\theta Y_a = a$ for all $a \in \mathbb{R}^d$ implies

$$\begin{aligned} P_{x_s} x_{s,t} &\simeq Y(x_s) z_{s,t} + \left(\nabla_{(\cdot)} Y\right) \theta(\cdot) [Y(x_s) \otimes Y(x_s) \mathbb{Z}_{s,t}] \\ &= Y(x_s) z_{s,t} + \left(\nabla_{Y(\cdot)} Y_{(\cdot)}\right) \mathbb{Z}_{s,t} \\ &= Y_{z_{s,t}}(x_s) + P_{x_s} (\partial_{Y_a} Y_b)(x_s)|_{a\otimes b = \mathbb{Z}_{s,t}}. \end{aligned} \tag{6.17}$$

It only remains to show

$$Q_{x_s} x_{s,t} \simeq Q_{x_s} (\partial_{Y_a} Y_b)(x_s)|_{a\otimes b = \mathbb{Z}_{s,t}} \tag{6.18}$$

since adding Eqs. (6.17) and (6.18) gives Eq. (4.2) while Eq. (6.15) is the same as Eq. (4.3) and these equations are equivalent to $\mathbf{X} \in WG_p(M, o)$ solving Eq. (6.12). However from Eq. (3.15) of Lemma 3.28,

$$Q_{x_s} x_{s,t} \simeq Q_{x_s} \left(\partial_{P_{x_s}(\cdot)} P\right) \mathbb{X}_{s,t} \simeq Q_{x_s} \left(\partial_{P_{x_s}(\cdot)} P\right) [Y(x_s) \otimes Y(x_s)] \mathbb{Z}_{s,t} = \left[Q_{x_s} \left(\partial_{Y(x_s)} P\right) Y(x_s)\right] \mathbb{Z}_{s,t}.$$

This gives Eq. (6.18) since $Q(\partial_Y P) Y = Q\partial_Y Y$ which is proved by applying $Q$ to the identity,

$$\partial_Y Y = \partial_Y [PY] = (\partial_Y P) Y + P\partial_Y Y.$$

■

## 6.3 Rolling via the frame bundle

We can specialize this result to $O(M)$. Making use of the notation in Example 6.3. we obtain the following.

**Corollary 6.12** *Fix $o \in M$ and $u_o$ an orthogonal frame at $o$. Then every $\mathbf{U} \in WG_p([0,T], O(M), u_o)$ determines an element of $WG_p\left([0,T], \mathbb{R}^d \times so(d), (0,0)\right)$ by the map*

$$\begin{array}{ccc} WG_p([0,T], O(M), u_o) & \longrightarrow & WG_p\left([0,T], \mathbb{R}^d \times so(d), (0,0)\right) \\ \mathbf{U} & \longrightarrow & \int_0^\cdot \theta^{Y^{O(M)}}(d\mathbf{U}_t) := \int_0^\cdot (\theta, \omega)(d\mathbf{U}_t) \end{array} \tag{6.19}$$

*Suppose that $\mathbf{Z} \in WG_p\left(\mathbb{R}^d \times so(d), (0,0)\right)$, and let $\mathbf{a}, \mathbf{A}$ denote the projections of $\mathbf{Z}$ to the elements of $WG_p\left(\mathbb{R}^d, 0\right)$ and $WG_p(so(d), 0)$ respectively. Define $\mathbf{U}$ to be the solution to the RDE*

$$d\mathbf{U} = Y^{O(M)}_{d\mathbf{Z}_t}(u_t) \text{ with } u_0 = u_o \text{ given,} \tag{6.20}$$

*which may explode in finite time $\tau := \tau(\mathbf{Z}) < T$. Then $\mathbf{U}$ is in $WG_p([0,\tau), O(M), u_o)$ and over $[0,\tau)$ we have*

$$\int_0^\cdot \theta^{Y^{O(M)}}(d\mathbf{U}) = \mathbf{Z}. \tag{6.21}$$

**Theorem 6.13** *The solution to (6.20) determines the inverse to the map (6.19) until explosion; that is, the solution to (6.20) satisfies (6.21), and any* $\mathbf{U} \in WG_p(O(M), u_o)$ *agrees with* $\mathbf{W}$, *the solution to the RDE*

$$d\mathbf{W} = Y^{O(M)}(w_t)\, d\left[\int_0^t \theta^{Y^{O(M)}}(d\mathbf{U}_s)\right] \text{ with } w_0 = u_o \in O(M),$$

*until the explosion time of this equation.*

**Definition 6.14** *We say a rough path* $\mathbf{U} \in WG_p([0,T], O(M), u_o)$ *is* ***horizontal*** *provided*

$$\int (\theta, \omega)(d\mathbf{U}_t) = \int \theta^{Y^{O(M)}}(d\mathbf{U}) \in WG_p\left([0,T], \mathbb{R}^d \times \left\{0_{so(d)}\right\}, (0,0)\right), \tag{6.22}$$

*where* $\omega$ *is the connection one-form defined in (5.17) and* $\theta$ *is the canonical one form on* $O(M)$ *defined in Eq. (5.16). We use* $\mathcal{H}WG_p([0,T], O(M), u_o)$ *to denote the horizontal rough paths* $WG_p([0,T], O(M), u_o)$.

**Remark 6.15** *Another way to state Eq. (6.22) is that* $\mathbf{U} \in WG_p([0,T], O(M), u_o)$ *is* ***horizontal*** *provided,*

$$\int (\theta, \omega)(d\mathbf{U}) = \int (\theta, \mathbf{0})(d\mathbf{U}_t)$$

*where* $\mathbf{0} \in \Omega^1(O(M), so(d))$ *is the identically zero one form on* $O(M)$ *with values in* $so(d)$. *Consequently* $\mathbf{U} \in WG_p([0,T], O(M), u_o)$ *is horizontal implies* $\int \omega(d\mathbf{U}) = 0$. *On the other hand it is not enough to assume* $\int \omega(d\mathbf{U}) = 0$ *in order to conclude* $\mathbf{U}$ *is horizontal because the condition* $\int \omega(d\mathbf{U}) = 0$ *does not rule out* $\left[\int (\theta, \omega)(d\mathbf{U})\right]^2$ *having cross term components, i.e. components in* $\mathbb{R}^d \otimes so(d) \oplus so(d) \otimes \mathbb{R}^d$.

**Proposition 6.16 (Parallel implies horizontal)** *If* $\mathbf{U} \in WG_p([0,T], O(M), u_o)$ *is parallel translation along* $\mathbf{X} := \pi_*(\mathbf{U}) \in WG_p(M, o)$, *then* $\mathbf{U}$ *is horizontal.*

**Proof.** Recall $\Gamma = dQ$ and that $\mathbf{U}$ solves (see Definition 5.13), $d\mathbf{U} = V^\nabla_{d\mathbf{X}}(u)$ where $V_a^\nabla(m,g) = (V_a(m), -\Gamma(V_a(m))\, g)$ and $V_a(m) = P_m a$ for all $a \in E$. Using these formulas we find for $u = (m,g) \in O(M)$ and $a, b \in E$ that,

$$\theta^{Y^{O(M)}}\left(V_b^\nabla(u)\right) = (\theta, \omega)\left(V_b^\nabla(u)\right) = (g^* V_b(m), 0)$$

and

$$\begin{aligned} V_a^\nabla(u)\left[\theta^{Y^{O(M)}}\left(V_b^\nabla\right)\right] &= V_a^\nabla(u)\left[(x,h) \to (h^* V_b(x), 0)\right] \\ &= \left(g^*(\partial_{V_a} V_b)(m) - g^*\Gamma(V_a(m))\, V_b(m), 0\right) = \left(g^* \nabla_{V_a(m)} V_b, 0\right), \end{aligned}$$

wherein in the last line we have used $Pg = g$ so that $g^* = g^* P$ and hence

$$g^*\Gamma(V_a(m))\, V_b(m) = g^* P_m dQ(V_a(m))\, P_m V_b(m) = 0.$$

From these identities and item 2. of Theorem 4.5 we conclude,

$$\left[\int \theta^{Y^{O(M)}}(d\mathbf{U})\right]^1_{s,t} \simeq \left(g_s^* V_{x_{st}}(x_s), 0\right) + \left(g_s^* \nabla_{V_a(x_s)} V_b, 0\right)|_{a\otimes b = \mathbb{X}_{s,t}}$$

and

$$\left[\int \theta^{Y^{O(M)}}(d\mathbf{U})\right]^2_{s,t} \simeq \theta^{Y^{O(M)}}\left(V_a^\nabla(u_s)\right) \otimes \theta^{Y^{O(M)}}\left(V_b^\nabla(u_s)\right)|_{a\otimes b = \mathbb{X}_{s,t}} = (g_s^* V_a(x_s), 0) \otimes (g_s^* V_b(x_s), 0)$$

from which it follows that

$$\int \theta^{Y^{O(M)}}(d\mathbf{U}) \in WG_p\left([0,T], \mathbb{R}^d \times \left\{0_{so(d)}\right\}, (0,0)\right).$$

■

**Theorem 6.17** *Let* $\mathbf{U} \in WG_p([0,T], O(M), u_o)$ *and* $\mathbf{X} := \pi_*(\mathbf{U}) \in WG_p(M, o)$ *be its push-forward under the projection* $\pi : O(M) \to M$*. Then the following are equivalent;*

1. $\mathbf{U}$ *is horizontal.*
2. *there exist* $\mathbf{a} \in WG_p([0,T], \mathbb{R}^d, 0)$ *such that*
$$d\mathbf{U} = B_{d\mathbf{a}_t}(u_t) \text{ with } u_0 = u_o \text{ given,} \tag{6.23}$$
3. *and* $\mathbf{U}$ *is parallel translation along* $\mathbf{X}$ *starting at* $u_0$*.*

**Proof.** From Theorem 5.17 we know 2. $\implies$ 3. and from Proposition 6.16 we know 3. $\implies$ 1. So to finish the proof it suffices to show 1. $\implies$ 2. For the proof of this assertion let

$$\mathbf{Z} := \int \theta^{Y^{O(M)}}(d\mathbf{U}) \in WG_p([0,T], \mathbb{R}^d \times so(d), (0,0))$$

and

$$\mathbf{a} := (a, \mathbb{A}) = \int P_{\mathbb{R}^d} d\mathbf{Z} = (P_{\mathbb{R}^d} z_{st}, P_{\mathbb{R}^d} \otimes P_{\mathbb{R}^d} \mathbb{Z}_{st})$$

where $P_{\mathbb{R}^d} : \mathbb{R}^d \times so(d) \to \mathbb{R}^d$ is the linear projection onto the first factor.

(1. $\implies$ 2.) By definition $\mathbf{U}$ is horizontal iff $\mathbf{Z} := \int \theta^{Y^{O(M)}}(d\mathbf{U}) \in WG_p([0,T], \mathbb{R}^d \times \{0\}, (0,0))$. Corollary 6.12 then asserts that

$$d\mathbf{U} = \mathcal{Y}^{O(M)}_{d\mathbf{Z}}(u_t) \text{ with } u_0 = u_o.$$

As $\mathbf{Z} \in WG_p([0,T], \mathbb{R}^d \times \{0\}, (0,0))$ one easily verifies that $\mathcal{Y}^{O(M)}_{\mathbf{Z}_{st}} = B_{\mathbf{a}_{st}}$ from which it follows that the previously displayed RDE is equivalent to the RDE in Eq. (6.23). ■

**Theorem 6.18** *Let* $M$ *be a Riemannian manifold with* $o \in M$ *and* $u_o \in O_o(M)$ *given. Then the map,*

$$\mathcal{H}WG_p([0,T], O(M), u_o) \ni \mathbf{U} \to \pi_*(\mathbf{U}) \in WG_p([0,T], M, o) \tag{6.24}$$

*is a bijection with inverse map given by,*

$$WG_p([0,T], M, o) \ni \mathbf{X} \to \mathbf{H}u_0 \in \mathcal{H}WG_p([0,T], O(M), u_o), \tag{6.25}$$

*where* $\mathbf{H}u_0 := \mathbf{U}$ *is parallel translation along* $\mathbf{X}$ *starting at* $u_0$ *as in Definition 5.13 and Proposition 5.15.*

**Corollary 6.19** *Let* $M$ *be a Riemannian manifold with* $o \in M$ *given. Then there exists a one-to-one correspondence between* $WG_p([0,T], M, o)$ *and* $WG_p(\mathbb{R}^d, 0)$ *determined for any choice of initial frame* $u_o \in O_o(M)$ *by*

$$\begin{array}{ccccc} WG_p([0,T], M, o) & \to & \mathcal{H}WG_p([0,T], O(M), u_o) & \to & WG_p(\mathbb{R}^d, 0) \\ \mathbf{X} & \to & \mathbf{U} = \mathbf{H}u_o & \to & \int_0^\cdot \theta(d\mathbf{H}u_o), \end{array} \tag{6.26}$$

*where* $\theta$ *is the canonical one-form.*

# A Some additional rough path results

In this section we gather some additional results and notation of the theory of rough paths on Banach spaces. The literature on Banach space valued rough paths is now so well-established as to be classical; the reader seeking more background has a great many choices: [29], [30], [17], [23], [19] and [15]. As in Section 2, let $V, W$ and $U$ denote Banach spaces. In addition we assume $p \in [2,3)$ is a fixed number and $\omega$ a control in the sense of Definition 2.3. Recall the definition of a $p-$rough path and $R_p(V)$, the set of $p$-rough paths on $V$ from Definition 2.4.

We can define a metric on $R_p(V)$ by setting

$$\rho_{p,\omega}(\mathbf{X},\mathbf{Y}) := \sup_{0\le s<t\le T} \frac{|x_{s,t}-y_{s,t}|_V}{\omega(s,t)^{1/p}} + \sup_{0\le s<t\le T} \frac{|\mathbb{X}_{s,t}-\mathbb{Y}_{s,t}|}{\omega(s,t)^{2/p}}, \tag{A.1}$$

for $\mathbf{X}=(x,\mathbb{X})$, $\mathbf{Y}=(y,\mathbb{Y})\in R_p(V)$. Note that endowed with this metric $R_p(V)$ is a complete metric space.

## A.1 Concatenation of local rough paths on $M$

Localisation plays an important role in the manifold setting, and we need results which will allow us to glue together locally constructed rough paths on $M$. The following elementary lemma (compare [6]) allows us to concatenate a finite number of rough paths.

**Lemma A.1 (Concatenating rough paths)** *Suppose that* $\Pi=\{0=t_0<t_1<\cdots<t_n=T\}$ *is a partition of* $[0,T]$*. For* $k\in\{1,...,n\}$ *let* $J_k := [t_{k-1},t_k]$*, and for each* $k$ *assume we are given* $\mathbf{X}(k)\in WG_p(J_k,W)$*. Then there exists a unique* $\mathbf{X}\in WG_p([0,T],W)$ *such that* $x(0)=0$ *and for all* $1\le k\le n$,

$$\mathbf{X}(k)_{s,t}=\mathbf{X}_{s,t} \text{ for all } s,t\in J_k. \tag{A.2}$$

**Proof.** Let $x(0)=0$. For $0\le s\le t\le T$ with $s\in J_k$ and $t\in J_\ell$, we define

$$\mathbf{X}_{s,t} := \mathbf{X}(k)_{s,t_k}\mathbf{X}(k+1)_{t_k,t_{k+1}}\ldots\mathbf{X}(\ell)_{t_{\ell-1},t} \tag{A.3}$$

where we now view $\mathbf{X}(k)_{u,v}\in 1\oplus W\oplus W\otimes W$ and the multiplication is the usual multiplication in the truncated tensor algebra (see e.g. [30]). We now need to check that $\mathbf{X}\in WG_p([0,T],W)$.

The multiplicative property of rough paths follows directly from Eq. (A.3). The weakly geometric property can either be verified by direct calculation or one just observes that a rough path is weakly geometric if and only if it has finite $p$-variation and takes values in the free nilpotent group of step $\lfloor p\rfloor$ (see e.g. [30] p. 53). We finally check that $\mathbf{X}$ satisfies the correct variation conditions. To this end observe that if $\omega$ is a control so that

$$|x_{u,v}|=|x_{u,v}(k)|\le\omega(u,v)^{1/p} \text{ and } \left|\mathbf{X}^2_{u,v}\right| \le\omega(u,v)^{2/p} \text{ for } u,v\in J_k,\ 1\le k\le n$$

a straightforward calculation shows that there exists a constant $C_{p,n}$ such that $C_{p,n}\omega(s,t)$ controls the concatenated path. ■

The following lemma allows us to compose the flows of rough differential equations (RDEs).

**Lemma A.2 (RDE Concatenation Lemma)** *Let* $\tau\in[0,T]$, $\mathbf{Z}\in WG_p(W,[0,T])$ *and* $Y:V\to \mathrm{Hom}(W,V)$ *be a smooth map. Suppose* $\widetilde{\mathbf{X}}\in WG_p(V)$ *solves*

$$d\mathbf{X}_t = Y_{d\mathbf{Z}_t}(x_t) \tag{A.4}$$

*with initial data* $\widetilde{x}_0=e\in V$ *for* $t\in[0,\tau]$ *and* $\widehat{\mathbf{X}}\in WG_p(V)$*. solves* (A.4) *with initial data* $\widehat{x}_\tau=\widetilde{x}_\tau$ *for* $t\in[\tau,T]$*. Then the rough path path obtained by concatenating* $\widetilde{\mathbf{X}}_t$ *and* $\widehat{\mathbf{X}}_t$ *in the sense of Lemma A.1 solves the rough differential equation* (A.4) *with initial data* $x_0=e$ *for* $t\in[0,T]$*.*

**Proof.** We only have to check the definition of an RDE solution for time $s<\tau<t$, i.e. times $s<t$ which straddle $\tau$. We write $\mathbf{X}=(x,\mathbb{X})$ for the concatenated path and $G(x)$ for $Y'(x)Y(x)$. We have

$$\begin{aligned}
x_{s,t} &= x_{s,\tau}+x_{\tau,t}\\
&\simeq Y(x_s)z_{s,\tau}+G(x_s)\mathbb{Z}_{s,\tau}+Y(x_\tau)z_{\tau,t}+G(x_\tau)\mathbb{Z}_{\tau,t}\\
&\simeq Y(x_s)z_{s,\tau}+[Y(x_s)+Y'(x_s)x_{s,\tau}]z_{\tau,t}+G(x_s)\mathbb{Z}_{s,\tau}+G(x_s)\mathbb{Z}_{\tau,t}\\
&\simeq Y(x_s)[z_{s,\tau}+z_{\tau,t}]+[Y'(x_s)Y(x_s)z_{\tau,t}]z_{\tau,t}+G(x_s)\mathbb{Z}_{s,\tau}+G(x_s)\mathbb{Z}_{\tau,t}\\
&= Y(x_s)z_{s,t}+G(x_s)[z_{\tau t}\otimes z_{\tau,t}+\mathbb{Z}_{s,\tau}+\mathbb{Z}_{\tau,t}]\\
&= Y(x_s)z_{s,t}+G(x_s)\mathbb{Z}_{s,t} \qquad \text{(Chen's identity)}.
\end{aligned}$$

The second order term is simpler, we have

$$\begin{aligned}\mathbb{X}_{s,t} &= \mathbb{X}_{s,\tau} + \mathbb{X}_{\tau,t} + x_{s,\tau} \otimes x_{\tau,t} \\ &\simeq Y(x_s) \otimes Y(x_s) \mathbb{Z}_{s,\tau} + Y(x_\tau) \otimes Y(x_\tau) \mathbb{Z}_{\tau,t} + Y(x_s) \otimes Y(x_\tau) [z_{s,\tau} \otimes z_{\tau,t}] \\ &\simeq Y(x_s) \otimes Y(x_s) [\mathbb{Z}_{s,\tau} + \mathbb{Z}_{\tau,t} + z_{s,\tau} \otimes z_{\tau,t}] = [Y(x_s) \otimes Y(x_s)] \mathbb{Z}_{s,t}\end{aligned}$$

as desired. ■

## A.2 Push forwards of rough paths

In this subsection introduce the notion of a push forward of a rough path between two Banach spaces and record its elementary properties (cf. also [6]).

**Definition A.3** *Suppose that $\varphi \in C^2(W,V)$ and $\mathbf{Z} \in WG_p(W)$, then the **push-forward** of $\mathbf{Z}$ by $\varphi$ is defined by*

$$\varphi_* \mathbf{Z} := \varphi(z_0) + \int d\varphi(d\mathbf{Z}).$$

*In more detail we are letting*

$$[\varphi_* \mathbf{Z}]_0^1 := \varphi(z_0), \quad [\varphi_* \mathbf{Z}]_{s,t}^1 = \left[\int d\varphi(d\mathbf{Z})\right]_{s,t}^1, \text{ and } [\varphi_* \mathbf{Z}]_{s,t}^2 = \left[\int d\varphi(d\mathbf{Z})\right]_{s,t}^2.$$

Note that $\varphi_* \mathbf{Z} \in WG_p(V)$. The first level of the push forward of a rough path has a more explicit representation.

**Lemma A.4** *For $\varphi \in C^2(W,V)$ and $\mathbf{Z} \in WG_p(W)$ we have $[\varphi_* \mathbf{Z}]_{s,t}^1 = \varphi(z_t) - \varphi(z_s)$. In particular, Definition A.3 may also be stated as;*

$$[\varphi_* \mathbf{Z}]_s^1 = \varphi(z_s) \text{ and } [\varphi_* \mathbf{Z}]_{s,t}^2 = \left[\int d\varphi(d\mathbf{Z})\right]_{s,t}^2 \simeq [\varphi'(z_s) \otimes \varphi'(z_s)] \mathbb{Z}_{s,t}.$$

**Proof.** From the symmetry of $\varphi''$ and the fact that $\mathbf{Z} \in WG_p(W)$ we may conclude that $\varphi''(z_s) \mathbb{Z}_{s,t} = \frac{1}{2}\varphi''(z_s)[z_{s,t} \otimes z_{s,t}]$. Using this observation along with Taylor's Theorem shows that

$$\begin{aligned}[\varphi_* \mathbf{Z}]_{s,t}^1 &= \varphi'(z_s) z_{s,t} + \varphi''(z_s) \mathbb{Z}_{s,t} \\ &= \varphi'(z_s) z_{s,t} + \frac{1}{2}\varphi''(z_s)[z_{s,t} \otimes z_{s,t}] = \varphi(z_t) - \varphi(z_s) + O\left(|z_{s,t}|^3\right) \\ &\simeq \varphi(z_t) - \varphi(z_s).\end{aligned}$$

As both ends of this equation are continuous additive functionals we may conclude using Remark 2.11 that $[\varphi_* \mathbf{Z}]_{s,t}^1 = \varphi(z_t) - \varphi(z_s)$. ■

**Theorem A.5 (Integration of push forwards)** *Suppose that $\mathbf{Z} \in WG_p(W)$, $\varphi \in C^2(W,V)$, and $\alpha \in C^2(V, \mathrm{End}(V,U))$ is a one form on $V$ with values in $U$. Then*

$$\int (\varphi^* \alpha)(d\mathbf{Z}) = \int \alpha(d[\varphi_* \mathbf{Z}]).$$

**Proof.** By definition $\beta := \varphi^* \alpha$ is a $U$-valued one form on $W$ which is determined by

$$\beta(z) v = \alpha(\varphi(z)) \varphi'(z) v \in U \text{ for all } z, v \in W.$$

Therefore,

$$\begin{aligned}
\left[\int(\varphi^{*}\alpha)(d\mathbf{Z})\right]_{s,t} &= \left[\int\beta(d\mathbf{Z})\right]_{s,t} \cong \left[\beta(z_s)Z^1_{s,t}+\beta'(z_s)\mathbb{Z}_{s,t}\right]\oplus\left[\beta(z_s)\otimes\beta(z_s)\mathbb{Z}_{s,t}\right] \\
&= \left[\alpha(\varphi(z_s))\varphi'(z_s)Z^1_{s,t}+\alpha'(\varphi(z_s))\varphi'(z_s)\otimes\varphi'(z_s)\mathbb{Z}_{s,t}+\alpha(\varphi(z_s))\varphi''(z_s)\mathbb{Z}_{s,t}\right] \\
&\qquad \oplus\left[\alpha(\varphi(z_s))\varphi'(z_s)\otimes\alpha(\varphi(z_s))\varphi'(z_s)\right]\mathbb{Z}_{s,t} \\
&\simeq \left[\alpha(\varphi(z_s))\left[\varphi_*\mathbf{Z}\right]^2_{s,t}+\alpha'(\varphi(z_s))\varphi'(z_s)\otimes\varphi'(z_s)\mathbb{Z}_{s,t}\right] \\
&\qquad \oplus\alpha(\varphi(z_s))\otimes\alpha(\varphi(z_s))\left[\varphi'(z_s)\otimes\varphi'(z_s)\right]\mathbb{Z}_{s,t} \\
&\simeq \left[\alpha(\varphi(z_s))\left[\varphi_*\mathbf{Z}\right]^2_{s,t}+\alpha'(\varphi(z_s))\left[\varphi_*\mathbf{Z}\right]^2_{s,t}\right]\oplus\alpha(\varphi(z_s))\otimes\alpha(\varphi(z_s))\left[\varphi_*\mathbf{Z}\right]^2_{s,t} \\
&\simeq \left[\int\alpha(d\left[\varphi_*\mathbf{Z}\right])\right]_{s,t}
\end{aligned}$$

which suffices to complete the proof. ■

**Corollary A.6 (Functoriality of push forwards.)** *Let* $\mathbf{Z}\in WG_p(W)$, $\varphi\in C^2(W,V)$, $\psi\in C^2(V,U)$, *then* $(\psi\circ\varphi)_*(\mathbf{Z})=\psi_*(\varphi_*(\mathbf{Z}))$.

**Proof.** By definition,

$$\left[\psi_*(\varphi_*(\mathbf{Z}))\right]^1_t=\psi\left(\left[\varphi_*(\mathbf{Z})\right]^1_t\right)=\psi(\varphi(z_t))=\left[(\psi\circ\varphi)_*(\mathbf{Z})\right]^1_t.$$

Moreover since

$$(\varphi^{*}d\psi)(v_x)=d\psi(\varphi_*v_x)=d\left[\psi\circ\varphi\right](v_x)$$

we have from Theorem A.5 that

$$\begin{aligned}
\left[\psi_*(\varphi_*(\mathbf{Z}))\right]_{s,t} &= \left[\int d\psi(d\varphi_*(\mathbf{Z}))\right]_{s,t}=\left[\int(\varphi^{*}d\psi)(d\mathbf{Z})\right]_{s,t} \\
&= \left[\int d\left[\psi\circ\varphi\right](d\mathbf{Z})\right]_{s,t}=\left[(\psi\circ\varphi)_*(\mathbf{Z})\right]_{s,t}.
\end{aligned}$$

■

# B Proof of Theorem 5.5

**Proof of Theorem 5.5..** The conditions in Eq. (5.4) may be restated as; $(m,g)\in M\times\mathrm{End}\left(\mathbb{R}^d,E\right)$ is in $O_m(M)$ iff $Q(m)g\equiv 0$ and $g^*g=I_{\mathbb{R}^d}$. This observation shows that the function, $G$, in Eq. (5.6) has been manufactured so that $\pi^{-1}(U)=G^{-1}(\{(0,0,0)\})$. So in order to finish the proof it suffices to show the differential, $G'$, of $G$ is surjective at all point $(m,g)\in\pi^{-1}(U)\subset O(M)$. In order to simplify notation, let $q:=Q(m)$, $p=P(m)$, and

$$\dot{q}:=(\partial_\xi Q)(m)=\frac{d}{dt}|_0 Q(m+t\xi).$$

Given $(\xi,h)\in\mathrm{Hom}\left(\mathbb{R}^d,E\right)$ and $(m,g)\in O(M)$ a simple computation shows

$$\begin{aligned}
G'(m,g)(\xi,h) &= \left(\partial_{(\xi,h)}G\right)(m,g)=\frac{d}{dt}|_0 G(m+t\xi,g+th) \\
&= ((\partial_\xi F)(m),(\partial_\xi Q)(m)g,0)+(0,Q(m)h,h^*g+g^*h) \\
&= (F'(m)\xi,\dot{q}g+qh,h^*g+g^*h).
\end{aligned}\tag{B.1}$$

Since $pg=g$ and $qg=0$ we know that $h^*g=h^*pg=(ph)^*g$ and hence

$$h^*g+g^*h=h^*g+(h^*g)^*=(ph)^*g+g^*(ph)$$

and so we may rewrite Eq. (B.1) as

$$\begin{aligned} G'(m,g)(\xi,h) &= \left(F'(m)\xi, \dot{q}pg + qh, (ph)^* g + g^*(ph)\right) \\ &= \left(F'(m)\xi, q\dot{q}g + qh, (ph)^* g + g^*(ph)\right). \end{aligned}$$

From this expression and the observations; 1) $ph$ and $qh$ may be chosen to be arbitrary linear transformation from $\mathbb{R}^d$ to $\tau_m M$ and $\tau_m M^{\perp}$ respectively, 2) $\operatorname{Nul}(g^*)^{\perp} = \operatorname{Ran}(g) = \operatorname{Ran}(p)$, and 3) $F'(m)$ is surjective, it is now easily verified (take $ph = gB$ where $B \in \mathcal{S}_d$) that $G'(m,g)$ is surjective as well. As a consequence of $O(M)$ being an embedded sub-manifold with local defining function $G$, it follows that

$$\begin{aligned} \tau_{(m,g)} O(M) &= \operatorname{Nul}(G'(m,g)) \\ &= \{(\xi,h) : (F'(m)\xi, (\partial_\xi Q)(m)g + Q(m)h, h^*g + g^*h) = (0,0,0)\} \\ &= \left\{(\xi,h)_{(m,g)} : \xi \in \tau_m M,\ Q(m)h = -(\partial_\xi Q)(m)g \text{ and } g^*h \in so(d)\right\}. \end{aligned}$$

■

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