BUFFER-OVERFLOWS: JOINT LIMIT LAWS OF UNDERSHOOTS AND OVERSHOOTS OF REFLECTED PROCESSES

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Abstract. Let $\tau(x)$ be the epoch of first entry into the interval (x,∞), $x>0$, of the reflected process Y of a Lévy process X, and define the overshoot $Z(x)=Y(\tau(x))-x$ and undershoot $z(x)=x-Y(\tau(x)-)$ of Y at the first-passage time over the level x. In this paper we establish, separately under the Cramér and positive drift assumptions, the existence of the weak limit of $(z(x),Z(x))$ as x tends to infinity and provide explicit formulas for their joint CDFs in terms of the Lévy measure of X and the renewal measure of the dual of X. Furthermore we identify explicit stochastic representations for the limit laws. We apply our results to analyse the behaviour of the classical M/G/1 queueing system at buffer-overflow, both in a stable and unstable case.

1. Introduction

Consider a classical single server M/G/1 queueing system, consisting of a stream of jobs of sizes given by positive IID random variables $U_1, U_2, \ldots$, arriving according to a standard Poisson process N with rate λ, and a server that processes these jobs at unit speed. At a given time t the workload Y in the system is given by

$$Y(t) = X(t) - X_*(t), \qquad X_*(t) = \inf_{s\leq t}\{X(s), 0\}, \tag{1.1}$$

$$X(t) = I(t) - O(t), \qquad I(t) = u_0 + \sum_{n=1}^{N_t} U_n, \qquad O(t) = t, \tag{1.2}$$

where $u_0 \geq 0$ is the workload in the system at time 0, $I(t)$ denotes the cumulative workload of all jobs that have arrived by time t and $O(t)$ the cumulative capacity at time t (*i.e.*, the amount of service that could have been provided if the server has never been idle up to time t). We refer to [1, 19] for background on queueing theory. In generalisations of the classical M/G/1 model it has been proposed to replace the compound Poisson process X in Eqn. (1.2) by a general Lévy process leading to the so-called Lévy-driven queues. In case the system in Eqns. (1.1)–(1.2) has a finite buffer of size $x>0$ for the storage of the workload, two quantities of interest are the under- and overshoots of the workload Y at the first time $\tau(x)$ of *buffer-overflow* (*i.e.*, $z(x) = x - Y(\tau(x)-)$ and $Z(x) = Y(\tau(x)) - x$ resp.), representing the level of the workload just before the buffer-overflow and the part of the job lost at $\tau(x)$ (see e.g. [7, 14]).

Our main result (Theorem 2 below) states that if a Lévy process X satisfies the Cramér assumption and a non-lattice condition, the joint limit of the distribution of the under- and overshoot $(z(x), Z(x))$ of Y at $\tau(x)$ (as $x\uparrow\infty$) exists and is explicitly given by the following formula:

$$\lim_{x\uparrow\infty} P\left[(z(x),Z(x))\in \mathrm{d}u\otimes\mathrm{d}v\right] = \frac{\gamma}{\phi(0)}\widehat{V}_\gamma(u)\,\nu(\mathrm{d}v+u)\,\mathrm{d}u, \qquad \text{for } u,v>0, \tag{1.3}$$

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where $\widehat{V}_\gamma(u) \doteq \int_{[0,u]} \mathrm{e}^{\gamma(u-y)}\widehat{V}(\mathrm{d}y)$, $\widehat{V}(\mathrm{d}y)$ is the renewal measure of the dual of X, ϕ the Laplace exponent of the ascending ladder-height process, ν the Lévy measure of X and γ the Cramér coefficient (see Section 2 for precise definitions). In addition, we show that there may be an atom at $(u,v) = (0,0)$, the size of which, given in (2.10) below, corresponds to the asymptotic probability of creeping of the reflected process. Note that, unlike (1.3), analogous limit results for the Lévy process X at the moment of its first passage over the level x require conditioning on the event that the process reaches the level x in finite time (see e.g. Lemma 1(iii) below and [12, Thm. 7.1], [15, Thm. 4.2]).

In the case $X(1)$ is non-lattice and has positive and finite mean we identify (in Theorem 1 below) the limiting joint law of the under- and overshoot of Y as

$$\lim_{x\uparrow\infty} P\left[(z(x),Z(x)) \in \mathrm{d}u\otimes\mathrm{d}v\right] = \frac{\widehat{V}(u)}{\phi'(0)}\,\nu(\mathrm{d}v+u)\,\mathrm{d}u, \qquad \text{for } u,v>0, \tag{1.4}$$

with a possibly non-zero atom at $(u,v) = (0,0)$ (given in (2.5)). Furthermore, we give a stochastic representation of the joint limit law $(z(\infty),Z(\infty)) \sim (S(\infty)C(\infty), S(\infty)(1-C(\infty)))$, both in the Cramér and the positive drift cases, where $(S(\infty),C(\infty))$, supported in $\mathbb{R}_+\times[0,1]$, is given explicitly in terms of the distribution of $S(\infty)$ and the conditional law of $C(\infty)$ given $S(\infty)$. In particular, unlike in the setting of subordinators (cf. [6]), the limit laws $S(\infty)$ and $C(\infty)$ are typically dependent (see Corollaries 1 and 2 below). The investigation of analogous questions in the case X has zero drift and/or heavy tails requires methods different to those employed in this paper and are hence left for future research.

In the classical queueing model given by (1.1)–(1.2), the random variable $X(1)$ is non-lattice if the distribution F of the job-sizes U_i is. Our results yield the joint limit law explicitly in terms of the distribution F and the arrival rate λ:

$$\lim_{x\uparrow\infty} P\left[(z(x),Z(x)) \in \mathrm{d}u\otimes\mathrm{d}v\right] = \frac{\lambda}{\lambda m_1 - 1}(1-\mathrm{e}^{r^*u})F(\mathrm{d}v+u)\mathrm{d}u, \qquad u,v>0, \tag{1.5}$$

where $m_1 \doteq \int_{(0,\infty)} xF(\mathrm{d}x)$ is the mean of F and r^* denotes the largest (resp. smallest) root s in $\mathbb{R}$ of the characteristic equation

$$\lambda\int_{(0,\infty)} \mathrm{e}^{sy}F(\mathrm{d}y) - \lambda - s = 0,$$

in the Cramér (resp. positive drift) case,[1] (*cf.* Remarks (i) and (ii) following Theorem 2).

While formulas (1.3)–(1.4) and (1.5) hold for any starting point $Y_0 = u_0 \geq 0$, it is not hard to see that it suffices to establish those relations just for the value $u_0 = 0$. In the Cramèr case, the probability that the first time of buffer-overflow over the level x occurs *before* the end of the *busy period* (*i.e.*, before the first time Y reaches zero) tends to zero as x tends to infinity. Hence we may assume, by the strong Markov property of Y applied at the end time of the busy period, that we have $Y_0 = 0$. In the nonnegative drift case (*i.e.*, when the queueing system is *unstable*), the proof of Theorem 1 (see Section 3.3.1) yields that the joint limit law in (1.4) is equal to the asymptotic distribution, as $x\uparrow\infty$, of the under- and overshoot of X (with $X_0 = u_0$) at the epoch of its first entrance into the set (x,∞). The latter limit law clearly does not depend on the starting level u_0, due to the spatial homogeneity of the process X.

The arguments outlined in the previous paragraph also imply that the limit distribution in (1.3) (and hence (1.5)) remains valid if Y is in its *steady state*, *i.e.*, the workload process Y was started according to its stationary distribution, which exists since, under the Cramér assumption, Y is an ergodic strong Markov process and the corresponding queueing system is *stable*. Furthermore, it is worth noting that, in the

[1]Note that in the Cramér case it holds $r^* > 0$, $\lambda m_1 < 1$ and in the (unstable) positive drift case we have $r^* < 0$, $\lambda m_1 > 1$.

Cramér case, the right-hand side of (1.3) (and hence that of (1.5)) is in fact also equal to the asymptotic distribution of the under- and overshoot *conditional* on the event that the buffer-overflow takes place in the busy period (this result follows directly by combining the proof of Theorem 2 below with the two-sided Cramér estimate for X, see e.g. [18, Prop. 7]).

Various aspects of the law of the reflected process have been studied recently in a number of papers. The exact asymptotic decay of the distribution of the maximum of an excursion, under the Itô-excursion measure, was identified in [9] under the Cramér condition. Also in the Cramér case, the joint asymptotic distributions of the overshoot, the maximum and the current value of the reflected process were obtained in [18]. In special cases a number of papers are devoted to the characterisation of the law of the reflected process at the moment of buffer-overflow. For example, in the case of spectrally negative Lévy processes, the joint Laplace-transform of the pair $(\tau(x), Y(\tau(x)))$ was obtained in [2]. A sex-tuple law extension of this result, centred around the epoch of the first-passage of the reflected process, was given in [17].

The remainder of the paper is organised as follows: the main results are stated in Section 2 and their proofs are given in Section 3. Section 3.1 defines the setting of the proof. Lemma 1, which plays an important role in the proofs of Theorems 1 and 2, is stated and proved in Section 3.2. Section 3.3 gives the proofs of the main results.

2. Joint limiting distributions

Let X be a Lévy process, that is, a stochastic process with independent and stationary increments and càdlàg paths, with $X(0) = 0$, and let $Y = \{Y(t), t \geq 0\}$ be the reflected process of X at its infimum, *i.e.*,

$$Y(t) \doteq X(t) - \inf_{0\leq s\leq t} X(s), \qquad t \geq 0. \tag{2.1}$$

To avoid trivialities, we assume throughout that X does not have monotone paths. Then the process Y crosses any positive level x in finite time almost surely, that is, the moment of first-passage

$$\tau(x) \doteq \inf\{t \geq 0 : Y(t) \in (x, \infty)\}$$

is finite with probability 1, for any $x > 0$. Denote by Ψ_x the joint complementary distribution function of the pair $(z(x), Z(x))$ of under- and overshoot of Y,

$$\Psi_x(u, v) \doteq P[z(x) > u, Z(x) > v], \qquad u, v \geq 0, \quad x > 0,$$

where we defined $z(x) \doteq x - Y(\tau(x)-)$ and $Z(x) \doteq Y(\tau(x)) - x$.

Recall that the renewal function $V : \mathbb{R}_+ \to \mathbb{R}_+$ of X is the unique non-decreasing right-continuous function with the Laplace transform given by

$$\int_0^\infty \mathrm{e}^{-\theta y} V(y)\mathrm{d}y \quad = \quad (\theta\phi(\theta))^{-1}, \qquad \text{where} \tag{2.2}$$

$$\phi(s) \quad \doteq \quad -\log E[\mathrm{e}^{-sH(1)}\mathbf{I}_{\{1<L(\infty)\}}], \qquad \text{for } s \geq 0, \tag{2.3}$$

$\mathbf{I}_{\{1<L(\infty)\}}$ denotes the indicator of the event $\{1 < L(\infty)\}$, L denotes a local time of X at its running supremum X^*, $X^*(t) \doteq \sup_{0\leq s\leq t} X(s)$, with $L(\infty) = \lim_{t\uparrow\infty} L(t)$, and H is the ascending ladder-height process of X. The corresponding measure $V(\mathrm{d}y)$ is the potential measure of H, *i.e.*, $V(\mathrm{d}y) = \int_0^\infty P[H(t) \in \mathrm{d}y]\mathrm{d}t$ and $V(x) = \int_{[0,x]} V(\mathrm{d}y)$. Similarly $\widehat{L}$, $\widehat{H}$, $\widehat{\phi}$ and $\widehat{V}$ denote the local time, the ladder process, its characteristic exponent and the renewal function of the dual process $\widehat{X} \doteq -X$ respectively. We assume throughout the paper that L and $\widehat{L}$ are normalised in such a way that $-\log E[\mathrm{e}^{\mathrm{i}\theta X(1)}] = \phi(-\mathrm{i}\theta)\widehat{\phi}(\mathrm{i}\theta)$,

$\theta \in \mathbb{R}$, holds, and denote by ν the Lévy measure of X and by $\overline{\nu}(a) \doteq \nu((a,\infty))$, $a > 0$, its tail function. For the background on ladder processes and fluctuation theory we refer to Bertoin [4, Ch. VI].

Throughout the paper we will use the following notation for the limiting probability (if it exists):

$$\Psi_\infty(u,v) \doteq \lim_{x\uparrow\infty} \Psi_x(u,v) \qquad \text{for and } u,v \geq 0.$$

The first limit result concerns the positive drift case:

Theorem 1. *Let the law $X(1)$ be non-lattice and suppose $E[|X(1)|] < \infty$. If $E[X(1)] \in (0,\infty)$, then the weak limit $(z(\infty), Z(\infty))$ of $(z(x), Z(x))$ exists as $x \to \infty$. More precisely, $\phi'(0) \in (0,\infty)$ and the limit law $(z(\infty), Z(\infty))$ is given by*

$$P[z(\infty) > u, Z(\infty) > v] \;=\; \Psi_\infty(u,v) \;=\; \frac{1}{\phi'(0)} \int_u^\infty \overline{\nu}(v+z)\widehat{V}(z)\mathrm{d}z, \quad u,v \geq 0, \tag{2.4}$$

$$P[z(\infty) = 0, Z(\infty) = 0] \;=\; 1 - \Psi_\infty(0,0) \;=\; \frac{m}{\phi'(0)}, \tag{2.5}$$

where $m \in [0,\infty)$ is given by $m \doteq \lim_{\theta\uparrow\infty} \phi(\theta)/\theta$.

Before turning to the negative drift case we note that the limit law admits the following stochastic representation:

Corollary 1. *Let the assumptions of Theorem 1 hold true. Let $C(\infty)$ and $S(\infty)$ be random variables taking values in $[0,1]$ and $\mathbb{R}_+$ respectively, with joint distribution specified by*

$$P[S(\infty) \in \mathrm{d}s] \;=\; \frac{1}{\phi'(0)} \int_0^s \widehat{V}(y)\mathrm{d}y\nu(\mathrm{d}s) + \frac{m}{\phi'(0)}\delta_0(\mathrm{d}s), \qquad s \geq 0, \tag{2.6}$$

$$P[C(\infty) \in \mathrm{d}c | S(\infty) = s] \;=\; \frac{s\widehat{V}(cs)}{\int_0^s \widehat{V}(u)\mathrm{d}u}\mathrm{d}c, \qquad c \in [0,1], s > 0, \tag{2.7}$$

and $P[C(\infty) \in \mathrm{d}c | S(\infty) = 0] = \mathrm{d}c$, $c \in [0,1]$, where δ_0 denotes the unit point mass at zero. Then we have the following: **(i)** *The pair $(C(\infty)\, S(\infty), (1 - C(\infty))\, S(\infty))$ is equal in distribution to $(z(\infty), Z(\infty))$.*
(ii) *$S(\infty)$ has the same law as $z(\infty) + Z(\infty)$ and is equal in distribution to the weak limit of the size of the jump $\Delta Y(\tau(x)) = Y(\tau(x)) - Y(\tau(x)-)$ of Y at the epoch $\tau(x)$, as $x \to \infty$.*
(iii) *It holds $P[C(\infty) \in \mathrm{d}c \,|S(\infty) > 0] = P[\frac{z(\infty)}{z(\infty)+Z(\infty)} \in \mathrm{d}c \,|z(\infty) + Z(\infty) > 0]$, $c \in [0,1]$.*
(iv) *For $0 < c \leq 1$, $s > 0$, we have $P[C(\infty) > c, S(\infty) > s] = \lim_{x\uparrow\infty} P[\frac{z(x)}{z(x)+Z(x)} > c, z(x) + Z(x) > s]$.*

Proof of Corollary 1. The proof of (i) follows by a direct conditioning argument: For any $u, v \in \mathbb{R}_+$, we have

$$\begin{aligned} P[C(\infty)\, S(\infty) > u, (1 - C(\infty))\, S(\infty) > v] &= \int_{(u+v,\infty)} P\left[\frac{u}{s} < C(\infty) < 1 - \frac{v}{s}\middle| S(\infty) = s\right] P[S(\infty) \in \mathrm{d}s] \\ &= \frac{1}{\phi'(0)} \int_{(u+v,\infty)} [F_s(1 - v/s) - F_s(u/s)] \int_0^s \widehat{V}(y)\mathrm{d}y\, \nu(\mathrm{d}s), \end{aligned} \tag{2.8}$$

where F_s denotes the CDF of $C(\infty)$ conditional on $S(\infty) = s$ and we used that $u/s \leq 1 - v/s \leq 1$ iff $s \geq u + v$. Inserting the form (2.6) of F_s shows that the RHS of (2.8) is equal to Ψ_∞. The statements in parts (ii), (iii) and (iv) are straightforward consequences of Theorem 1, part (i) and the continuous mapping theorem. $\square$

In the negative drift case we will restrict ourselves to the classical Cramér setting:

Assumption 1. Suppose that the Cramér-assumption holds, *i.e.*, there exists a $\gamma \in (0,\infty)$ such that $E[\mathrm{e}^{\gamma X(1)}] = 1$, $X(1)$ is non-lattice with a finite mean and $E[|X(1)|\mathrm{e}^{\gamma X(1)}] < \infty$.

Note that if a non-trivial Lévy process X satisfies As. 1, then $E[X(1)] \in (-\infty, 0)$ (since $u \mapsto E[\mathrm{e}^{uX(1)}]$ is strictly convex on $[0,\gamma]$). In the case of negative drift the limiting distribution is given as follows:

Theorem 2. *Let As. 1 hold. Then the weak limit $(z(\infty), Z(\infty))$ of $(z(x), Z(x))$ exists as $x \to \infty$. More precisely, $\phi(0) \in (0,\infty)$ and the limit law $(z(\infty), Z(\infty))$ is given by*

$$P[z(\infty) > u, Z(\infty) > v] = \Psi_\infty(u,v) = \frac{\gamma}{\phi(0)} \int_u^\infty \overline{\nu}(v+z)\widehat{V}_\gamma(z)\mathrm{d}z, \quad u, v \geq 0, \tag{2.9}$$

$$P[z(\infty) = 0, Z(\infty) = 0] = 1 - \Psi_\infty(0,0) = \frac{\gamma m}{\phi(0)}, \tag{2.10}$$

where we denote $\widehat{V}_\gamma(z) \doteq \int_{[0,z]} \mathrm{e}^{\gamma(z-y)}\widehat{V}(\mathrm{d}y)$ and $m \in [0,\infty)$ is as defined in Theorem 1.

Reasoning as above we obtain the following analogous stochastic representation:

Corollary 2. *Let As. 1 hold. Let $C(\infty)$ and $S(\infty)$ be random variables taking values in $[0,1]$ and $\mathbb{R}_+$ respectively with joint distribution specified by*

$$P[S(\infty) \in \mathrm{d}s] = \frac{\gamma}{\phi(0)} \int_0^s \widehat{V}_\gamma(y)\mathrm{d}y\nu(\mathrm{d}s) + \frac{\gamma m}{\phi(0)}\delta_0(\mathrm{d}s), \qquad s \geq 0, \tag{2.11}$$

$$P[C(\infty) \in \mathrm{d}c | S(\infty) = s] = \frac{s\widehat{V}_\gamma(cs)}{\int_0^s \widehat{V}_\gamma(u)\mathrm{d}u}\mathrm{d}c, \qquad c \in [0,1], s > 0, \tag{2.12}$$

and $P[C(\infty) \in \mathrm{d}c|S(\infty) = 0] = \mathrm{d}c$, $c \in [0,1]$. Then the pair $(C(\infty)\,S(\infty), (1 - C(\infty))\,S(\infty))$ is equal in distribution to $(z(\infty), Z(\infty))$ and the analogues of (ii), (iii) and (iv) of Corollary 1 hold.

Remarks. (i) If X is spectrally positive (*i.e.*, $\nu((-\infty,0)) = 0$) with $\psi(\theta) = \log E[\mathrm{e}^{-\theta X(1)}]$ and satisfies As. 1, the ladder process of the dual is a deterministic drift $\widehat{H}(t) = \widehat{\phi}'(0)t$ (see [4, Ch.VII.1]) and hence

$$\widehat{V}(y) = \frac{y}{\widehat{\phi}'(0)}, \qquad \phi(0) = \frac{\psi'(0)}{\widehat{\phi}'(0)}, \qquad \widehat{V}_\gamma(z) = (\gamma\widehat{\phi}'(0))^{-1}(\mathrm{e}^{\gamma z} - 1),$$

where $\widehat{\phi}'(0) > 0$ and $\gamma > 0$ is the largest root of $\psi(-\theta) = 0$. The second equality follows from the Wiener-Hopf factorisation $-\psi(\theta) = \phi(\theta)\widehat{\phi}(-\theta)$, for $\theta \geq -\gamma$, and $\widehat{\phi}(0) = 0$, see [4, Ch.VI.4]. We find

$$\Psi_\infty(u,v) = \frac{1}{\psi'(0)} \int_u^\infty (\mathrm{e}^{\gamma z} - 1)\overline{\nu}(v+z)\mathrm{d}z \qquad \text{for } u, v \geq 0.$$

The Wiener-Hopf factorisation also implies $m = \lim_{\theta\to\infty} \psi(\theta)/(\theta^2\widehat{\phi}'(0)) = \sigma^2/(2\widehat{\phi}'(0))$, where σ^2 is the Gaussian component of X. Hence the atom at zero has mass

$$P[z(\infty) = 0, Z(\infty) = 0] = \frac{\gamma\sigma^2}{2\psi'(0)}.$$

The law of $(S(\infty), C(\infty))$, defined in (2.11)–(2.12), is explicitly given by

$$P[S(\infty) \in \mathrm{d}s] = \frac{1}{\gamma\psi'(0)}(\mathrm{e}^{\gamma s} - 1 - \gamma s)\nu(\mathrm{d}s) + \frac{\gamma\sigma^2}{2\psi'(0)}\delta_0(\mathrm{d}s), \quad s \in \mathbb{R}_+,$$

$$P[C(\infty) \in \mathrm{d}c|S(\infty) = s] = \frac{\gamma s(\mathrm{e}^{\gamma cs} - 1)}{\mathrm{e}^{\gamma s} - 1 - \gamma s}\mathrm{d}c, \quad c \in [0,1], s > 0,$$

with $P[C(\infty) \in \mathrm{d}c|S(\infty) = 0] = \mathrm{d}c$, $c \in [0,1]$.

(ii) If X is spectrally positive with $E[X(1)] \in (0,\infty)$, we have the identities (*cf.* [4, p. 191])

$$\phi'(0) = -\frac{\psi'(0)}{\widehat{\phi}(0)}, \quad \widehat{V}(y) = \frac{1 - \mathrm{e}^{-\Phi(0)y}}{\widehat{\phi}(0)},$$

where $\psi(\theta) = \log E[\mathrm{e}^{-\theta X(1)}]$ (hence $\psi'(0) = -E[X(1)] \in (-\infty, 0)$) and $\Phi(0) = \widehat{\phi}(0)/\widehat{\phi}'(0) > 0$ is the largest root of the equation $\psi(\theta) = 0$. The joint asymptotic distribution of under- and overshoot is in this case given explicitly by the formula

$$\Psi_\infty(u,v) = -\frac{1}{\psi'(0)} \int_u^\infty \overline{\nu}(v+y)(1 - \mathrm{e}^{-\Phi(0)y})\mathrm{d}y \qquad \text{for } u, v \geq 0. \tag{2.13}$$

Since $\widehat{\phi}(\theta) = (\Phi(0) + \theta)\widehat{\phi}'(0)$, the Wiener-Hopf factorisation $-\psi(\theta) = \phi(\theta)\widehat{\phi}(-\theta)$ implies that the atom at zero exists if and only if σ^2 (Gaussian component of X) is non-zero and takes the form

$$P[z(\infty) = 0, Z(\infty) = 0] = -\frac{\sigma^2 \Phi(0)}{2\psi'(0)}. \tag{2.14}$$

The law of $(S(\infty), C(\infty))$, given in (2.6)–(2.7), is explicitly described by

$$\begin{aligned} P[C(\infty) \in \mathrm{d}c | S(\infty) = s] &= \frac{\Phi(0)s(1 - \mathrm{e}^{-\Phi(0)cs})}{\mathrm{e}^{-\Phi(0)s} - 1 + \Phi(0)s}\mathrm{d}c, \quad c \in [0,1], s > 0, \\ P[S(\infty) \in \mathrm{d}s] &= -\frac{1}{\psi'(0)\Phi(0)}(\mathrm{e}^{-\Phi(0)s} - 1 + \Phi(0)s)\nu(\mathrm{d}s) - \frac{\sigma^2\Phi(0)}{2\psi'(0)}\delta_0(\mathrm{d}s), \quad s \in \mathbb{R}_+, \end{aligned}$$

and $P[C(\infty) \in \mathrm{d}c | S(\infty) = 0] = \mathrm{d}c$, $c \in [0,1]$.

(iii) In Corollary 2(ii) of [18], the marginal law of $Z(\infty)$ was identified and the following expression for the overshoot was given:

$$P[Z(\infty) > v] = \frac{\gamma}{\phi(0)} \mathrm{e}^{-\gamma v} \int_{(v,\infty)} \mathrm{e}^{\gamma z} \overline{\nu}_H(z)\mathrm{d}z, \qquad v \geq 0, \tag{2.15}$$

where $\overline{\nu}_H(a) \doteq \nu_H((a,\infty))$, $a > 0$, is the tail of the Lévy measure ν_H of H. Combining this with Theorem 2, we find that $\Psi_\infty(0,v) = P[Z(\infty) > v]$ for all $v \geq 0$. Indeed,

$$\begin{aligned} \frac{\phi(0)}{\gamma}\Psi_\infty(0,v) &= \int_{[0,\infty)} \widehat{V}(\mathrm{d}y) \int_y^\infty \mathrm{e}^{\gamma(z-y)}\overline{\nu}(v+z)\mathrm{d}z = \int_{[0,\infty)} \widehat{V}(\mathrm{d}y) \int_0^\infty \mathrm{e}^{\gamma z}\overline{\nu}(v+z+y)\mathrm{d}z \\ &= \int_0^\infty \mathrm{e}^{\gamma z}\mathrm{d}z \int_{[0,\infty)} \overline{\nu}(v+z+y)\widehat{V}(\mathrm{d}y) = \int_v^\infty \mathrm{e}^{\gamma(z-v)}\mathrm{d}z \int_{[0,\infty)} \overline{\nu}(z+y)\widehat{V}(\mathrm{d}y), \end{aligned}$$

which is equal to $\frac{\phi(0)}{\gamma}P[Z(\infty) > v]$ by (2.15) and Vigon's identity (2.16) (established in [21] and relating the tail $\overline{\nu}_H$ of the Lévy measure ν_H of H to the dual renewal function $\widehat{V}$ and the upper tail $\overline{\nu}(a) = \nu((a,\infty))$, $a > 0$, of the Lévy measure ν of X):

$$\overline{\nu}_H(a) = \int_{[0,\infty)} \overline{\nu}(a+y)\widehat{V}(\mathrm{d}y). \tag{2.16}$$

In (2.16) the local times L and $\widehat{L}$ are normalised such that $-\log E[\mathrm{e}^{\theta X(1)}] = \phi(-\theta)\widehat{\phi}(\theta)$ for $\theta \in [0,\gamma]$ (see e.g. [21, Thm. 2.1] and the remark that follows the theorem), as is assumed to hold throughout this paper. We stress that (2.16) and the statements in Theorems 1 and 2 above hold for any choice of L (resp. $\widehat{L}$) as long as $\widehat{L}$ (resp. L) is defined such that this factorisation holds.

(iv) The assumption (in Theorems 1 and 2) that $X(1)$ is non-lattice is satisfied if the Lévy measure ν of X is non-lattice or if the Gaussian coefficient of X is non-zero.

(v) Theorems 1 and 2 imply that $P[z(\infty)>0, Z(\infty)=0] = P[z(\infty)=0, Z(\infty)>0] = 0$. Put differently, the limit law $(z(\infty), Z(\infty))$ is supported in the open quadrant and possibly at the origin, but not on the coordinate axes away from the origin.

(vi) It follows from the proofs of Theorems 1 and 2 in the next section that the joint limiting undershoot and overshoot for the reflected process Y is the same as that of the original Lévy process X in the case X has positive drift (*cf.* Lemma 1(ii)). Hence, the reflection at zero does not play a role as far as the limiting under- and overshoot is concerned. Under As. 1, the situation is different: the joint limit law for Y is equal to that of process X conditioned to drift to $+\infty$ (see Section 3.2 for the precise form of this conditioning).

3. Proofs

3.1. **Setting.** We next describe the setting of the remainder of the paper, and refer to [4, Ch. I] for further background on Lévy processes. Let $(\Omega, \mathcal{F}, \{\mathcal{F}(t)\}_{t\geq 0}, P)$ be a filtered probability space that carries a Lévy process X. The sample space $\Omega \doteq D(\mathbb{R})$ is taken to be the Skorokhod space of real-valued functions that are right-continuous on $\mathbb{R}_+$ and have left-limits on $(0,\infty)$, $\{\mathcal{F}(t)\}_{t\geq 0}$ denotes the completed filtration generated by X, which is right-continuous, and $\mathcal{F}$ is the completed sigma-algebra generated by $\{X(t), t\geq 0\}$. For any $x\in\mathbb{R}$ denote by P_x the probability measure on $(\Omega,\mathcal{F})$ under which the shifted process $X-x$ has the same law as X under P and by E_x the expectation under P_x. Throughout we identify $P\equiv P_0$ and $E\equiv E_0$ and let $\mathbf{I}_A$ denote the indicator of a set A. θ denotes the shift operator on Ω (for a definition see e.g. [4, Ch. I.2]).

3.2. **Undershoots and overshoots of X.** An important step in the proofs of Theorems 1 and 2 consists in the identification of the limiting joint distribution of the under- and overshoot of X, given in Lemma 1 below. Let $T(x) \doteq \inf\{t\geq 0 : X(t) > x\}$ denote the first-passage time of X over the level x. On the set $\{T(x)<\infty\}$, the overshoot $K(x)$ (resp. undershoot $k(x)$) of the process X at the level x is the distance between x and the positions of X at (resp. just before) the epoch $T(x)$:

$$k(x) \doteq x - X(T(x)-), \qquad K(x) \doteq X(T(x)) - x. \tag{3.1}$$

For any $x>0$ the joint (complementary) distribution of the pair $(k(x), K(x))$ is denoted by Φ_x, viz.

$$\Phi_x(u,v) \doteq P[k(x)>u, K(x)>v, T(x)<\infty], \qquad u,v\geq 0.$$

In the case $P[T(x)<\infty]<1$, or equivalently, when X tends to $-\infty$ (as is the case under As. 1), the distribution function Φ_x is defective for any $x>0$. When $P[T(x)<\infty]>0$ (as is the case under As. 1), we consider the conditioned distribution $\Phi_x^{\#}$ defined as follows:

$$\Phi_x^{\#}(u,v) \doteq P[k(x)>u, K(x)>v | T(x)<\infty], \qquad u,v\geq 0. \tag{3.2}$$

Lemma 1. (i) *Recall the definition of ϕ in Eqn.* (2.3). *Suppose that $X(1)$ is integrable with $E[X(1)]\in(0,\infty)$. Then it holds $\phi'(0)=E[H(1)]\in(0,\infty)$.*

(ii) *Let the law of $X(1)$ be non-lattice and suppose that $E[|X(1)|]<\infty$ and $E[X(1)]\in(0,\infty)$. Then $(k(x),K(x))$ converges weakly to $(k(\infty),K(\infty))$ as $x\to\infty$ and the joint limit law is given by*

$$P[k(\infty)>u, K(\infty)>v] \;=\; \Phi_\infty(u,v) \;=\; \frac{1}{\phi'(0)}\int_u^\infty \overline{\nu}(v+z)\widehat{V}(z)\mathrm{d}z, \quad u,v\geq 0, \tag{3.3}$$

$$P[k(\infty)=0, K(\infty)=0] \;=\; 1-\Phi_\infty(0,0) \;=\; \frac{m}{\phi'(0)}, \tag{3.4}$$

where we denote $\Phi_\infty(u,v) \doteq \lim_{x\to\infty}\Phi_x(u,v)$ *and* $m = \lim_{\theta\uparrow\infty}\phi(\theta)/\theta$ *is as defined in Theorem 1.*

(iii) *Let As. 1 hold. Then* $(k(x), K(x))$*, under the conditional law* $\Phi_x^{\#}(\mathrm{d}u\otimes\mathrm{d}v)$ *given in* (3.2)*, converges weakly to* $(k(\infty), K(\infty))$ *as* $x\to\infty$*. The joint limit law is given by*

$$P[k(\infty)>u, K(\infty)>v] \;=\; \Phi_\infty^{\#}(u,v) \;=\; \frac{\gamma}{\phi(0)}\int_u^\infty \overline{\nu}(v+z)\widehat{V}_\gamma(z)\mathrm{d}z, \quad u,v\geq 0, \tag{3.5}$$

$$P[k(\infty)=0, K(\infty)=0] \;=\; 1-\Phi_\infty^{\#}(0,0) \;=\; \frac{\gamma m}{\phi(0)}, \tag{3.6}$$

where we denote $\Phi_\infty^{\#}(u,v) \doteq \lim_{x\to\infty}\Phi_x^{\#}(u,v)$ *and use the notation* $m = \lim_{\theta\uparrow\infty}\phi(\theta)/\theta$ *and* $\widehat{V}_\gamma(z) = \int_{[0,z]}\mathrm{e}^{\gamma(z-y)}\widehat{V}(\mathrm{d}y)$ *as in Theorem 2.*

Remark. The marginal asymptotic distributions of the overshoot and undershoot of X over a positive level under As. 1 (*cf.* part (iii) of the lemma) were derived in [13, Thm. 4.1]. The authors herein also identify the asymptotic creeping probability for each of the two variables. A direct proof of the existence of the joint limit law $(k(\infty), K(\infty))$, including the asymptotic creeping probability of the pair and its explicit description (3.5)–(3.6), is given below.

Proof. **(i)** $E[X(1)] > 0$ implies $X(t)\to\infty$ P-a.s. as $t\uparrow\infty$. Hence $E[H(1)]\in(0,\infty]$ and $E[\widehat{L}(\infty)] = 1/\widehat{\phi}(0) < \infty$. Since $E[X(1)]<\infty$, we have $\int_{[1,\infty)} y\nu(\mathrm{d}y) < \infty$. By definition we have $\widehat{V}(\infty) \doteq \lim_{y\to\infty}\widehat{V}(y) = E[\widehat{L}(\infty)]$. Fubini's theorem, the estimate $\int_{[1,\infty)} z\nu(y+\mathrm{d}z) \leq \int_{[1,\infty)}(z+y)\nu(y+\mathrm{d}z) \leq \int_{[1,\infty)} x\nu(\mathrm{d}x) < \infty$ for any $y\geq 0$ and (2.16) imply

$$\int_{[1,\infty)} y\nu_H(\mathrm{d}y) \;=\; \int_{[0,\infty)}\widehat{V}(\mathrm{d}y)\int_{[1,\infty)} z\nu(y+\mathrm{d}z) \leq \widehat{V}(\infty)\int_{[1,\infty)} x\nu(\mathrm{d}x) < \infty,$$

and part (i) of the lemma follows.

(ii) The compensation formula [4, Ch. O.5] applied to the Poisson point process $\{\Delta X(t), t\geq 0\}$ (here $X(0-)\doteq 0$ and $\Delta X(t)\doteq X(t)-X(t-)$ for $t\geq 0$) and the form of the resolvent of X killed upon entering (x,∞) (see [4, p. 176]) imply the following identity (recall $\overline{\nu}(a) = \nu((a,\infty))$, $a>0$, is the tail of the Lévy measure ν):

$$\begin{aligned}
\Phi_x(u,v) &= E\sum_{t>0}\mathbf{I}_{\{X^*(t-)<x,\ x-X(t-)>u,\ X(t-)+\Delta X(t)-x>v\}} \\
&= E\int_0^\infty \overline{\nu}(v+x-X(t-))\mathbf{I}_{\{x-X(t-)>u, X^*(t-)<x\}}\mathrm{d}t = E\int_0^{T(x)}\overline{\nu}(v+x-X(t))\mathbf{I}_{\{x-X(t)>u\}}\mathrm{d}t \\
&= \int_{[0,x]}\overline{F}(x-z)V(\mathrm{d}z),
\end{aligned} \tag{3.7}$$

where as before $\mathbf{I}_A$ denotes the indicator of a set A and $\overline{F}$ is given by the expression

$$\overline{F}(z) = \int_{[0,\infty)}\overline{\nu}(v+z+y)\mathbf{I}_{\{z+y>u\}}\widehat{V}(\mathrm{d}y), \qquad \text{for any } z\geq 0, \tag{3.8}$$

and the function V is defined in (2.2) (alternatively, the identity (3.7) can be established via an argument based on the quintuple law from [8, Thm. 3]).

Claim 1. The limit in (3.3) holds in the case $\max\{u,v\}>0$ (*i.e.*, we allow either of u and v to be zero but not both).

Proof of Claim 1. The idea is to apply the renewal theorem to (3.7). With this in mind we need to show that the function in (3.8) is directly Riemann integrable, as defined in [11, Definition on p. 362]. It is sufficient to show that $\overline{F}$ is bounded above by a bounded integrable function on $\mathbb{R}_+$ (see the comment below Definition on page 362 in [11]). If $v > 0$, the function $z \mapsto \overline{\nu}(z+v)$ is bounded (by $\overline{\nu}(v)$) and, by the assumption that $E[|X(1)|] < \infty$, integrable on $\mathbb{R}_+$ (by [20, Thm. 25.3]). Furthermore, the inequality $\widehat{V}(\infty) < \infty$ holds (see e.g. proof of Lemma 1(i) above). Hence we have

$$0 \leq \overline{F}(z) \leq \overline{\nu}(z+v)\widehat{V}(\infty) \leq \overline{\nu}(v)\widehat{V}(\infty), \quad \text{for all } z \geq 0, \tag{3.9}$$

making the function $\overline{F}$ directly Riemann integrable if $v > 0$. In the case $u > 0$, we have

$$\overline{F}(z) \leq \int_{[(u-z)^+,\infty)} \overline{\nu}(z+y)\widehat{V}(\mathrm{d}y) \leq \overline{\nu}(z+(u-z)^+)\widehat{V}(\infty) \leq \overline{\nu}(u)\widehat{V}(\infty), \qquad \text{for any } z \geq 0, \tag{3.10}$$

where as usual $(u-z)^+ = \max\{u-z, 0\}$, again yielding direct Riemann integrablility of $\overline{F}$.

Let Θ be independent of H and exponentially distributed with $E[\Theta] = 1$. We next verify that $H(\Theta)$ is a non-lattice random variable. Indeed, if on the contrary the random variable $H(\Theta)$ has lattice support, Theorem 30.10 in [20] yields that H is a compound Poisson process, necessarily with a Lévy measure that has lattice support. Hence the bivariate subordinator (L^{-1}, H) is also compound Poisson (L^{-1} is the ascending ladder-time process defined as the right inverse of the local time L), implying that the maxima $X^*(L^{-1}(t)) = H(t)$ of X are contained in a lattice and are finite in number on any finite time interval. Hence X must be a compound Poisson process with a Lévy measure that has lattice support. As this case is excluded by the assumption that $X(1)$ is non-lattice, we conclude thus that the law $P[H(\Theta) \in \mathrm{d}z]$, with mean $\phi'(0)$, is not supported on a lattice. It follows by a straightforward calculation that $V(\mathrm{d}z)$ is the renewal measure corresponding to $P[H(\Theta) \in \mathrm{d}z]$.

An application of the renewal theorem in [11, Thm. on p. 363] in combination with (3.7) implies then that $\Phi_\infty(u,v) = \lim_{x\to\infty} \Phi_x(u,v)$ exists and is equal to

$$\Phi_\infty(u,v) = \frac{1}{\phi'(0)} \int_{[0,\infty)} \overline{F}(z)\mathrm{d}z \qquad \text{for any } u, v \geq 0 \text{ such that } \max\{u,v\} > 0. \tag{3.11}$$

It remains to verify that the RHS of (3.11) is equal to that of (3.3). The definition of $\overline{F}$ in (3.8) and several applications of Fubini's theorem yield the following:

$$\begin{aligned}
\int_{[0,\infty)} \overline{F}(z)\mathrm{d}z &= \int_{[0,u]} \mathrm{d}z \int_{(u-z,\infty)} \overline{\nu}(v+z+y)\widehat{V}(\mathrm{d}y) + \int_{(u,\infty)} \mathrm{d}z \int_{[0,\infty)} \overline{\nu}(v+z+y)\widehat{V}(\mathrm{d}y) \\
&= \int_{[0,\infty)} \widehat{V}(\mathrm{d}y) \int_{((u-y)^+,u]} \overline{\nu}(v+z+y)\mathrm{d}z + \int_{[0,\infty)} \widehat{V}(\mathrm{d}y) \int_{(u,\infty)} \overline{\nu}(v+z+y)\mathrm{d}z \\
&= \int_{[0,\infty)} \widehat{V}(\mathrm{d}y) \int_{[(u-y)^+,\infty)} \overline{\nu}(v+z+y)\mathrm{d}z, \qquad (3.12)
\end{aligned}$$

where as usual $(u-y)^+ = \max\{u-y, 0\}$. The equality in (3.12) and further applications of the Fubini theorem yield

$$\begin{aligned}
\int_{[0,\infty)} \overline{F}(z)\mathrm{d}z &= \int_{[0,u]} \widehat{V}(\mathrm{d}y) \int_{[u-y,\infty)} \overline{\nu}(v+z+y)\mathrm{d}z + \int_{(u,\infty)} \widehat{V}(\mathrm{d}y) \int_{[0,\infty)} \overline{\nu}(v+z+y)\mathrm{d}z \\
&= \int_{[0,u]} \widehat{V}(\mathrm{d}y) \int_{[u,\infty)} \overline{\nu}(v+z)\mathrm{d}z + \int_{(u,\infty)} \widehat{V}(\mathrm{d}y) \int_{[y,\infty)} \overline{\nu}(v+z)\mathrm{d}z \\
&= \widehat{V}(u) \int_{[u,\infty)} \overline{\nu}(v+z)\mathrm{d}z + \int_{[u,\infty)} \overline{\nu}(v+z)\mathrm{d}z \int_{(u,z]} \widehat{V}(\mathrm{d}y) \\
&= \widehat{V}(u) \int_{[u,\infty)} \overline{\nu}(v+z)\mathrm{d}z + \int_{[u,\infty)} \left(\widehat{V}(z) - \widehat{V}(u)\right) \overline{\nu}(v+z)\mathrm{d}z \\
&= \int_{[u,\infty)} \widehat{V}(z)\overline{\nu}(v+z)\mathrm{d}z.
\end{aligned}$$

This equality together with (3.11) prove Claim 1.

Claim 2. The following limit holds

$$\lim_{x\to\infty} P[K(x)=0, k(x)=0] = 1 - \frac{1}{\phi'(0)} \int_0^\infty \overline{\nu}_H(z)\,\mathrm{d}z, \tag{3.13}$$

where ν_H is the Lévy measure of H.

Proof of Claim 2. Under the assumptions of Lemma 1(i), the subordinator H has infinite lifetime. Hence we can define the following quantities P-a.s. for any $x>0$:

$$\rho(x) \doteq \inf\{t \geq 0 : H(t) \in (x,\infty)\}, \quad K_H(x) \doteq H(\rho(x)) - x, \quad k_H(x) \doteq x - H(\rho(x)-). \tag{3.14}$$

It is clear that, while the undershoots of X and H at the level x do not generally coincide, *i.e.*, $P[k(x) \neq k_H(x)] > 0$, the following equalities hold P-a.s. for any $x>0$:

$$K(x) = K_H(x) \qquad \text{and} \qquad \{k(x)=0\} = \{k_H(x)=0\}. \tag{3.15}$$

We therefore have $P[K(x)=0, k(x)=0] = P[K_H(x)=0, k_H(x)=0]$. Since H is a subordinator with finite mean equal to $\phi'(0)$, the equality in (3.13) follows by [6, Cor. 1 & Rem. 2]. This proves Claim 2.

Claim 2 now allows us to prove the equalities in (3.3) in the case $u=v=0$ and (3.4). For the former, note that the following holds as a consequence of the Fubini theorem, the change of variables formula and Vigon's identity (2.16):

$$\begin{aligned}
\int_0^\infty \overline{\nu}(z)\widehat{V}(z)\mathrm{d}z &= \int_0^\infty \overline{\nu}(z) \int_{[0,z]} \widehat{V}(\mathrm{d}y)\mathrm{d}z = \int_{[0,\infty)} \widehat{V}(\mathrm{d}y) \int_y^\infty \overline{\nu}(z)\mathrm{d}z \\
&= \int_{[0,\infty)} \widehat{V}(\mathrm{d}y) \int_0^\infty \overline{\nu}(z+y)\mathrm{d}z = \int_0^\infty \mathrm{d}z \int_{[0,\infty)} \overline{\nu}(z+y)\widehat{V}(\mathrm{d}y) \\
&= \int_0^\infty \overline{\nu}_H(z)\,\mathrm{d}z. \qquad (3.16)
\end{aligned}$$

Furthermore, by (3.15) and [6, Remark 2] we have as $x\to\infty$

$$P[K(x)=0, k(x)>0] = P[K_H(x)=0, k_H(x)>0] \quad \to \quad 0, \tag{3.17}$$

$$P[K(x)>0, k(x)=0] = P[K_H(x)>0, k_H(x)=0] \quad \to \quad 0. \tag{3.18}$$

Therefore Claim 2, together with (3.16) and (3.17)–(3.18), implies

$$\Phi_\infty(0,0) = \lim_{x\to\infty} P[K(x)>0, k(x)>0] = 1 - \lim_{x\to\infty} P[K(x)=0, k(x)=0] = \frac{1}{\phi'(0)} \int_0^\infty \overline{\nu}(z)\widehat{V}(z)\mathrm{d}z.$$

Hence (3.3) holds in the case $u = v = 0$ and, in view of Claim 1, for all $u, v \geq 0$ as stated in the lemma.

The Lévy-Khinchine representation of ϕ and the Fubini theorem yield

$$\phi'(0) = m + \int_{(0,\infty)} x\nu_H(\mathrm{d}x) = m + \int_0^\infty \mathrm{d}z \int_{(z,\infty)} \nu_H(\mathrm{d}x) = m + \int_0^\infty \overline{\nu}_H(z)\mathrm{d}z,$$

so that (3.4) holds by Claim 2 and (3.3) (for $u = v = 0$). This proves Lemma 1(ii).

(iii) Let $P^{(\gamma)}$ be the Cramér measure on $(\Omega, \mathcal{F})$, the restriction of which to $\mathcal{F}(t)$ is defined by $P^{(\gamma)}(A) \doteq E[\mathrm{e}^{\gamma X(t)}\mathbf{I}_A]$ for any $A \in \mathcal{F}(t), t \in \mathbb{R}_+$. Under $P^{(\gamma)}$ it holds $E^{(\gamma)}[|X(1)|] = E[|X(1)|\mathrm{e}^{\gamma X(1)}] < \infty$ and $E^{(\gamma)}[X(1)] > 0$ and hence $P^{(\gamma)}(T(x) < \infty) = 1$. Define $\Phi_x^{(\gamma)}(u, v) \doteq P^{(\gamma)}[k(x) > u, K(x) > v, T(x) < \infty]$ for any $u, v \geq 0$. Changing measure yields

$$\Phi_x^{\#}(u, v)P[T(x) < \infty] = \mathrm{e}^{-\gamma x}E^{(\gamma)}[\mathrm{e}^{-\gamma K(x)}\mathbf{I}_{\{k(x)>u,K(x)>v,T(x)<\infty\}}] = \mathrm{e}^{-\gamma x} \int_{(v,\infty)} \mathrm{e}^{-\gamma w}\Phi_x^{(\gamma)}(u, \mathrm{d}w). \tag{3.19}$$

By part (ii) of the lemma, the limit $\Phi_x^{(\gamma)}(u, v) \to \Phi_\infty^{(\gamma)}(u, v)$, as $x \uparrow \infty$, exists for all $u, v \geq 0$. Assume first $\Phi_\infty^{(\gamma)}(u, v) > 0$ and note that the probability measures $\mathbf{I}_{\{w>v\}}\Phi_x^{(\gamma)}(u, \mathrm{d}w)/\Phi_x^{(\gamma)}(u, v)$ on $\mathbb{R}$ converge weakly to the probability measure $\mathbf{I}_{\{w>v\}}\Phi_\infty^{(\gamma)}(u, \mathrm{d}w)/\Phi_\infty^{(\gamma)}(u, v)$ as $x \uparrow \infty$. Cramér's asymptotic formula from [5],

$$\lim_{x\to\infty} \frac{\mathrm{e}^{-\gamma x}}{P[T(x) < \infty]} = C_\gamma^{-1}, \qquad \text{where} \qquad C_\gamma \doteq \frac{\phi(0)}{\gamma\phi'(-\gamma)}, \tag{3.20}$$

Theorem 3.9.1(vi) in [10] applied to the bounded function $w \mapsto \mathbf{I}_{\{w>v\}}\mathrm{e}^{-w\gamma}$, and Lemma 1 (ii) imply

$$\lim_{x\to\infty} \Phi_x^{\#}(u, v) = C_\gamma^{-1} \int_{(v,\infty)} \mathrm{e}^{-\gamma w}\Phi_\infty^{(\gamma)}(u, \mathrm{d}w) = \frac{C_\gamma^{-1}}{\phi^{(\gamma)\prime}(0)} \int_{(v,\infty)} \mathrm{e}^{-\gamma w} \int_{(u,\infty)} \nu^{(\gamma)}(y + \mathrm{d}w)\widehat{V}^{(\gamma)}(y)\mathrm{d}y. \tag{3.21}$$

In the case $\Phi_\infty^{(\gamma)}(u, v) = 0$ we note $\int_{(v,\infty)} \mathrm{e}^{-\gamma w}\Phi_x^{(\gamma)}(u, \mathrm{d}w) \leq \Phi_x^{(\gamma)}(u, v)$. Hence by (3.20) and Lemma 1 (ii) we find $\lim_{x\to\infty} \Phi_x^{\#}(u, v) = \lim_{x\to\infty} \frac{\mathrm{e}^{-\gamma x}}{P[T(x)<\infty]} \int_{(v,\infty)} \mathrm{e}^{-\gamma w}\Phi_x^{(\gamma)}(u, \mathrm{d}w) = 0$. Therefore the first equality in (3.21) holds also in the case $\Phi_\infty^{(\gamma)}(u, v) = 0$.

The Wiener-Hopf factorisation [4, p. 166, Eqn. (4)] implies $\phi^{(\gamma)}(\theta) = \phi(\theta - \gamma)$ and $\widehat{\phi}^{(\gamma)}(\theta) = \widehat{\phi}(\theta + \gamma)$ for all $\theta \geq 0$. The elementary equality $\nu^{(\gamma)}(\mathrm{d}y) = \mathrm{e}^{\gamma y}\nu(\mathrm{d}y)$ and the form of C_γ given in (3.20) therefore yield

$$\lim_{x\to\infty} \Phi_x^{\#}(u, v) = \frac{\gamma}{\phi(0)} \int_u^\infty \overline{\nu}(v + y)\mathrm{e}^{\gamma y}\widehat{V}^{(\gamma)}(y)\mathrm{d}y \qquad \text{for any } u, v \geq 0.$$

Recalling that the Laplace transform of $\widehat{V}^{(\gamma)}$ is given by $[\theta\widehat{\phi}^{(\gamma)}(\theta)]^{-1} = [\theta\widehat{\phi}(\theta + \gamma)]^{-1}$ (*cf.* (2.2)), we observe that the Laplace transforms of the function $y \mapsto \mathrm{e}^{\gamma y}\widehat{V}^{(\gamma)}(y)$ and the convolution $y \mapsto \widehat{V}_\gamma(y) = \int_{[0,y]} \mathrm{e}^{\gamma(y-z)}\widehat{V}(\mathrm{d}z)$ are both equal to $[(\theta - \gamma)\widehat{\phi}(\theta)]^{-1}$ (recall that $\int_{[0,\infty)} \mathrm{e}^{-\theta z}\widehat{V}(\mathrm{d}z) = 1/\widehat{\phi}(\theta)$). Hence the two functions can only differ on a countable set, which has Lebesgue measure zero. Therefore (3.5) follows.

The final task is to establish (3.6). Change of measure as in (3.19) yields

$$P[k(x) = 0, K(x) = 0|T(x) < \infty] = \frac{\mathrm{e}^{-\gamma x}}{P[T(x) < \infty]}P^{(\gamma)}[k(x) = 0, K(x) = 0].$$

Since $\phi^{(\gamma)}(\theta) = \phi(\theta - \gamma)$ for all $\theta \geq 0$, we have $m = \lim_{\theta\to\infty} \phi^{(\gamma)}(\theta)/\theta = \lim_{\theta\to\infty} \phi(\theta - \gamma)/\theta$ and $\phi^{(\gamma)\prime}(0) = \phi'(-\gamma)$. The limit in (3.4) (applied for the measure $P^{(\gamma)}[\cdot]$) and (3.20) yield

$$\lim_{x\to\infty} P[k(x) = 0, K(x) = 0|T(x) < \infty] = \frac{m}{\phi'(-\gamma)}C_\gamma^{-1} = \frac{m\gamma}{\phi(0)}.$$

This proves the second equality in (3.6). For the first equality, note that the limits in (3.17)–(3.18), under the measure $P^{(\gamma)}[\cdot]$, imply

$$1 - \Phi_\infty(0,0) = 1 - \lim_{x\to\infty} P[k(x) > 0, K(x) > 0 | T(x) < \infty] = \lim_{x\to\infty} P[k(x) = 0, K(x) = 0 | T(x) < \infty].$$

This concludes the proof of (3.6) and of the lemma. □

3.3. **Proofs of Theorems 1 and 2.** In this section we establish our main results.

3.3.1. *Proof of Theorem 1:* Fix $u, v \geq 0$ and let $A(x)$ denote either $A_{u,v}(x) \doteq \{z(x) > u, Z(x) > v\}$ or $A_0(x) \doteq \{z(x) = 0, Z(x) = 0\}$. Furthermore, define $\Psi_x \doteq P[A(x)]$. Note that $\Psi_x = \Psi_x(u,v)$ if $A(x) = A_{u,v}(x)$.

For any $M \in (0, x)$, the strong Markov property of Y at $\tau(M)$ implies the following:

$$\begin{aligned} \Psi_x &= E[\mathbf{I}_{\{Y(\tau(M))<x\}} P[A(x), \tau(x) < \tau_0 | Y(\tau(M))]] \\ &+ P[Y(\tau(M)) \geq x, A(x)] + E[\mathbf{I}_{\{Y(\tau(M))<x\}} P[A(x), \tau(x) > \tau_0 | Y(\tau(M))]], \end{aligned} \tag{3.22}$$

where $\tau_0 \doteq \inf\{t \geq 0 : Y(t) = 0\}$. Define $\widehat{T}(y) \doteq \inf\{t \geq 0 : X(t) < -y\}$, for any $y \geq 0$, and note that for any $y > 0$, the processes $\{Y(t), Y(0) = y, t \leq \tau_0\}$ and $\{X(t), X(0) = y, t \leq \widehat{T}(0)\}$ are equal in law. If $A(x) = A_{u,v}(x)$, let $B(x)$ denote $\{k(x) > u, K(x) > v, T(x) < \infty\}$ (*cf.* equation (3.1)), and in the case $A(x) = A_0(x)$ define $B(x) \doteq \{k(x) = 0, K(x) = 0, T(x) < \infty\}$. In either case let $\Phi_x \doteq P[B(x)]$ and recall that $P_z[\cdot] = P[\cdot | X_0 = z]$. Then, for any $z \in [M, x)$, we have:

$$P[A(x), \tau(x) < \tau_0 | Y(\tau(M)) = z] = \Phi_{x-z} - P_z[B(x), \widehat{T}(0) < T(x)],$$

$$P[A(x), \tau(x) > \tau_0 | Y(\tau(M)) = z] \leq P_z[\widehat{T}(0) < T(x)] \leq P_z[\widehat{T}(0) < \infty] \leq P_M[\widehat{T}(0) < \infty], \tag{3.23}$$

$$P_z[B(x), \widehat{T}(0) < T(x)] \leq P_z[\widehat{T}(0) < T(x)] \leq P[\widehat{T}(z) < \infty] \leq P[\widehat{T}(M) < \infty], \tag{3.24}$$

Since $\{Y(\tau(M)) \geq x, A(x)\} \subset \{\tau(M) = \tau(x)\}$ and $P[\widehat{T}(M) < \infty] = P_M[\widehat{T}(0) < \infty]$, the inequalities above and (3.22) yield the following estimate:

$$|\Psi_x - \Phi_x| \leq 2P[\widehat{T}(M) < \infty] + P[\tau(M) = \tau(x)] + \left| E\left[\mathbf{I}_{\{Y(\tau(M))<x\}} \Phi_{x - Y(\tau(M))}\right] - \Phi_x \right|.$$

Lemma 1 (ii) implies the following P-a.s. limit:

$$\lim_{x\to\infty} \mathbf{I}_{\{Y(\tau(M))<x\}} \Phi_{x-Y(\tau(M))} = \lim_{x\to\infty} \Phi_x = \begin{cases} \Phi_\infty(u,v), & \text{if } A(x) = A_{u,v}(x), \\ 1 - \Phi_\infty(0,0), & \text{if } A(x) = A_0(x). \end{cases}$$

Since $\lim_{x\to\infty} \mathbf{I}_{\{\tau(M)=\tau(x)\}} = 0$ P-a.s., the dominated convergence theorem yields the estimate

$$\limsup_{x\to\infty} |\Psi_x - \Phi_x| \leq 2P[\widehat{T}(M) < \infty] \qquad \text{for any } M > 0.$$

Since $E[X(1)] > 0$, X drifts to $+\infty$. Hence $P[\widehat{T}(M) < \infty] \to 0$ as $M \to \infty$ and the proof of the positive drift case is complete.

3.3.2. *Proof of Theorem 2:* We assume henceforth that As. 1 is satisfied. The proof is based on Itô-excursion theory. Refer to [4, Chs O, IV] for a treatment of Itô-excursion theory for Lévy processes and for further references.

Denote by $\epsilon = \{\epsilon(t), t \geq 0\}$ the excursion process of Y away from zero. Since, under As. 1, Y is a recurrent strong Markov process under P, Itô's characterisation yields that ϵ is a Poisson point process under P. Its intensity measure under P will be denoted by n. Let $\zeta(\varepsilon)$ denote the lifetime of a generic excursion ε and let $\rho(x, \varepsilon)$ denote the first time that the excursion ε enters (x, ∞), viz.

$$\rho(x, \varepsilon) = \inf\{t \geq 0 : \varepsilon(t) > x\}. \tag{3.25}$$

In the sequel we will drop the dependence of $\rho(x, \varepsilon)$ and $\zeta(\varepsilon)$ on ε, and write ζ and $\rho(x)$, respectively.

Theorem 2 follows directly by combining Lemmas 2 and 3 below.

Lemma 2. *For any $u, v \geq 0$ and $x > 0$ the following holds true:*

$$P[z(x) > u, Z(x) > v] = n(E_{u,v}(x)|\rho(x) < \zeta) \doteq \frac{n(E_{u,v}(x))}{n(\rho(x) < \zeta)}, \tag{3.26}$$

$$P[z(x) = 0, Z(x) = 0] = n(E_0(x)|\rho(x) < \zeta) \doteq \frac{n(E_0(x))}{n(\rho(x) < \zeta)}, \tag{3.27}$$

where $E_{u,v}(x) \doteq \{x - \varepsilon(\rho(x)-) > u, \varepsilon(\rho(x)) - x > v, \rho(x) < \zeta\}$ and $E_0(x) \doteq \{x - \varepsilon(\rho(x)-) = 0, \varepsilon(\rho(x)) - x = 0, \rho(x) < \zeta\}$.

Proof of Lemma 2: By [4, Ch. O, Prop. 2], for sets A, B with $n(A) \in (0, \infty)$, we have $P[\epsilon(T_A) \in B] = n(B|A) = n(A \cap B)/n(A)$ where $T_A = \inf\{t \geq 0 : \epsilon(t) \in A\}$. The lemma now follows by noting that the left-hand sides of (3.26) and (3.27) are the probabilities that the first excursion in $A = \{\rho(x) < \zeta\}$ is in $E_{u,v}(x)$ and $E_0(x)$, respectively. □

Lemma 3. *Let $u, v \geq 0$ and recall $\widehat{V}_\gamma(z) = \int_{[0,z]} \mathrm{e}^{\gamma(z-y)} \widehat{V}(\mathrm{d}y)$. The following holds true:*

$$\lim_{x \to \infty} n(E(x)|\rho(x) < \zeta) = \begin{cases} \frac{\gamma}{\phi(0)} \int_u^\infty \overline{\nu}(v+z) \widehat{V}_\gamma(z) \mathrm{d}z, & \text{if } E(x) = E_{u,v}(x), \\ 1 - \frac{\gamma}{\phi(0)} \int_u^\infty \overline{\nu}(v+z) \widehat{V}_\gamma(z) \mathrm{d}z, & \text{if } E(x) = E_0(x), \end{cases}$$

where the events $E_{u,v}(x)$ and $E_0(x)$ are as defined in Lemma 2.

Remarks. **(i)** The proof of Lemma 3 uses the following facts, which hold by [5] and [9], respectively, if 0 is regular for $(0, \infty)$ under the law of X and As. 1 is satisfied:

$$P[T(x) < \infty] \sim C_\gamma \mathrm{e}^{-\gamma x} \quad \text{as } x \to \infty, \quad \text{where} \quad C_\gamma = \frac{\phi(0)}{\gamma \phi'(-\gamma)}, \tag{3.28}$$

$$n(\rho(x) < \zeta) \sim C_\gamma \widehat{\phi}(\gamma) \mathrm{e}^{-\gamma x} \quad \text{as } x \to \infty. \tag{3.29}$$

Here and in the rest of the paper we write $f(x) \sim g(x)$ as $x \to \infty$ if $\lim_{x \to \infty} f(x)/g(x) = 1$.

(ii) A further ingredient of the proof of Lemma 3 are the following asymptotic identities, established in [18, Lemma 9]:

$$n(\mathrm{e}^{\gamma \varepsilon(\rho(x))} \mathbf{I}_{\{\rho(x) < \zeta\}}) \sim \widehat{\phi}(\gamma) \quad \text{as } x \to \infty, \tag{3.30}$$

$$\mathrm{e}^{\gamma x} n(\varepsilon(\rho(z)) > x, \rho(x) < \zeta) = o(1) \quad \text{as } x \to \infty, \text{ for any } z > 0. \tag{3.31}$$

(iii) The key observation in [5] is that under the Cramér-assumption ($E[\mathrm{e}^{\gamma X(1)}] = 1$), $V^{(\gamma)}(\mathrm{d}z) \doteq \exp(\gamma z) V(\mathrm{d}z)$ is a renewal measure corresponding to the probability distribution $P^{(\gamma)}[H(\Theta) \in \mathrm{d}z] \doteq$

$\exp(\gamma z)P[H(\Theta)\in \mathrm{d}z]$, where Θ is an exponential random variable with $E[\Theta]=1$, independent of X and hence of H, and the measure $P^{(\gamma)}$ is the h-transform of P with $h(z)\doteq\exp(\gamma z)$. Under $P^{(\gamma)}$ the process X tends to $+\infty$ and hence As. 1 implies the assumptions of Lemma 1(ii) above, making $H(\Theta)$ a non-lattice random variable under $P^{(\gamma)}$ by the argument in the proof of Lemma 1(ii). Estimate (3.28) then follows from the classical renewal theorem for non-lattice random walks with the step-size distribution $H(\Theta)$ under $P^{(\gamma)}$ (see the conclusion of the proof of the theorem in [5] for this argument and [11, p. 363] for the statement of the renewal theorem). The assumption in [5] ensuring $H(\Theta)$ is non-lattice stipulates that 0 be regular for $(0,\infty)$ under the law of X, which in particular implies the non-lattice condition in As. 1. However, since the argument in [5] only requires $H(\Theta)$ to be non-lattice, the estimate in (3.28) holds in our setting under As. 1 (without assuming the regularity of 0). Likewise, the argument in [9] relies solely on the fact that $V^{(\gamma)}(\mathrm{d}z)$ is a renewal measure of the non-lattice law $P^{(\gamma)}[H(\Theta)\in\mathrm{d}z]$. Hence estimate (3.29) also holds under As. 1.

Proof of Lemma 3: Fix $M>0$ and pick $u,v\geq 0$. Throughout the proof $E(x)$ denotes either of the sets $E_{u,v}(x)$ and $E_0(x)$ defined in Lemma 2. We start from the elementary observation that relates the following two conditional n-measures:

$$n(E(x)|\rho(x)<\zeta)=n(E(x)|\rho(M)<\zeta)\cdot\frac{n(\rho(M)<\zeta)}{n(\rho(x)<\zeta)},\qquad x>M. \tag{3.32}$$

Recall that the coordinate process under the probability measure $n(\cdot|\rho(M)<\zeta)$ has the same law as the first excursion of Y away from zero with height larger than M. The strong Markov property under $n(\cdot|\rho(M)<\zeta)$ implies that $\varepsilon\circ\theta_{\rho(M)}$ has the same law under $n(\cdot|\rho(M)<\zeta)$ as the coordinate process of X under P, with entrance law $\mu_M(\mathrm{d}y)\doteq n(\varepsilon(\rho(M))\in\mathrm{d}y|\rho(M)<\zeta)$, that is killed upon its first entrance into $(-\infty,0)$. Recall $\widehat{T}(y)=\inf\{t\geq0: X(t)<-y\}$, for $y\geq 0$, and note that for every $x>M$ we have:

$$n(E(x)|\rho(M)<\zeta)=\int_{[M,x]}P_z[B(x)]\mu_M(\mathrm{d}z)+U(x), \tag{3.33}$$

where

$$B(x)=\begin{cases}\{k(x)>u, K(x)>v, T(x)<\infty\} & \text{if } E(x)=E_{u,v}(x),\\ \{k(x)=0, K(x)=0, T(x)<\infty\} & \text{if } E(x)=E_0(x),\end{cases}$$

and $U(x)$ is given by the following expression:

$$U(x)=n(E(x),\varepsilon(\rho(M))>x|\rho(M)<\zeta)-\int_{[M,x]}P_z[B(x),\widehat{T}(0)<T(x)<\infty]\mu_M(\mathrm{d}z).$$

Note that $P_z[B(x),\widehat{T}(0)<T(x)<\infty]\leq P_z[\widehat{T}(0)<T(x)<\infty]$ for any $z\in(0,x)$ and hence we find by [18, Prop. 7 (i)] that the following holds (the constant C_γ is given in (3.28)):

$$P_z[\widehat{T}(0)<T(x)<\infty]=P[\widehat{T}(z)<T(x-z)<\infty]\sim C_\gamma \mathrm{e}^{-\gamma x}E[\mathrm{e}^{\gamma(X(\widehat{T}(z))+z)}]\qquad \text{as } x\to\infty.$$

The following facts hold: $X(\widehat{T}(z))+z\leq 0$, the measure $\mu_M(\mathrm{d}y)$ is concentrated on $[M,\infty)$ with $\mu_M([M,\infty))=1$ for any $M>0$ and the equality in (3.31) is satisfied. Hence, for a fixed $M>0$, we have

$$-C_\gamma\mathrm{e}^{-\gamma x}\leq U(x)\leq \mathrm{e}^{-\gamma x}o(1)\qquad \text{as } x\to\infty. \tag{3.34}$$

Note that in the case $E(x)=E_0(x)$, the upper bound for $U(x)$ is in fact equal to zero, since it holds $n(E_0(x),\varepsilon(\rho(M))>x,\rho(M)<\zeta)=0$.

By Lemma 1 (iii), for any fixed $z \geq 0$, the following limit holds:

$$L_\infty \doteq \lim_{x\to\infty} P_z[B(x)|T(x)<\infty] = \begin{cases} \frac{\gamma}{\phi(0)} \int_u^\infty \overline{\nu}(v+z)\widehat{V}_\gamma(z)\mathrm{d}z, & \text{if } E(x) = E_{u,v}(x), \\ 1 - \frac{\gamma}{\phi(0)} \int_u^\infty \overline{\nu}(v+z)\widehat{V}_\gamma(z)\mathrm{d}z, & \text{if } E(x) = E_0(x). \end{cases} \tag{3.35}$$

Furthermore, from the asymptotic relation in (3.28) we find $P_z[T(x) < \infty] = P[T(x-z) < \infty] = \mathrm{e}^{-\gamma x} C_\gamma \mathrm{e}^{\gamma z}(1 + r(x-z))$ as $x \to \infty$ for any $z \geq 0$, where $r : \mathbb{R}_+ \to \mathbb{R}$ is bounded and measurable with $\lim_{x'\to\infty} r(x') = 0$. By (3.30) the function $z \mapsto \mathrm{e}^{\gamma z}$, $z \in [M, \infty)$, is in $L^1(\mu_M)$ for all large M. The dominated convergence theorem and (3.29) therefore imply:

$$\begin{aligned} \lim_{x\to\infty} \int_{[M,x]} \frac{P_z[B(x)]}{n(\rho(x)<\zeta)} \mu_M(\mathrm{d}z) &= \lim_{x\to\infty} \int_{[M,x]} P_z[B(x)|T(x)<\infty] \frac{P_z[T(x)<\infty]}{n(\rho(x)<\zeta)} \mu_M(\mathrm{d}z) \\ &= n(\mathrm{e}^{\gamma\varepsilon(\rho(M))}|\rho(M)<\zeta) \cdot L_\infty \cdot \widehat{\phi}(\gamma)^{-1}. \end{aligned} \tag{3.36}$$

Recall that $E[X(1)] < 0$ by As. 1 and hence $\widehat{\phi}(\gamma) > 0$ since $\widehat{H}$ is a non-trivial subordinator. Hence (3.29), (3.32), (3.33), (3.34) and (3.36) imply the following inequalities for any fixed $M > 0$:

$$\begin{aligned} -\widehat{\phi}(\gamma)^{-1} n(\rho(M)<\zeta) &\leq \liminf_{x\to\infty} n(E(x)|\rho(x)<\zeta) - n(\mathrm{e}^{\gamma\varepsilon(\rho(M))}\mathbf{I}_{\{\rho(M)<\zeta\}}) \cdot \widehat{\phi}(\gamma)^{-1} \cdot L_\infty \\ &\leq \limsup_{x\to\infty} n(E(x)|\rho(x)<\zeta) - n(\mathrm{e}^{\gamma\varepsilon(\rho(M))}\mathbf{I}_{\{\rho(M)<\zeta\}}) \cdot \widehat{\phi}(\gamma)^{-1} \cdot L_\infty \leq 0. \end{aligned}$$

Since these inequalities hold for all large $M > 0$, taking the limit as $M \to \infty$ and deploying (3.30) yields $\lim_{x\to\infty} n(E(x)|\rho(x) < \zeta) = L_\infty$. This fact, together with (3.35), concludes the proof of the theorem. □

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