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Journal:	*Inverse Problems*
Manuscript ID:	IP-100255.R1
Manuscript Type:	Paper
Date Submitted by the Author:	n/a
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Article Keywords:	STOCHASTIC TRANSPORT EQUATIONS, partial observations, inverse problem
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INVERSE PROBLEMS FOR STOCHASTIC TRANSPORT EQUATIONS

DAN CRISAN, YOSHIKI OTOBE, AND SZYMON PESZAT

Abstract. Inverse problems for stochastic linear transport equations driven by a temporal or spatial white noise are discussed. We analyse stochastic linear transport equations which depend on an unknown potential and have either additive noise or multiplicative noise. We show that one can approximate the potential with arbitrary small error when the solution of the stochastic linear transport equation is observed over time at some fixed point in the state space.

1. Introduction

Let u be the solution to the transport equation with additive noise

$$\begin{cases} \partial_t u(t,x) = \partial_x u(t,x) + V(x)u(t,x) + \dot{W}, & t>0,\ x\in\mathbb{R},\\ u(0,x) = u_0(x) \end{cases} \tag{1}$$

or to its version with multiplicative noise

$$\begin{cases} \partial_t u(t,x) = \partial_x u(t,x) + (V(x) + \dot{W})u(t,x), & t>0,\ x\in\mathbb{R},\\ u(0,x) = u_0(x), \end{cases} \tag{2}$$

where $\dot{W}$ denotes either a temporal or spatial Gaussian white noise. Informally, in the case of temporal noise $\dot{W}$ depends on the time variable t whereas in the spatial case it depends on the space variable x, for more details see Section 2.1.

The present paper is concerned with the following inverse problem. Namely, we assume that the initial value u_0 is known and that we can observe the value of the solution u at a certain point $a\in\mathbb{R}$ in space. Without any loss of generality we choose that $a=0$. Our goal is to estimate the potential V, which is assumed to be a deterministic bounded measurable function. In other words, we a looking to construct a "constant" mapping $(u_0, u(\cdot,0)) \mapsto V$. We emphasise that $u(\cdot,0)$ is random while V is deterministic. In the present paper, we propose two methods to estimate the potential V by using the random observations: the first is based on the use of the quadratic variation process corresponding to V and the second is based on the law of large numbers.

Inverse problems for partial differential equations have been studied in various contexts of the engineering and physics. Important tasks for such studies are not only to ensure the existence and uniqueness of the solution to the given inverse problem but also to ensure its stability, in other words the continuity of the solution with respect to the observation data in order to cope

Yoshiki Otobe's research was supported by JSPS KAKENHI Grant Number 22740086.
Szymon Peszat's research was supported by Polish National Science Center grant DEC2013/09/B/ST1/03658.

with observation errors. However, even for linear PDE, it is known that the inverse problems are ill-posed in the sense of Hadamard. To avoid this difficulty, we adopt here a different approach: we model the observation errors as being part of the system by adding a random force to the partial differential equations itself. Of course, the choice of the randomness will depend on the actual physical systems they model and we consider here PDEs with multiplicative noise and PDEs with additive noise. In both cases, we choose (Gaussian) white noise as the randomness: this choice is both natural and mathematically interesting.

In this paper, we treat stochastic linear transport equations on one-dimensional space, see e.g., [13, pp. 2–3] for the inverse problem for the deterministic transport equation. In the case of the transport equation, the analysis is facilitated as the solution u of the direct problem is given in a very explicit form. We have found that the case of the additive spatial noise is much simpler than the case of the additive temporal noise. Surprisingly in the case of the multiplicative temporal and spatial noise the same estimator works. We hope that the observation made here will prove useful in further studies of the corresponding inverse problem for wider class of stochastic PDEs including stochastic wave equations including those driven by space-time white noise. It should be also interesting to study multi-dimensional transport equations. If the driving noise is temporal white noise, a similar study could be possible (see Section 8), while it could not be expected to go to the direction of the case of spatial white noise because if the noise is white and multi-dimensional, it cannot be expected to have a usual function as a solution to the stochastic PDE. Inverse problems for other stochastic problems are studied by, e.g., [1] for diffusion equations, and also for partial diffusion equations e.g., [2, 3, 4, 6, 10, 14].

The paper is organized as follows: In Section 2 we will summarise the results we obtained for the inverse problems without proofs. We consider four types of equations (noise is added to the equation additively or multiplicatively, and the noise is temporal or spatial). After we introduce the notion of white noise, we include two subsections to describe the additive spatial noise and the additive temporal noise. The multiplicative cases are handled in a single subsection. In Section 3 we cover several background results for the direct problem related to transport equations. The proofs of the main results stated in Section 2 are covered in Sections 4–7. We complete the paper with some remarks on the extension of our result to multidimensional case. These are incorporated in Section 8.

Acknowledgements. The authors are grateful for the financial support from the Isaac Newton Institute for Mathematical Sciences in Cambridge. A part of the study was done during our stay there. The second author thanks Professor M. Yamamoto of the University of Tokyo and J.-S. Choi to arise his interest in the stability of the inverse problems and lead him to this direction with many valuable discussions and comments when he was a student. He also thanks Professor J. Zabczyk for the discussions during his stay at the University of Warwick.

2. Results for inverse problems

In this section, we first give a rigorous mathematical description of the notion of Gaussian white noise and then present our main results. The proofs are postponed to the following sections.

2.1. **Gaussian white noise.** Let $W_1 = (W_1(r),\, r \geq 0)$ and $W_2 = (W_2(r),\, r \geq 0)$ be two independent standard Brownian motions defined on a probability space $(\Omega, \mathfrak{F}, \mathbb{P})$. Let

$$W(r) = \begin{cases} W_1(-r) & \text{if } r \leq 0, \\ W_2(r) & \text{otherwise,} \end{cases}$$

be the so-called two sided Brownian motion. Let $\dot{W}$ be the distributional derivative of W, that is

$$\dot{W}(\psi) = \int_{-\infty}^{\infty} \psi(r) dW(r), \qquad \psi \in C_0^\infty(\mathbb{R}).$$

We call $\dot{W}$ white noise. $\dot{W}$ can be uniquely extended to the linear mapping $\dot{W}\colon L^2(\mathbb{R}) \mapsto L^2(\Omega, \mathfrak{F}, \mathbb{P})$ such that for arbitrary $\psi, \phi \in L^2(\mathbb{R})$, the random vector $(\dot{W}(\psi), \dot{W}(\phi))$ is Gaussian, with mean zero, and with the covariance matrix

$$\begin{bmatrix} \langle \psi, \psi \rangle & \langle \psi, \phi \rangle \\ \langle \psi, \phi \rangle & \langle \phi, \phi \rangle \end{bmatrix}.$$

Clearly $W(r) = W(\mathbf{1}_{[0,r]})$ if $r \geq 0$ and $W(r) = W(\mathbf{1}_{[r,0]})$ if $r \leq 0$, where $\mathbf{1}_A$ denotes the indicator function of a set A.

The white noise $\dot{W}$ can be interpreted as an infinite series

$$\dot{W} = \sum_{k=1}^{\infty} \gamma_k e_k, \tag{3}$$

where (e_k) is an orthonormal basis of $L^2(\mathbb{R})$, and $\gamma_k = W(e_k)$, $k \in \mathbb{N}$, is a sequence of independent $N(0,1)$ random variables. The series, however, does not converge in $L^2(\mathbb{R})$ but in any Hilbert space H for which the embedding $L^2(\mathbb{R}) \hookrightarrow H$ is Hilbert–Schmidt. Then (see e.g. [12])

$$W(\phi) = \sum_{k=1}^{\infty} \gamma_k \langle \phi, e_k \rangle_{L^2(\mathbb{R})}, \qquad \phi \in L^2(\mathbb{R}),$$

where the series converges in $L^2(\Omega, \mathfrak{F}, \mathbb{P})$. Note that following on from (3), we get that

$$W(r) = \sum_{k=1}^{\infty} \gamma_k \int_0^r e_k(s) ds, \qquad r \geq 0,$$

and

$$W(r) = \sum_{k=1}^{\infty} \gamma_k \int_r^0 e_k(s) ds, \qquad r < 0.$$

The variable r can be either time or space and so, depending of the set-up, the white noise can be either temporal or spatial.

2.2. **The case of additive spatial noise.** Our first result provides the form of the weak solution u to (1) observed at point 0. Let us start with the definition of the weak solution.

Definition 2.1. We say that $u \in L^1_{\rm loc}$ is a *weak solution* to the stochastic partial differential equation (1) driven by spatial white noise if and only if for any test function $\phi \in C^1_0(\mathbb{R})$ and for almost all $t \geq 0$,

$$\langle u(t), \phi\rangle = \langle u_0, \phi\rangle + \int_0^t \langle u(s), -\phi' + V\phi\rangle ds + tW(\phi).$$

Proposition 2.2. *Let u be the weak solution to* (1)*. Then there is a one-dimensional standard Brownian motion B such that*

$$u(t,0) = u_0(t)e^{\int_0^t V(s)ds} + \int_0^t e^{\int_0^s V(r)dr} dB(s), \qquad t \geq 0.$$

We will estimate the potential V using the quadratic variation of the observation process $u(t,0)$, $t \geq 0$. We introduce next some notation.

Let (I_n) be a sequence of sets such that $I_n \subset \{k/2^n \colon k = 0, \dots, 2^{n+1}\}$, $n \in \mathbb{N}$. Define

$$\mathbf{u}(I_n;t) := \sum_{k\in I_n} \left[u\left(\frac{t(k+1)}{2^n}, 0\right) - u\left(\frac{tk}{2^n}, 0\right)\right]^2, \qquad t > 0. \tag{4}$$

Let

$$I^1_n = \left\{ \frac{k}{2^n} \colon k = 0, \dots, 2^n - 1\right\},$$
$$I^2_n = \left\{ \frac{k}{2^n} \colon k = 0, \dots, 2^n + 2^{[5n/6]} - 1\right\},$$
$$I^3_n = \left\{ \frac{k}{2^n} \colon k = 0, \dots, 2^n - 2^{[5n/6]} - 1\right\}.$$

Write

$$\hat{V}_n(t) := \frac{2^{n-1}}{2^{[5n/6]}t} \frac{\mathbf{u}(I^2_n;t) + \mathbf{u}(I^3_n;t) - 2\mathbf{u}(I^1_n;t)}{\mathbf{u}(I^2_n;y) - \mathbf{u}(I^1_n;t)}, \qquad n \in \mathbb{N},\ t > 0. \tag{5}$$

Theorem 2.3. *Assume that the initial value u_0 is locally Hölder continuous with exponent $\gamma > 1/2$. Then for any $t > 0$, $\hat{V}_n(t)$ converges to $V(t)$, $\mathbb{P}$-a.s. Moreover for all $t > 0$ and $\varepsilon > 0$, there is a $\mathbb{P}$-a.s. finite random variable $C(\varepsilon, t)$ independent of n such that*

$$\left| V(t) - \hat{V}_n(t)\right| \leq C(\varepsilon, t) 2^{-\left(\frac{1}{6} - \varepsilon\right)n}. \tag{6}$$

Remark 2.4. For practical purpose, it would be interesting to know the behaviour of the constant appearing in equation (6). The constant is arrived at through a series of nonlinear transformations and we can't gauge its decay or blow up as a function of time or ε. The only noteworthy property of the constant is that it is independent of n and therefore equation (6) shows that the estimating error $V(t) - \hat{V}_n(t)$ vanishes exponential fast with n.

2.3. **The case of additive temporal noise.** As above, we define the weak solution to (1) with temporal noise in the following way:

Definition 2.5. An L^2_ρ-valued random process u is called a weak solution to (1) if it is adapted to the filtration given by W and for any $\psi \in C_0^1(\mathbb{R})$, and for almost all t we have $\mathbb{P}$-a.s.

$$\langle u(t), \psi\rangle = \langle u_0, \psi\rangle + \int_0^t \langle u(s), -\psi' + V\psi\rangle ds + W(t) \int_{\mathbb{R}} \psi(x)dx.$$

As in the case of spatial noise (subsection 2.2), the weak solution u and the observation process $u(\cdot, 0)$ are given explicitly (see Proposition 3.4). However, in the case of the additive temporal noise, the quadratic variation processes have no information about the coefficients (see Remark 3.5), hence we cannot apply the same method. Therefore we will reduce our consideration to a parameter estimation problem, namely we will assume that $V(x) = aq(x)$, $x \in \mathbb{R}$, where $q\colon \mathbb{R} \mapsto \mathbb{R}$ is a known bounded measurable function. The inverse problem is now restricted to finding/estimating the parameter a. As a result we can solve a simplified version of the inverse problem. Note that the parameter estimation approach works (without any changes) also in the case of the previous subsection. We distinguish two cases: the periodic and indicator potential cases.

2.3.1. *Periodic potential case.* Assume first that q is a bounded measurable periodic function with period 1 such that $\int_0^1 q(x)dx = 0$. Let $\iota\colon \mathbb{R} \mapsto (0, \infty)$ be the function defined as

$$\iota(w) = \int_0^1 e^{2w \int_0^s q(r)dr} ds.$$

Obviously, ι is continuously differentiable from $\mathbb{R}$ to $(0, \infty)$, and

$$\iota'(w) = \int_0^1 e^{2w \int_0^s q(r)dr} \int_0^s q(r)drds.$$

In the following we will assume that $\iota'(w)$ does not change sign. In particular, this implies that ι is a homeomorphism from $\mathbb{R}$ to

$$\mathrm{Range}\,(\iota) := \left(\inf_{x\in\mathbb{R}} \iota(x),\ \sup_{x\in\mathbb{R}} \iota(x) \right).$$

Remark 2.6. The condition that the derivative of ι does not change sign holds true if for example the function

$$s \mapsto \int_0^s q(r)dr, \qquad s \in [0, 1],$$

does not change sign, see Example 2.9 below.

Let $\hat{u}_n$ be the following (observable) quantity

$$\hat{u}_n = \frac{1}{n} \sum_{k=0}^{n-1} \left(u(k+1, 0) - u(k, 0) - (u_0(k+1) - u_0(k))\right)^2.$$

In the proposition below $\hat{a}_n = \iota^{-1}(\hat{u}_n)$ if $\hat{u}_n \in \mathrm{Range}\,(\iota)$ and $\hat{a}_n = 1$ otherwise.

Proposition 2.7. *For any $\varepsilon > 0$ there exists a random variable $C(\varepsilon)$ satisfying $\mathbb{E}\, C(\varepsilon)^{2/\varepsilon} < \infty$, such that*

$$|\hat{a}_n - a| \le n^{-\frac{1}{2}+\varepsilon} C(\varepsilon), \qquad \forall\, n \in \mathbb{N},\ \mathbb{P}\text{-a.s.} \tag{7}$$

Remark 2.8. Using a standard t-test one can also obtain confidence intervals for a.

Example 2.9. Assume that $q(x) = \sin 2\pi x$, $x \in \mathbb{R}$. Then

$$\iota(w) = \int_0^1 e^{2w\int_0^s q(r)dr} ds = \int_0^1 e^{\frac{w(1-\cos 2\pi s)}{\pi}} ds,$$

with the derivative

$$\iota'(w) = \int_0^1 e^{\frac{w(1-\cos 2\pi s)}{\pi}} (1 - \cos 2\pi s) ds,$$

which is obviously strictly positive. Hence the condition $\iota' \neq 0$ which ensures the asymptotic rate of convergence is satisfied. Note that in this case the range of ι is $(0, \infty)$.

Let $q(x) = \cos 2\pi x$, then $\iota \colon \mathbb{R} \mapsto (0, \infty)$ is given by

$$\begin{aligned}\iota(w) &= \int_0^1 e^{2w\int_0^s q(r)dr} ds = \int_0^1 e^{\frac{w \sin 2\pi s}{\pi}} ds = \frac{1}{2\pi}\int_0^{2\pi} e^{\frac{w\sin x}{\pi}} dx \\ &= \frac{1}{2\pi} J(w/\pi),\end{aligned}$$

where $J(a) := \int_0^{2\pi} e^{a\sin x} dx$, $a \in \mathbb{R}$, is a Bessel function of the second kind with the derivative

$$\begin{aligned}J'(a) &= \int_0^{2\pi} e^{a\sin x} \sin x dx = \int_0^{\pi} e^{a\sin x} \sin x dx + \int_{\pi}^{2\pi} e^{a\sin x} \sin x dx \\ &= 2\int_0^{\pi} \sinh(a \sin x) \sin x dx.\end{aligned}$$

In this case the function changes sign at 0. As a result, the above analysis enables us to identify the absolute value of a but not its sign.

2.3.2. *Indicator case.* Assume that $q(x) = 1_{[0,b]}(x)$ with a certain (known) $b > 0$.

Let $j_b \colon \mathbb{R} \mapsto (0, \infty)$ be the function defined as

$$j_b(x) := \begin{cases} \left(b - \frac{1}{x}\right) e^{2bx} + \frac{1}{x}\left(2e^{bx} - 1\right), & x \neq 0, \\ b, & x = 0. \end{cases}$$

Note that j_b is a continuously differentiable bijection.

Let $\hat{u}_n$ be the following (observable) quantity

$$\hat{u}_n = \frac{1}{n}\sum_{k=0}^{n-1} \left(u((2k+2)b, 0) - u((2k+1)b, 0)\right)^2.$$

Define $\hat{a}_n = j_b^{-1}(\hat{u}_n)$.

Proposition 2.10. *Assume that $u_0 \equiv 0$. Then, for any $\varepsilon > 0$ there is a random variable $C(\varepsilon)$ satisfying $\mathbb{E}\, C(\varepsilon)^{2/\varepsilon} < \infty$, such that*

$$|\hat{a}_n - a| \le n^{-\frac{1}{2}+\varepsilon} C(\varepsilon), \qquad \forall\, n \in \mathbb{N},\ \mathbb{P}\text{-}a.s. \tag{8}$$

2.4. **The case of multiplicative spatial/temporal noise.** The following proposition provides the form of the observation process. It is a direct consequence of Lemmas 3.7 and 3.9.

Proposition 2.11. *If the noise is temporal and the equation* (2) *is considered in the Stratonovich sense then the observation process is given by*

$$u(t,0) = u_0(t) \exp\left\{ \int_0^t V(r)dr + W(t) \right\}, \qquad t \ge 0. \tag{9}$$

If the noise is spatial then there is a standard Brownian motion B such that

$$u(t,0) = u_0(t) \exp\left\{ \int_0^t V(r)dr + B(t) \right\}, \qquad t \ge 0. \tag{10}$$

Assume that $u_0(t) \neq 0$ for any $t > 0$. Let

$$Z(t) := \log \frac{u(t,0)}{u_0(t)}, \qquad t \ge 0,$$

be the observable process. Assume that V is periodic with a known period T. Given $\varepsilon > 0$, set

$$h_n^\varepsilon = 2^{\frac{3}{4n} + \varepsilon n - n} = 2^{-(\frac{1}{4} - \varepsilon)n}, \qquad n \in \mathbb{N},$$

and

$$\hat{V}_n^\varepsilon(t) := \frac{1}{h_n^\varepsilon} \sum_{k=0}^{2^n - 1} \left(Z(kT + t + h_n^\varepsilon) - Z(kT + t) \right), \qquad n \in \mathbb{N}$$

Theorem 2.12. *For any $\varepsilon > 0$ and any $t > 0$, there is a random variable $C(\varepsilon, t)$ satisfying $\mathbb{E}\, C(\varepsilon, t) < \infty$, such that*

$$\left| V(t) - \hat{V}_n^\varepsilon(t) \right| \le C(\varepsilon, t) 2^{-(\frac{1}{4} - \varepsilon)n}, \qquad \mathbb{P}\text{-}a.s.\ \forall n \in \mathbb{N}. \tag{11}$$

Remark 2.13. Similarly to (6), the arguments used in deducing equation (11) do no allow as to study its decay or blow up as a function of t or ε.

3. Results for direct problems

In the following we will work with the weighted space $L^2_\rho := L^2(\mathbb{R}, \mathcal{B}(\mathbb{R}), \rho(x)dx)$, where $\rho(x) := (1 + |x|^2)^{-1}$, $x \in \mathbb{R}$.

Given a bounded measurable function $V \colon \mathbb{R} \mapsto \mathbb{R}$ and an initial value $u_0 \in L^2_\rho$, let us consider the linear deterministic transport problem

$$\partial_t u(t,x) = \partial_x u(t,x) + V(x) u(t,x), \qquad u(0,x) = u_0(x). \tag{12}$$

Let $C_0(\mathbb{R})$ and $C_0^1(\mathbb{R})$ be the spaces of compactly supported continuous and continuously differentiable functions on $\mathbb{R}$. Given $w \in L^2_\rho$ and $\phi \in C_0(\mathbb{R})$ write

$$\langle w, \phi\rangle := \int_{\mathbb{R}} w(x)\phi(x)dx.$$

To shorten the notation, we denote $L^1_{\mathrm{loc}} := L^1_{\mathrm{loc}}([0,\infty), \mathcal{B}([0,\infty)), dt; L^2_\rho(\mathbb{R}))$.

Definition 3.1. We say that $u \in L^1_{\mathrm{loc}}$ is a *weak solution* to (12) if for any test function $\phi \in C_0^1(\mathbb{R})$ and for almost all $t \geq 0$,

$$\langle u(t), \phi\rangle = \langle u_0, \phi\rangle + \int_0^t \langle u(s), -\phi' + V\phi\rangle ds.$$

Let $S = (S(t), t \geq 0)$ be the following C_0-group defined on L^2_ρ,

$$S(t)v(x) = v(t+x)e^{\int_0^t V(x+s)ds}, \qquad t,\, x \in \mathbb{R},\ v \in L^2_\rho. \tag{13}$$

The proof of the following result is standard and left to the reader.

Proposition 3.2. *For any $u_0 \in L^2_\rho(\mathbb{R})$ there is a unique weak solution u to* (12) *and $u(t) = S(t)u_0$, for almost all t.*

3.1. **The case of additive spatial noise.** We first show the existence and uniqueness of the solution.

Proposition 3.3. *For any $u_0 \in L^2_\rho$ there is a unique weak solution (see Definition 2.1) u to* (1) *driven by a spatial noise, and for any orthonormal basis (e_k) of $L^2(\mathbb{R})$, $u(t)$ is expressed by the following formula:*

$$u(t) = S(t)u_0 + \sum_{k=1}^\infty \int_0^t S(t-s)e_k ds W(e_k),$$

where the series converges in $L^2(\Omega, \mathfrak{F}, \mathbb{P}; L^2_\rho)$ for every t . Moreover, if the initial value u_0 is continuous, then u has a continuous version in time and space.

Proof. First we show the convergence of the series. By the definition of S we have

$$\begin{aligned}\int_0^t S(t-s)e_k(x)ds &= \int_0^t e^{\int_0^{t-s} V(x+r)dr} e_k(x+t-s)ds \\ &= \int_0^t e^{\int_0^{s} V(x+r)dr} e_k(x+s)ds \\ &= \int_x^{t+x} e^{\int_0^{y-x} V(x+r)dr} e_k(y)dy \\ &= \langle e_k, f(x,t,\cdot)\rangle,\end{aligned}$$

where

$$f(x,t,y) := 1_{[x,t+x]}(y)e^{\int_0^{y-x} V(x+r)dr}.$$

We have

$$\sum_{k=1}^{\infty}\langle e_k, f(x,t,\cdot)\rangle^2 = \int_{\mathbb{R}} f^2(x,t,y)dy = \int_x^{x+t} e^{2\int_0^{y-x} V(r)dr}dy \le te^{2t\|V\|_\infty},$$

hence (recall that ρ is the weight)

$$\begin{aligned}\sum_{k=1}^{\infty}\int_{\mathbb{R}}\langle e_k, f(x,t,\cdot)\rangle^2\rho(x)dx &= \int_{\mathbb{R}}\int_{\mathbb{R}} f^2(x,t,y)\rho(x)dxdy \\ &\le te^{2t\|V\|_\infty}\int_{\mathbb{R}}\rho(x)dx < \infty,\end{aligned}$$

and the required convergence holds by the Parseval theorem and that $\{W(e_k)\}$ is i.i.d. Gaussian. To show that u solves the equation, note that by Proposition 3.2, and the so-called Duhamel or variation of constants formula, for any n,

$$u^n(t) = S(t)u_0 + \sum_{k=1}^{n} W(e_k)\int_0^t S(t-s)e_k ds,$$

is a weak solution to

$$\begin{cases}\partial_t u(t,x) = \partial_x u(t,x) + V(x)u(t,x) + \sum_{k=1}^n W(e_k)e_k(x), \\ u(0,x) = u_0(x).\end{cases}$$

Thus for any test function $\phi \in C_0^1(\mathbb{R})$,

$$\langle u^n(t),\phi\rangle = \langle v,\phi\rangle + \int_0^t \langle u^n(s), -\phi' + V\phi\rangle ds + t\sum_{k=1}^{n} W(e_k)\langle e_k,\phi\rangle.$$

Taking the limit as $n\to\infty$ we obtain the desired conclusion. The uniqueness follows from the uniqueness of the (homogeneous) linear equation. The existence of a continuous version follows from the Kolmogorov–Loève–Chentsov theorem. Indeed, assume for simplicity that $u_0 \equiv 0$. Then

$$\begin{aligned}\mathbb{E}\left|u(t_1,x_1) - u(t_2,x_2)\right|^2 &= \sum_{k=1}^{\infty}\langle e_k, f(x_1,t_1,\cdot) - f(x_2,t_2,\cdot)\rangle^2 \\ &= \int_{\mathbb{R}} |f(x_1,t_1,y) - f(x_2,t_2,y)|^2 dy.\end{aligned}$$ □

We now give a proof of Proposition 2.2.

Proof of Proposition 2.2. Recall that (e_k) is an orthonormal basis of $L^2(\mathbb{R})$. Then,

$$B(t) := \sum_{k=1}^{\infty} W(e_k)\int_0^t e_k(s)ds, \qquad t\ge 0,$$

is a Wiener process on $\mathbb{R}$ (see Section 2.1). We have

$$\begin{aligned}\sum_{k=1}^{\infty}\int_0^t S(t-s)e_k(0)ds W(e_k)\\ &= \sum_{k=1}^{\infty} W(e_k)\int_0^t e^{\int_0^{t-s}V(r)dr}e_k(t-s)ds\\ &= \sum_{k=1}^{\infty} W(e_k)\int_0^t e^{\int_0^{s}V(r)dr}e_k(s)ds\\ &= \int_0^t e^{\int_0^s V(r)dr}dB(s).\end{aligned}$$

□

3.2. **The case of additive temporal noise.** Consider (1) with temporal noise $\dot{W}$. Then W is a standard Brownian motion in t-variable. Using the variation of constants formula we obtain the following:

Proposition 3.4. *For any measurable bounded initial data u_0, the weak solution to* (1) *with temporal nose is given by the formula*

$$u(t,x) = u_0(x+t)e^{\int_0^t V(x+r)dr} + \int_0^t e^{\int_0^{t-s}V(x+r)dr}dW(s). \tag{14}$$

In particular

$$u(t,0) = u_0(t)e^{\int_0^t V(r)dr} + \int_0^t e^{\int_0^{t-s}V(r)dr}dW(s), \qquad t\geq 0. \tag{15}$$

Remark 3.5. Assume that u_0 is continuously differentiable. Then u is in fact a strong solution to (1), that is

$$u(t,x) = u_0(x) + \int_0^t \left(\partial_x u(s,x) + V(x)u(s,x)\right)ds + W(t), \qquad t\geq 0,\ x\in\mathbb{R}.$$

Thus in particular we have

$$u(t,0) = u_0(0) + \int_0^t \left(\partial_x u(s,0) + V(0)u(s,0)\right)ds + W(t), \qquad t\geq 0.$$

Hence the quadratic variation of the observation process $u(\cdot,0)$ on the interval $[0,t]$ equals t and will not provide any information on the potential V.

3.3. **The case of multiplicative temporal noise.** Assume that $\dot{W}$ is a temporal white noise, that is W is a standard Wiener process defined on a filtered probability space $(\Omega,\mathfrak{F},(\mathfrak{F}_t),\mathbb{P})$.

Definition 3.6. We call an $(\mathfrak{F}_t)$-adapted L^2_ρ-valued process u a solution to equation (2) if for any test function $\phi\in C_0^1(\mathbb{R})$,

$$\langle u(t),\phi\rangle = \langle u_0,\phi\rangle + \int_0^t \langle u(s), -\phi' + V\phi\rangle\, ds + \int_0^t \langle u(s),\phi\rangle \circ dW(s),$$

where the last term is understood as the Stratonovich integral.

Recall that S is the C_0-group on L^2_ρ corresponding to the linear equation (12).

Lemma 3.7. *The solution to* (2) *driven by a temporal noise is unique and given by*

$$u(t,x) = S(t)u_0(x)\exp\{W(t)\}. \tag{16}$$

In particular the observation process is given by

$$u(t,0) = u_0(t)\exp\left\{\int_0^t V(r)dr + W(t)\right\}, \qquad t \geq 0. \tag{17}$$

Proof. The uniqueness follows form the general theory of SPDEs with Lipschitz diffusion terms. To see that u defined in (16) solves (2) take a test function $\phi \in C^1_0(\mathbb{R})$. We have

$$\begin{aligned} d\langle u(t),\phi\rangle &= d\exp\{W(t)\}\langle S(t)u_0,\phi\rangle \\ &= \exp\{W(t)\}\langle S(t)u_0,\phi\rangle d_S W(t) + \exp\{W(t)\}\langle S(t)u_0, -\phi' + V\phi\rangle dt \\ &= \langle u(t), -\phi' + V\phi\rangle dt + \langle u(t),\phi\rangle d_S W(t). \end{aligned}$$

□

3.4. **The case of multiplicative spatial noise.** Assume now that $\dot{W}$ is a spatial noise.

Definition 3.8. We call an $L^2_\rho(\mathbb{R})$-valued process u a solution to equation (2) with a spatial white noise if for any test function $\phi \in C^2_0(\mathbb{R})$,

$$\langle u(t),\phi\rangle = \langle u_0,\phi\rangle + \int_0^t \left[\langle u(s), -\phi' + V\phi\rangle + \sigma W(u(s)\phi)\right]\, ds.$$

Recall that (e_k) is an orthonormal basis of $L^2(\mathbb{R})$, see Section 2.1.

Lemma 3.9. *The solution to* (2) *driven by the spatial noise is unique and given by*

$$u(t,x) = S(t)u_0(x)\exp\left\{\sigma\sum_{k=1}^\infty W(e_k)\int_0^t e_k(x+s)ds\right\}, \tag{18}$$

where the series

$$\sum_{k=1}^\infty W(e_k)\int_0^t e_k(x+s)ds$$

converges in $L^2(\Omega,\mathfrak{F},\mathbb{P})$. Moreover, there is a standard Brownian motion B such that the observation process $u(t,0)$, $t\geq 0$, is given by

$$u(t,0) = u_0(t)\exp\left\{\int_0^t V(s)ds + B(t)\right\}, \qquad t \geq 0. \tag{19}$$

Proof. The convergence of the series follows from the fact that

$$\sum_{k=1}^\infty \left(\int_0^t e_k(x+s)ds\right)^2 = \sum_{k=1}^\infty \langle e_k 1_{[x,x+t]}\rangle^2 = t.$$

Since for any n,

$$u^n(t,x) = S(t)u_0(x)\exp\left\{\sigma\sum_{k=1}^n W(e_k)\int_0^t e_k(x+s)ds\right\},$$

solves

$$\partial_t u^n = \partial_x u^n + \left(V+\sigma\sum_{k=1}^n W(e_k)e_k\right)u^n, \quad u^n(0)=u_0,$$

one can show as in the proof of Proposition 3.2, that the right hand side of (18) defines a solution to (2). Finally, (19) follows, see Section 2.1 and the proof of Proposition 2.2. □

4. Proof of Theorem 2.3

Let us fix a value $t>0$. We will estimate the potential $V(t)$ using the quadratic variation of the observation process $t\mapsto u(t,0)$. Note that since u_0 is Hölder continuous with exponent $\gamma>\frac{1}{2}$, for any sequence of partitions (t_k^n) of $[0,t]$ satisfying

$$\delta_n = \max\{t_{k+1}^n - t_k^n,\ k=0,\ldots,m_k-1\}\to 0 \quad \text{as} \quad n\to\infty,$$

we have

$$\lim_{n\to\infty}\sum_{k=0}^{m_n-1}\left(u_0(t_{k+1}^n)e^{\int_0^{t_{k+1}^n}V(s)ds} - u_0(t_k^n)e^{\int_0^{t_k^n}V(s)ds}\right)^2 = 0.$$

Therefore the quadratic variation of the process $t\mapsto u(t,0)$ coincides with that of the stochatic integral $t\to\int_0^t e^{\int_0^s V(r)dr}dB(s)$. We then have that

$$\lim_{n\to\infty}\sum_{k=0}^{m_n-1}\left(u(t_{k+1}^n,0)-u(t_k^n,0)\right)^2 = \int_0^t e^{2\int_0^s V(r)dr}ds,$$

where (t_k^n), $n\in\mathbb{N}$, $k=0,\ldots,m_n-1$, is a carefully chosen sequence of partitions of the interval $[0,t]$.

Let

$$f(s) := e^{\int_0^s V(r)dr}, \quad s\geq 0,$$

and

$$F(t) := \int_0^t f^2(s)\,ds = \int_0^t e^{2\int_0^s V(r)\,dr}\,ds, \quad t\geq 0.$$

Note that

$$V(t) = \left(\frac{1}{2}\log F'(t)\right)' = \frac{1}{2}\frac{F''(t)}{F'(t)}.$$

Define

$$F(I_n;t) := \sum_{k\in I_n}\int_{t_k^n}^{t_{k+1}^n} f^2(s)ds,$$

where $t_k^n = kt/2^n$.

Note that, given a sequence (I_n) there is a constant C_1 such that for any n,

$$A_n := \mathbb{E}\left\{\sum_{k\in I_n}\left[\left(u(t^n_{k+1},0)-u(t^n_k,0)\right)^2-\int_{t^n_k}^{t^n_{k+1}} f^2(s)ds\right]\right\}^2 \le \frac{C_1}{2^n}.$$

For, we have

$$\left(u(t^n_{k+1},0)-u(t^n_k,0)\right)^2-\int_{t^n_k}^{t^n_{k+1}} f^2(s)ds = 2\int_{t^n_k}^{t^n_k}\left(u(s,0)-u(t^n_k,0)\right) f(s)dB(s).$$

Let $C_2 := \sup\{f^2(s); 0\le s\le 2t\}$. Thus

$$\begin{aligned}A_n &= 4\sum_{k\in I_n}\mathbb{E}\left(\int_{t^n_k}^{t^n_{k+1}}\left(u(s,0)-u(t^n_k,0)\right) f(s)dB(s)\right)^2\\ &= 4\sum_{k\in I_n}\int_{t^n_k}^{t^n_{k+1}}\mathbb{E}\left(u(s,0)-u(t^n_k,0)\right)^2 f^2(s)ds\\ &\le 4C_2\sum_{k\in I_n}\int_{t^n_k}^{t^n_{k+1}}\mathbb{E}\left(u(s,0)-u(t^n_k,0)\right)^2 ds\\ &\le 4C_2\sum_{k\in I_n}\int_{t^n_k}^{t^n_{k+1}}\int_{t^n_k}^{s} f^2(r)drds \le \frac{4C_2^2 2t}{2^n}.\end{aligned}$$

Thus for any $\delta<1$,

$$\mathbb{E}\sum_{n=1}^{\infty} 2^{\delta n}A_n < \infty.$$

Consequently for any $\delta\in(0,1)$ and for any sequence (I_n) there is a random variable $c(\delta,(I_n))$ with $\mathbb{E}\, c(\delta,(I_n))<\infty$ such that for any n,

$$\left(\mathbf{u}(I_n;t)-F(I_n;t)\right)^2 \le 2^{-\delta n}c(\delta,(I_n)), \quad \mathbb{P}\text{-a.s.},$$

where $\mathbf{u}(I_n;t)$ is given by (4).

By the Taylor formula there is a constant C such that for any $0\le h\le 1$,

$$\left|\frac{F(t+h)+F(t-h)-2F(t)}{h^2}-F''(t)\right| + \left|\frac{F(t+h)-F(t)}{h}-F'(t)\right| \le Ch.$$

Recall the sequences (I^i_n), $i=1,2,3$ introduced in Section 2.2. Let $k_n = 2^{[5n/6]}$ and let $h_n := k_n 2^{-n}$. Note that

$$F(I^1_n;t)=F(t), \quad F(I^2_n;t)=F(t+th_n), \quad F(I^3_n;t)=F(t-th_n).$$

Write

$$\hat{F}'_n(t) := \frac{\mathbf{u}(I^2_n;t)-\mathbf{u}(I^1_n;t)}{h_n t}, \qquad \hat{F}''_n(t) := \frac{\mathbf{u}(I^2_n;t)+\mathbf{u}(I^3_n;t)-2\mathbf{u}(I^1_n;t)}{h_n^2 t^2}.$$

Note that $\hat{V}_n(t)$ defined by (5) is such that

$$\hat{V}_n(t) = \frac{1}{2}\frac{\hat{F}''_n(t)}{\hat{F}'_n(t)}.$$

Then for any δ there is a $\mathbb{P}$-a.s. finite random variable C_δ such that

$$\begin{aligned}\left|\hat{F}'_n(t) - F'(t)\right| &\le Ch_n + C_\delta 2^{-\delta n/2}(h_n t)^{-1}\\ &\le Ck_n 2^{-n}t + C_\delta t^{-1}2^{-\delta n/2}2^n k_n^{-1}\\ &\le Ck_n 2^{-n}t + C_\delta t^{-1}2^{(2-\delta)n/2}k_n^{-1}\end{aligned}$$

and

$$\begin{aligned}\left|\hat{F}''_n(t) - F''(t)\right| &\le Ch_n + \ C_\delta 2^{-\delta n/2}(h_n t)^{-2}\\ &\le Ck_n 2^{-n}t + C_\delta t^{-2}2^{-\delta n/2}2^{2n} k_n^{-2}\\ &\le Ck_n 2^{-n}t + C_\delta t^{-2}2^{(4-\delta)n/2}k_n^{-2}.\end{aligned}$$

Then it is easy to see that

$$2^{(2-\delta)n/2}k_n^{-1} \le 2\times 2^{n(1-3\delta)/6} \text{ and } 2^{(4-\delta)n/2}k_n^{-2} \le 4\times 2^{n(2-3\delta)/6}.$$

Since δ is an arbitrary number from $(0,1)$, for any $\varepsilon > 0$, there is a $\mathbb{P}$-a.s. finite random variable $\tilde{C}_\varepsilon \ge 0$ such that

$$\left|\hat{F}'_n - F'(t)\right| + \left|\hat{F}''_n - F''(t)\right| \le \tilde{C}_\varepsilon 2^{-n\left(\frac{1}{6}-\varepsilon\right)}. \tag{20}$$

Finally rRecall that the unknown potential $V(t)$ is given by

$$V(t) = \frac{1}{2}\frac{F''(t)}{F'(t)}.$$

Thus

$$\begin{aligned}2\left|V(t) - \frac{1}{2}\frac{\hat{F}''_n}{\hat{F}'_n}\right| &= \left|\frac{F''(t)}{F'(t)} - \frac{\hat{F}''_n}{\hat{F}'_n}\right|\\ &\le \frac{1}{F'(t)}\left|F''(t) - \hat{F}''_n\right| + \frac{1}{F'(t)}\frac{\left|\hat{F}''_n\right|}{\left|\hat{F}'_n\right|}\left|F'(t) - \hat{F}'_n\right|.\end{aligned}$$

Since $F'(t) > 0$ and $\frac{|F''_n|}{|F'_n|}$ converges to $2\,|V(t)|$, the desired estimate (6) follows from (20). $\square$

5. Proof of Proposition 2.7

From (15), we have

$$\begin{aligned} u(k+1,0) &= u_0(k+1) + \int_0^{k+1} e^{a\int_0^{k+1-s} q(r)dr} dW(s) \\ &= u_0(k+1) + \int_0^{k} e^{\int_0^{k-s} aq(r)dr} dW(s) + \int_k^{k+1} e^{a\int_0^{k+1-s} q(r)dr} dW(s), \end{aligned}$$

where we use the fact that $\int_0^1 q(r)dr = 0$. It follows that the (observable) random variables

$$x_k := (u(k+1,0) - u(k,0) - (u_0(k+1) - u_0(k))$$

have the representation

$$x_k = \int_k^{k+1} e^{a\int_0^{k+1-s} q(r)dr} dW(s)$$

and therefore they are i.i.d. Gaussian, with mean 0 and covariance

$$\int_k^{k+1} e^{2a\int_0^{k+1-s} q(r)dr} ds = \int_0^{1} e^{2a\int_0^{1-s} q(r)dr} ds = \iota(a).$$

Hence, by the law of large numbers,

$$\lim_{n\to\infty} \hat{u}_n := \lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} x_k^2 = \iota(a).$$

The convergence result follows by the continuity property of ι^{-1}.

To prove the convergence rate (7), observe first that, for any $p > 2$, there exists a constant $c = c_p$ such that

$$\mathbb{E}\,|\hat{u}_n - \iota(a)|^p \le \frac{c}{n^{p/2}}.$$

It follows that

$$\begin{aligned} \mathbb{E}\left[\sum_{n=1}^{\infty} n^{\frac{p}{2}-2}|\hat{u}_n - \iota(a)|^p\right] &\le \sum_{n=1}^{\infty} \mathbb{E}\left[n^{\frac{p}{2}-2}|\hat{u}_n - \iota(a)|^p\right] \\ &\le \sum_{n=1}^{\infty} \frac{c}{n^2} = \frac{c\pi^2}{6} < \infty. \end{aligned}$$

Hence the positive random variable

$$C_p := \sum_{n=1}^{\infty} n^{\frac{p}{2}-2}|\hat{u}_n - \iota_\sigma(a)|^p$$

satisfies $\mathbb{E}\,C_p < \infty$. Hence, $\mathbb{P}$-almost surely,

$$n^{\frac{1}{2}-\frac{2}{p}}|\hat{u}_n - \iota(a)| \le (C_p)^{\frac{1}{p}}$$

for any $n \ge 1$. In particular,

$$n^{\frac{1}{2}-\varepsilon}|\hat{u}_n - \iota(a)| \le (C_{\frac{2}{\varepsilon}})^{\frac{\varepsilon}{2}}.$$

Finally since ι' does not change sign and is continuous it follows that it is bounded away from 0 on an open neighbourhood around a. Hence the inverse of ι is Lipschitz on that neighbourhood. If L is the Lipschitz constant of ι^{-1} on that neighbourhood, we get that, for sufficiently large n, $\hat{u}_n$ will belong to that neighbourhood and hence

$$n^{\frac{1}{2}-\varepsilon}\left|\hat{a}_n - a\right| \le L n^{\frac{1}{2}-\varepsilon}\left|\hat{u}_n - \iota(a)\right| \le L(C_{\frac{2}{\varepsilon}})^{\frac{\varepsilon}{2}},$$

which gives the claim. □

6. Proof of Proposition 2.10

We examine only the more difficult case of temporal noise (compare Propositions 2.2 and 3.4). For $t \ge b$ we have

$$\begin{aligned} u(t,0) &= \int_0^t e^{a\int_0^{t-s}\mathbf{1}_{[0,b]}dr} dW(s) \\ &= \int_0^{t-b} e^{ab} dW(s) + \int_{t-b}^t e^{a(t-s)} dW(s) \\ &= e^{ab} W(t-b) + \int_{t-b}^t e^{a(t-s)} dW(s). \end{aligned}$$

Therefore we have

$$u(nb,0) = e^{ab} W((n-1)b) + \int_{(n-1)b}^{nb} e^{a(t-s)} dW(s).$$

In particular, for any $k \in \mathbb{N}$,

$$\begin{aligned} &u((2k+2)b,0) - u((2k+1)b,0) \\ &\quad = e^{ab}\left[W((2k+1)b) - W(2kb)\right] \\ &\qquad + \int_{(2k+1)b}^{(2k+2)b} e^{a((2k+2)b-s)} dW(s) - \int_{2kb}^{(2k+1)b} e^{a((2k+1)b-s)} dW(s) \\ &\quad = \int_{2kb}^{(2k+2)b} f_{k,a}(s) dW(s), \end{aligned}$$

where

$$f_{k,a}(s) := \begin{cases} e^{ab} - e^{a((2k+1)b-s)}, & s \in [2kb, (2k+1)b), \\ e^{a((2k+2)b-s)}, & s \in [(2k+1)b, (2k+2)b]. \end{cases}$$

Therefore

$$\mathbb{E}\left[u((2k+2)b,0) - u((2k+1)b,0)\right]^2 = \int_{(2kb}^{(2k+2)b} f_{k,a}(s)^2 ds = j_b(a).$$

Summing up,

$$\hat{u}_n = \frac{1}{n}\sum_{k=0}^{n-1}\left(u((2k+2)b,0) - u((2k+1)b,0)\right)^2,$$

where $(u((2k+2)b,0)-u((2k+1)b,0))^2$, are i.i.d random variables with mean $j_b(a)$, and the result follows from the law of large numbers. Finally, one can show (8) using identical arguments to those in the proof of Proposition 2.7. $\square$

7. Proof of Theorem 2.12

Recall that the potential V is periodic with a known period T. Note that for any $t \in [0,T)$, the random variables

$$Z(kT+t)-Z(kT) = \int_{kT}^{kT+t} V(s)ds + B(kT+t) - B(kT), \qquad k \in \mathbb{N},$$

have the same Gaussian distribution $\mathcal{N}(m_t, t)$, where

$$m_t := \int_{kT}^{kT+t} V(s)ds = \int_0^t V(s+kT)ds = \int_0^t V(s)ds.$$

Define

$$\mathcal{X}_n(t) := \frac{\sum_{k=0}^{n-1}(Z(kT+t)-Z(kT))}{n}, \quad n \in \mathbb{N},\ t \in [0,T).$$

By the law of large numbers

$$\lim_{n\to\infty} \mathcal{X}_n(t) = \int_0^t V(s)ds, \quad \mathbb{P}\text{-a.s.}$$

Let $\varepsilon \in (0,1/4)$. Recall that $h_n^\varepsilon = 2^{-(1/4-\varepsilon)n} \downarrow 0$. Then for any $n \in \mathbb{N}$, the random variable

$$\xi_n^\varepsilon := \mathcal{X}_{2^n}(t+h_n^\varepsilon) - \int_0^{t+h_n^\varepsilon} V(s)ds$$

is normally distributed with mean 0 and variance $(t+h_n^\varepsilon)/2^n$. Hence there is a constant C_1 such that

$$\mathbb{E}\,(\xi_n^\varepsilon)^2 \le C_1 2^{-n}.$$

In particular, for $\delta = 1-4\varepsilon < 1$,

$$\sum_{n=1}^{\infty} 2^{\delta n}\mathbb{E}\,(\xi_n^\varepsilon)^2 < \infty,$$

and consequently there is a square integrable random variable $C_1(\delta)$ such that for any n,

$$\left|\mathcal{X}_{2^n}(t+h_n^\varepsilon) - \int_0^{t+h_n^\varepsilon} V(s)ds\right| = |\xi_n^\varepsilon| \le C_1(\delta)2^{-\delta n/2}, \quad \mathbb{P}\text{-a.s.}$$

Clearly we can assume that also

$$\left|\mathcal{X}_{2^n}(t) - \int_0^t V(s)ds\right| \le C_1(\delta)2^{-\delta n/2}, \quad \mathbb{P}\text{-a.s.}$$

To obtain $V(t)$ we have to differentiate $\int_0^t V(s)ds$. There is a constant C_2 such that

$$\left|V(t) - \frac{1}{h}\left(\int_0^{t+h} V(s)ds - \int_0^t V(s)ds\right)\right| \le C_2 h.$$

Thus

$$\begin{aligned}\left|V(t)-\hat V_n^\varepsilon(t)\right| &= \left|V(t)-\frac{1}{h_n^\varepsilon}\left(\mathcal{X}_{2^n}(t+h_n^\varepsilon)-\mathcal{X}_{2^n}(t)\right)\right|\\ &\le C_2h_n^\varepsilon+2(h_n^\varepsilon)^{-1}2^{-\delta n/2}C_1(\delta).\end{aligned}$$

Note that

$$\begin{aligned}&C_2h_n^\varepsilon+2(h_n^\varepsilon)^{-1}2^{-\delta n/2}C_1(\delta)\\ &\quad\le (C_2+2C_1(\delta))\left(2^{-(1/4-\varepsilon)n}+2^{(1/4-\varepsilon)-(1-4\varepsilon)n/2}\right)\\ &\quad\le (C_2+2C_1(\delta))\left(2^{-(1/4-\varepsilon)n}+2^{n(1/4-\varepsilon-1/2+2\varepsilon)}\right)\\ &\quad\le 2(C_2+2C_1(\delta))2^{-(1/4-\varepsilon)n},\end{aligned}$$

which completes the proof. □

8. Multidimensional case

Let us re-state first the stochastic transport equation in the multidimensional case. Let $b=(b^1,\dots,b^d)\in\mathbb{R}^d$, and let $V\colon\mathbb{R}^d\to\mathbb{R}$ be a measurable and locally bounded function. Let A_V be the following operator

$$A_Vv(x):=\sum_{k=1}^d b^k\partial_{x_k}v(x)+V(x)v(x),\quad x\in\mathbb{R}^d.$$

Then the weak solution to the deterministic transport equation

$$\partial_t u(t,x)=A_Vu(t,x),\qquad u(0,x)=u_0(x),\qquad t>0,\ x\in\mathbb{R}^d,$$

is given by $u(t,x)=S(t)u_0(x)$, where

$$S(t)v(x)=v(x+bt)e^{\int_0^t V(x+bs)ds}$$

is a C_0-group on the weighted space $L^2_\rho:=L^2(\mathbb{R}^d,\mathcal{B}(\mathbb{R}^d),(1+|x|^2)^{-d}dx)$.

Let $\dot W(t), t\ge 0$, be a temporal white noise. The (direct) solution to the multilinear stochastic transport equation with the additive noise $\dot W$

$$\partial_t u(t,x)=A_Vu(t,x)+\dot W(t),\qquad u(0,x)=u_0(x),$$

is given by

$$u(t,x)=S(t)u_0(x)+\int_0^t S(t-s)\mathbf{1}(x)dW(s)=u_0(x+tb)e^{\int_0^t V(x+br)dr}+\int_0^t e^{\int_0^{t-s}V(x+br)dr}dW(s).$$

Consequently, if $V(x)=aq(x)$, where either $s\mapsto q(bs)$ is a known periodic or indicator function and a is an unknown constant, then the corresponding inverse problem can be solved by applying the same arguments from Sections 2.3.1 and 2.3.2. In the case of multiplicative temporal noise, the observation process is given by

$$u(t,0)=u_0(tb)\exp\left\{\int_0^t V(rb)dr+W(t)\right\}.$$

Therefore only the value of the potential along the line $r \mapsto rb$ can be determined. To do this one can use an obvious modification of the estimator constructed in Section 2.4.

The case of a spatial noise is different. Namely a function-valued solution exists only if $d = 1$. This is a consequence (see e.g. [11]) of the fact that the operator

$$L^2(\mathbb{R}^d) \ni \psi \mapsto \int_0^t S(t-s)\psi ds \in L^2_\rho$$

is Hilbert–Schmidt if and only if $d = 1$.

References

[1] S. Albeverio, P. Blanchard, S. Kusuoka, and L. Streit, *An inverse problem for stochastic differential equations*, J. Statist. Phys. 57 (1989), 347–356.

[2] A.S. Blagoveščenskiĭ, *Inverse problem of the theory of wave propagation in a stochastic medium*, Mathematical questions in the theory of wave propagation, 10, Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI), vol. 89, 1979, Russian, pp. 63–70.

[3] P.L. Chow and R.Z. Khasminski, *Statistical approach to dynamical inverse problems*, Commun. Math. Phys. 189 (1997), 521–531.

[4] P.L. Chow and L. Maestrello, *Stochastic inverse problem in the radiation of noise*, SIAM J. Appl. Math. 35 (1978), 665–677.

[5] G. Da Prato and J. Zabczyk, *Stochastic Equations in Infinite Dimensions*, Cambridge University Press, Cambridge 1992.

[6] G.K. Galubev and R.Z. Khasminski, *A statistical approach to some inverse problems for partial differential equations*, Problems Inform. Transmission 35 (1999), 136–149.

[7] N. Ikeda and S. Watanabe, *Stochastic Differential Equations and Diffusion Processes*, second ed., North Holland–Kodansha, Amsterdam, 1989.

[8] I. Karatzas and S.E. Shreve, *Brownian Motion and Stochastic Calculus*, second ed., GTM 113, Springer–Verlag, 1991.

[9] O. Kallenberg, *Foundations of Modern Probability*, Springer-Verlag, New York, 2002.

[10] P. Kazimierczyk, *On the stochastic inverse problem for the heat conduction equation*, Rep. Math. Phys. 26 (1988), 245–259.

[11] S. Peszat and J. Zabczyk, *Stochastic Partial Differential Equations with Lévy Noise (an Evolution Equation Approach)*, Cambridge University Press, Cambridge 2007.

[12] A.M. Stuart, *Inverse problems: A Bayesian perspective*, Acta Numer. 19 (2010), 451–559.

[13] V.G. Romanov, *Inverse Problems of Mathematical Physics*, VNU Science Press, 1987.

[14] B.A. Ton, *An inverse problem for a nonlinear stochastic wave equation*, Stochastic Anal. Appl. 22 (2004), 19–42.

[15] J. B. Walsh, *An introduction to stochastic partial differential equations*, École d'Été de Probabilités de Saint-Flour XIV– 1984, LNM, vol. 1180, Springer-Verlag, 1986, pp. 265–439.

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