# An explicit Euler scheme with strong rate of convergence for non-Lipschitz SDEs

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### Abstract

We consider the approximation of stochastic differential equations (SDEs) with non-Lipschitz drift or diffusion coefficients. We present a modified explicit Euler-Maruyama discretisation scheme that allows us to prove strong convergence, with a rate. Under some regularity conditions, we obtain the *optimal* strong error rate. We consider SDEs popular in the mathematical finance literature, including the Cox-Ingersoll-Ross (CIR), the 3/2 and the Ait-Sahalia models, as well as a family of mean-reverting processes with locally smooth coefficients.

**Key words:** Stochastic differential equations, non-Lipschitz coefficients, explicit Euler-Maruyama scheme with projection, CIR model, Ait-Sahalia model.

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# 1   Introduction

One of the main tasks in mathematical finance is pricing of option derivatives. Typically, the underlying assets are modelled by multi-dimensional SDEs, which rarely admit closed-form solutions and need to be numerically simulated. Therefore, Monte Carlo techniques are used to approximate the prices of options, by simulating sample paths of the underlying assets and estimating functionals to price the derivatives of interest (see [12] for a comprehensive overview of such methods with applications to financial engineering).

Classical weak and strong convergence results for discretisation schemes of SDEs assume that the drift and the diffusion coefficients driving the SDEs are globally Lipschitz continuous (see [24]); however many popular models in the literature violate this assumption e.g. CIR, CEV, Ait-Sahalia models. Typically, in financial derivative pricing weak error is sufficient for applications. Strong convergence rates are important when using Multilevel Monte Carlo methods, as the strong rate of convergence can be used to optimise computation of functionals [10, 11].

In recent years there has been a strong interest in convergence results for discretisation schemes for SDEs with non-Lipschitz continuous coefficients in a restricted domain [2, 3, 4, 17, 20, 26]. A classical Euler-Maruyama discretisation scheme defines approximations which can potentially escape the domain of the true solution of the SDE. To prevent such an escape, several modifications have been introduced such as the drift-implicit [8] and the increment-tamed explicit Euler schemes [18, Theorem 3.15]. Modified Itô-Taylor schemes of order $\psi > 0$ have been shown to have pathwise convergence of order $\psi - \varepsilon$ for arbitrarily small $\varepsilon > 0$, provided that the drift and diffusion functions are sufficiently differentiable [21]. The method relies on a localisation argument similar to the one used in [13], with an auxiliary drift and diffusion function chosen upon the discretised process exiting a sub-domain. Recently, there have been results for strong approximations of scalar SDEs with one-sided Lipschitz continuous drift, constant diffusion, and values in some domain, using implicit Euler and Milstein schemes [26]. Strong rates of convergence for SDEs with irregular coefficients have been proven under more restrictive conditions imposed on the drift and diffusion functions [27], improving on results in [13, 14, 29].

A review for convergence of numerical methods specific to finance is provided by Kloeden and Neuenkirch [23].

Motivated by these varying approaches, we present an explicit Euler scheme with a projection, which has a computational cost of the same order as the explicit Euler-Maruyama scheme. We prove strong rates of convergence for this modified scheme under some regularity assumptions and integrability conditions for the true solution.

If necessary, a transformation can be applied to the process in order to shift the non-Lipschitz behaviour from the diffusion to the drift function, before using the modified scheme. The remainder of the paper is structured as follows. In Section 2, the modified Euler-Maruyama scheme is introduced. In Section 3, the main convergence result is proven for the scheme. In Section 4, the scheme is applied to families of SDEs commonly used in mathematical finance, including the CIR, the 3/2 and the Ait-Sahalia models. In Section 5, numerical results for the rates of convergence obtained are shown and discussed.

**Notations**: In the sequel, $D$ shall always denote an interval, such that $D \subseteq \mathbb{R}$. We denote by $\bar{D}_\eta$ the domain $[\eta, \infty)$, and $\bar{D} := \bar{D}_0$. Furthermore, we define the interval $\breve{D}_\zeta := [0, \zeta]$. We denote by $\mathcal{C}^2(D)$ the space of twice differentiable functions with continuous derivatives on $D$, and by $\mathcal{C}_b^2(D)$ the space of functions in $\mathcal{C}^2(D)$ with first and second bounded derivatives. We shall denote by $\mathbb{N}^+$ the set of strictly positive integers.

## 2 Definitions and assumptions

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ be a filtered probability space, and $W = (W_t)_{t \geq 0}$ a standard $(\mathcal{F}_t)$-adapted Brownian motion. Consider a stochastic differential equation of the form

$$\mathrm{d}Y_t = f(Y_t)\mathrm{d}t + \gamma(Y_t)\mathrm{d}W_t, \qquad Y_0 = y_0. \tag{2.1}$$

Throughout this article, we shall assume the following:
- the SDE (2.1) admits a unique strong solution in $D = (0, \infty)$;
- the drift $f$ is locally Lipschitz continuous and globally one-sided Lipschitz continuous on $D$, namely there exist $\alpha, \beta \geq 0$ and $K > 0$, such that for all $(x, y) \in D^2$:

$$|f(x) - f(y)| \leq K(1 + |x|^\alpha + |y|^\alpha + \frac{1}{|x|^\beta} + \frac{1}{|y|^\beta})|x - y|, \tag{2.2}$$

$$(x - y)(f(x) - f(y)) \leq K|x - y|^2. \tag{2.3}$$

- the diffusion function $\gamma$ is K-Lipschitz continuous on $\bar{D}$ for some $K > 0$: for all $(x, y) \in \bar{D}^2$, the inequality $|\gamma(x) - \gamma(y)| \leq K|x - y|$ holds.

**Remark 2.1.** *The function $\gamma$ could as well be defined on $D$. However, assuming the Lipschitz continuity of $\gamma$ on $D$ would lead to a natural extension of $\gamma$ on $\bar{D}$.*

**Remark 2.2.** *In many models used in practice (in particular the Feller/CIR diffusion in mathematical finance, see Section 4.1), these assumptions are not met. A suitable*

*change of variables allows us to bypass this issue: consider indeed an SDE of the form*

$$\mathrm{d}X_t = \mu(X_t)\mathrm{d}t + \sigma(X_t)\mathrm{d}W_t, \qquad X_0 = x_0, \tag{2.4}$$

*where the process $X$ takes values in some domain $D_X \subseteq \mathbb{R}$. If $\sigma(x) > 0$ for all $x \in D_X$, the Lamperti transformation of $X$ is defined as $F(x) \equiv \int^x \sigma(z)^{-1}\mathrm{d}z$, and Itô's Lemma then implies that the process defined pathwise by $Y := F(X)$ satisfies (2.1) with $f \equiv F'\mu + \frac{1}{2}F''\sigma^2$ and $\gamma \equiv F'\sigma$.*

Let $n \in \mathbb{N}^+$ be a fixed positive integer and $T > 0$ a fixed time horizon. Define the partition of the interval $[0, T]$ by $\pi := \{0 = t_0 < t_1 < \ldots < t_n = T\}$, with $\max_{i=0,\ldots,n-1}(t_{i+1} - t_i) \leq h = \mathcal{O}(1/n)$.

For a closed interval $\mathcal{C} \subset \mathbb{R}$, we define $p_\mathcal{C} : \mathbb{R} \to \mathcal{C}$ as the projection operator onto $\mathcal{C}$. We introduce a domain $D_n = [n^{-k}, n^{k'}] \subseteq D$ with strictly positive (and possibly infinite) real numbers $k, k'$. The constants $k$ and $k'$ will be chosen optimally later on. For ease of notation, we define $p_n := p_{D_n}$ and

$$p_n(x) \equiv n^{-k} \vee x \wedge n^{k'}. \tag{2.5}$$

In the following, we denote by $C$ a constant that depends only on $K$, $T$, $\alpha$, $\beta$, $y_0$, but whose value may change from line to line. We denote it by $C_p$ if it depends on an extra parameter $p$.

We now introduce our explicit scheme for the discretisation process $\hat{Y}$:

**Definition 2.1.** *Set $\hat{Y}_0 := Y_0$ and for $i = 0, \ldots, n-1$,*

$$\hat{Y}_{t_{i+1}} := \hat{Y}_{t_i} + f_n(\hat{Y}_{t_i})h_{i+1} + \bar{\gamma}(\hat{Y}_{t_i})\Delta W_{i+1},$$

*with $h_{i+1} := t_{i+1} - t_i$, $\Delta W_{i+1} := W_{t_{i+1}} - W_{t_i}$, $f_n := f \circ p_n$ and $\bar{\gamma} := \gamma \circ p_{\bar{D}}$.*

**Remark 2.3.** *For some applications, it may be interesting to force the scheme to take values in a domain, e.g. intervals $\bar{D}$, $\bar{D}_\eta$ or even $\breve{D}_\zeta$. To this end, we introduce some extensions of the previous scheme. For all $i \leq n$, we define $\bar{Y}_{t_i} := p_{\bar{D}}(\hat{Y}_{t_i})$, $\tilde{Y}_{t_i} := p_{\bar{D}_\eta}(\hat{Y}_{t_i})$ and $\breve{Y}_{t_i} := p_{\breve{D}_\zeta}(\hat{Y}_{t_i})$, for some $\eta, \zeta > 0$ to be determined later on, see Corollary 3.1 for details. In Proposition 3.3, we prove finite moments and finite inverse moments for these modifications.*

We have the following result whose proof is postponed to the appendix.

**Lemma 2.1.** *The composition $f_n$ is Lipschitz continuous with Lipschitz constant $L(n) = C(n^{k\beta} + n^{k'\alpha})$. Moreover, for any $n \in \mathbb{N}^+$, the function $f_n$ is one-sided Lipschitz continuous, with the same constant $K$ as the one-sided Lipschitz continuous constant of function $f$.*

**Remark 2.4.** *Since $f_n$ and $\gamma$ are Lipschitz continuous, an easy induction shows that the scheme given in Definition 2.1 satisfies $\max_{i=0,\dots,n} \mathbb{E}\left[|\hat{Y}_{t_i}|^2\right] < \infty$, for all $n \in \mathbb{N}^+$. The bound is* a priori *non-uniform in $n$, since the Lipschitz constant of $f_n$ depends on $n$.*

We now introduce the following assumption, which implies that $L(n)^2 h \leq C$, for all $n \in \mathbb{N}^+$, and which relates the locally Lipschitz exponents $\alpha$ and $\beta$ to the size of the truncated domain, $D_n$:

(**H**$p$): the strictly positive constants $k$, $k'$ satisfy $2\beta k \leq 1$ and $2\alpha k' \leq 1$.

We require additional assumptions to prove the strong convergence rate of our scheme: below (**H**$y1$) imposes a condition on the moments of the process $Y$ in terms of the locally Lipschitz exponents $\alpha$ and $\beta$, to obtain a minimal convergence rate. We shall further impose regularity conditions on $f$ and $\gamma$ to obtain a better rate of convergence.

(**H**$y1$): assume that (**H**$p$) holds and that there exist $q' > 2(\alpha + 1)$ and $q > 2\beta$ such that $\mathbb{E}(|Y_t|^{q'})$ and $\mathbb{E}(|Y_t|^{-q})$ are finite for all $t \in [0, T]$.

(**H**$y2$): assume that (**H**$y1$) holds, that the drift function $f$ is of class $\mathcal{C}^2(D)$, and that

$$\sup_{t \in [0,T]} \mathbb{E}|\gamma(Y_t)f'(Y_t)|^2 + \sup_{t \in [0,T]} \mathbb{E}\left|f'(Y_t)f(Y_t) + \frac{\gamma^2(Y_t)}{2}f''(Y_t)\right|^2 < \infty. \tag{2.6}$$

For an implicit scheme, strong rates of convergence have been derived in [26] assuming (**H**$y2$); inspired by their paper, our motivation is to recover strong rates of convergence for the explicit scheme in Definition 2.1.

## 3 Convergence results

In this section we prove strong rate of convergence for the scheme in Definition 2.1 under some of the assumptions stated above; this results follows from estimates for the regularity of the processes $Y$ and $f(Y)$ and the discretisation error of the scheme.

### 3.1 Preliminary estimates

Throughout this section, we shall always assume that (**H**$y1$) holds. Our first two results concern the error due to projecting $Y$ on $D_n$.

**Lemma 3.1.** *For any $t \in [0, T]$, the following inequality holds:*

$$\mathbb{E}\left[|Y_t - p_n(Y_t)|^2\right] \leq C_{q,q'}\left(\frac{1}{n^{k(q+2)}} + \frac{1}{n^{k'(q'-2)}}\right) =: K_1(n, q, q').$$

**Proof.** For any $t \in [0, T]$, we can write

$$\mathbb{E}\big[|Y_t - p_n(Y_t)|^2\big] \leq \frac{1}{n^{2k}} \mathbb{P}\{Y_t < \frac{1}{n^k}\} + \mathbb{E}\Big[|Y_t|^2 \mathbf{1}_{\{Y_t > n^{k'}\}}\Big] .$$

Set $\eta = q'/2$ and $\theta = q'/(q'-2)$, its conjugate exponent. Hölder's inequality yields

$$\mathbb{E}\Big[|Y_t|^2 \mathbf{1}_{\{Y_t > n^{k'}\}}\Big] \leq \mathbb{E}\Big[|Y_t|^{q'}\Big]^{1/\eta} \mathbb{P}\{Y_t > n^{k'}\}^{1/\theta}.$$

Using ($\mathbf{H}y1$) and the set equality $\{Y_t > n^{k'}\} = \{Y_t^{q'} > n^{k'q'}\}$, Markov's inequality implies $\mathbb{E}\Big[|Y_t|^2 \mathbf{1}_{\{Y_t > n^{k'}\}}\Big] \leq C_{q'} n^{-k'(q'-2)}$. Likewise, since $\{Y_t < n^{-k}\} = \{Y_t^{-q} > n^{kq}\}$, Markov's inequality yields $\mathbb{P}(Y_t < n^{-k}) \leq C_q n^{-kq}$. The proof then follows by combining the previous inequalities. $\qquad\square$

**Lemma 3.2.** *For any $t \in [0, T]$, the following upper bound holds:*

$$\mathbb{E}\big[|f(Y_t) - f_n(Y_t)|^2\big] \leq C_{q,q'} \left( \frac{1}{n^{k(q-2(\beta-1))}} + \frac{1}{n^{k'(q'-2(\alpha+1))}} \right) =: K_2(n, q, q').$$

**Proof.** Using (2.2), we observe that

$$
\begin{aligned}
|f(Y_t) - f_n(Y_t)|^2 &\leq C \left( 1 + |Y_t|^{-2\beta} + |Y_t|^{2\alpha} \right) |Y_t - p_n(Y_t)|^2 \\
&\leq C \left( 1 + |Y_t|^{-2\beta} \right) \frac{1}{n^{2k}} \mathbf{1}_{\{Y_t < n^{-k}\}} + C \left( 1 + |Y_t|^{2\alpha} \right) |Y_t|^2 \mathbf{1}_{\{Y_t > n^{k'}\}} \\
&:= A_1 + A_2.
\end{aligned}
$$

Set $\eta := q/(2\beta)$ and $\theta := q/(q-2\beta)$. Hölder's inequality then yields

$$\mathbb{E}(A_1) \leq \frac{C_q}{n^{2k}} \mathbb{E}\big[|Y_t|^{-q}\big]^{1/\eta} \mathbb{P}\{Y_t < n^{-k}\}^{1/\theta},$$

and ($\mathbf{H}y1$) together with Markov's inequality imply $\mathbb{E}(A_1) \leq C_q n^{-k(q-2(\beta-1))}$. Setting $\eta' := \frac{q'}{2(\alpha+1)}$ and $\theta' := \frac{q'}{q'-2(\alpha+1)}$, a similar computation gives $\mathbb{E}(A_2) \leq C_{q'} n^{-k'(q'-2(\alpha+1))}$. $\qquad\square$

The following lemma provides a regularity result for the process $Y$ and shall be required for the main convergence result. For a given stochastic process $X$ on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ and the partition $\pi$, we define its regularity by

$$\mathcal{R}_\pi[X] := \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \mathbb{E}\big[|X_t - X_{t_i}|^2\big] \, \mathrm{d}t . \tag{3.1}$$

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**Lemma 3.3.** *The regularity of $Y$ satisfies $\mathcal{R}_\pi[Y] \leq C_{q,q'}h$.*

**Proof.** For $t \in (t_i, t_{i+1}]$, since $\gamma$ is $K$-Lipschitz, $(\mathbf{H}y1)$ implies

$$\mathbb{E}\big[|Y_t - Y_{t_i}|^2\big] \leq C\mathbb{E}\left[\left(\int_{t_i}^t f(Y_s)\mathrm{d}s\right)^2 + \int_{t_i}^t (|Y_s|^2 + 1)\mathrm{d}s\right] \leq Ch\left(1 + \frac{1}{h}\mathbb{E}\left[\left(\int_{t_i}^t f(Y_s)\mathrm{d}s\right)^2\right]\right).$$

For $t \in (t_i, t_{i+1}]$, we now compute

$$\frac{1}{h}\mathbb{E}\left[\left(\int_{t_i}^t f(Y_s)\mathrm{d}s\right)^2\right] \leq \mathbb{E}\left[\int_{t_i}^{t_{i+1}} |f(Y_s)|^2 \mathrm{d}s\right]$$

$$\leq \int_{t_i}^{t_{i+1}} \mathbb{E}\big[|f(Y_s) - f_n(Y_s)|^2\big]\,\mathrm{d}s + \int_{t_i}^{t_{i+1}} \mathbb{E}\big[|f_n(Y_s)|^2\big]\,\mathrm{d}s$$

$$\leq Ch\left(K_2(n, q, q') + L(n)^2 \sup_{t \in [t_i, t_{i+1}]} \mathbb{E}\big[1 + |Y_t|^2\big]\right).$$

Using $(\mathbf{H}y1)$ and the inequality $L(n)^2h \leq C$, which holds under $(\mathbf{H}p)$, we obtain that $\mathbb{E}\big[|Y_t - Y_{t_i}|^2\big] \leq C_{q,q'}h$ holds for $t \in (t_i, t_{i+1}]$, and the lemma follows from the following upper bound:

$$\mathcal{R}_\pi[Y] = \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \mathbb{E}\big[|Y_t - Y_{t_i}|^2\big]\,\mathrm{d}t \leq C \max_{i=0,\dots,n-1} \sup_{t \in [t_i, t_{i+1}]} \mathbb{E}\big[|Y_t - Y_{t_i}|^2\big] \leq C_{q,q'}h\ .$$

$\square$

We now compute upper bounds for the regularity of $f(Y)$.

**Lemma 3.4.**

(i) *Under $(\mathbf{H}y1)$, the inequality $\mathcal{R}_\pi[f(Y)] \leq C\left(K_2(n, q, q') + L(n)^2h\right)$ holds;*

(ii) *Under $(\mathbf{H}y2)$, the inequality $\mathcal{R}_\pi[f(Y)] \leq Ch$ holds.*

**Proof.** The inequality in statement (i) is a direct consequence of the following computation:

$$\int_{t_i}^{t_{i+1}} \mathbb{E}\big[|f(Y_t) - f(Y_{t_i})|^2\big]\,\mathrm{d}t \leq C\Big(\int_{t_i}^{t_{i+1}} \mathbb{E}\big[|f(Y_t) - f_n(Y_t)|^2\big]\,\mathrm{d}t$$

$$+ \int_{t_i}^{t_{i+1}} \mathbb{E}\big[|f_n(Y_t) - f_n(Y_{t_i})|^2\big]\,\mathrm{d}t$$

$$+ h\mathbb{E}\big[|f_n(Y_{t_i}) - f(Y_{t_i})|^2\big]\Big) \leq Ch\left(K_2(n, q, q') + L(n)^2h\right),$$

where we used Lemma 3.2, Lemma 3.3, and $(\mathbf{H}p)$. Let us now prove statement (ii). The drift function $f$ is of class $\mathcal{C}^2(D)$ by $(\mathbf{H}y2)$; Itô's Formula on the interval $[t_i, t_{i+1}]$ reads

$$f(Y_{t_{i+1}}) - f(Y_{t_i}) = \int_{t_i}^{t_{i+1}} \left(f'(Y_t)f(Y_t) + \frac{1}{2}f''(Y_t)\gamma(Y_t)^2\right)\mathrm{d}t + \int_{t_i}^{t_{i+1}} f'(Y_t)\gamma(Y_t)\mathrm{d}W_t;$$

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squaring and applying the Cauchy-Schwarz inequality implies

$$\mathbb{E}\big[|f(Y_{t_{i+1}}) - f(Y_{t_i})|^2\big] \leq \int_{t_i}^{t_{i+1}} \mathbb{E}\left[|\gamma(Y_t)f'(Y_t)|^2 + h\left|f'(Y_t)f(Y_t) + \frac{\gamma^2(Y_t)}{2}f''(Y_t)\right|^2\right] \mathrm{d}t,$$

and statement (ii) follows from (2.6), direct integration on $[t_i, t_{i+1}]$ and summation. $\square$

## 3.2 Convergence result

We now consider the discretisation error between the true process $Y$ and the discretised process $\hat{Y}$. Let us introduce the following notations,

$$\delta Y_i := Y_{t_i} - \hat{Y}_{t_i}, \qquad \delta_n f_i := f_n(Y_{t_i}) - f_n(\hat{Y}_{t_i}), \qquad \delta\gamma_i := \gamma(Y_{t_i}) - \bar{\gamma}(\hat{Y}_{t_i}). \qquad (3.2)$$

We now state and prove a key result, which provides a bound on the squared differences $|\delta Y_i|^2$. This bound depends on both the partition size and the regularity (in the sense of (3.1)), and is refined further in Theorem 3.1 below.

**Proposition 3.1.** *Assume that* $(\mathbf{H}y1)$ *holds, then*

$$\max_{i=0,\dots,n} \mathbb{E}\big[|\delta Y_i|^2\big] \leq C\left(K_2(n, q, q') + \mathcal{R}_\pi[f(Y)] + \mathcal{R}_\pi[Y]\right). \qquad (3.3)$$

**Proof.** 1. We first show that the global error between the scheme and the solution is controlled by the sum of local truncation errors defined below. Indeed, we observe that, for $i \leq n - 1$,

$$Y_{t_{i+1}} = Y_{t_i} + f_n(Y_{t_i})h_{i+1} + \bar{\gamma}(Y_{t_i})\Delta W_{i+1} + \zeta_{i+1}^d + \zeta_{i+1}^w,$$

where

$$\zeta_{i+1}^d := \int_{t_i}^{t_{i+1}} (f(Y_t) - f_n(Y_{t_i}))\,\mathrm{d}t,$$

$$\zeta_{i+1}^w := \int_{t_i}^{t_{i+1}} (\gamma(Y_t) - \bar{\gamma}(Y_{t_i}))\,\mathrm{d}W_t = \int_{t_i}^{t_{i+1}} (\gamma(Y_t) - \gamma(Y_{t_i}))\,\mathrm{d}W_t.$$

The last equality comes from the fact that $Y$ takes its values in $D$ and then $\bar{\gamma}(Y_{t_i}) = \gamma(Y_{t_i})$, for all $i \leq n$. Therefore, squaring the difference $\delta Y_{i+1}$ gives

$$|\delta Y_{i+1}|^2 = |\delta Y_i|^2 + 2\delta Y_i \delta_n f_i h_{i+1} + 2\delta Y_i \delta\gamma_i \Delta W_{i+1} + 2\delta Y_i \zeta_{i+1}^d + 2\delta Y_i \zeta_{i+1}^w$$
$$+ |\delta_n f_i h_{i+1} + \delta\gamma_i \Delta W_{i+1} + \zeta_{i+1}^d + \zeta_{i+1}^w|^2.$$

Using the simple identity $\mathbb{E}_{t_i}\big[2\delta Y_i\delta\gamma_i\Delta W_{i+1} + 2\delta Y_i\zeta_{i+1}^w\big] = 0$ and an application of Young's inequality yields

$$\mathbb{E}\big[|\delta Y_{i+1}|^2\big] \le (1+Ch)\mathbb{E}\big[|\delta Y_i|^2\big] + C\mathbb{E}\left[|\delta_n f_i h_{i+1}|^2 + |\delta\gamma_i|^2 h_{i+1} + \frac{|\mathbb{E}_{t_i}[\zeta_{i+1}^d]|^2}{h} + |\zeta_{i+1}^w|^2\right]$$

$$\le \big(1 + Ch + CL(n)^2 h^2\big)\mathbb{E}\big[|\delta Y_i|^2\big] + C\mathbb{E}\left[\frac{\big(\mathbb{E}_{t_i}[\zeta_{i+1}^d]\big)^2}{h} + |\zeta_{i+1}^d|^2 + |\zeta_{i+1}^w|^2\right],$$

where we have used the fact that $f_n$ is one-sided Lipschitz continuous (Lemma 2.1), locally Lipschitz continuous with Lipschitz constant $L(n)$ and $\gamma$ is Lipschitz continuous. Since $(\mathbf{H}p)$ holds, $L(n)^2 h \le C$ and iteration yields

$$\max_{i=0,\dots,n} \mathbb{E}\big[|\delta Y_i|^2\big] \le C\sum_{j=1}^{n}\mathbb{E}\left[\frac{\big(\mathbb{E}_{t_j}[\zeta_j^d]\big)^2}{h} + |\zeta_j^d|^2 + |\zeta_j^w|^2\right] \tag{3.4}$$

$$\le C\sum_{j=1}^{n}\mathbb{E}\left[\frac{|\zeta_j^d|^2}{h} + |\zeta_j^w|^2\right]. \tag{3.5}$$

2. We now provide explicit errors for the global truncation. As $\gamma$ is $K$-Lipschitz, we have $\mathbb{E}\big[|\zeta_{i+1}^w|^2\big] \le C\int_{t_i}^{t_{i+1}}\mathbb{E}\big[|Y_t - Y_{t_i}|^2\big]\,\mathrm{d}t$, and hence

$$\sum_{i=1}^{n}\mathbb{E}\big[|\zeta_i^w|^2\big] \le C\mathcal{R}_\pi[Y]. \tag{3.6}$$

We now compute an upper bound for $\mathbb{E}\big[|\zeta_{i+1}^d|^2\big]$. Since

$$\zeta_{i+1}^d := \int_{t_i}^{t_{i+1}}(f(Y_t) - f_n(Y_{t_i}))\mathrm{d}t = \int_{t_i}^{t_{i+1}}(f(Y_t) - f(Y_{t_i}))\mathrm{d}t + \int_{t_i}^{t_{i+1}}(f(Y_{t_i}) - f_n(Y_{t_i}))\mathrm{d}t, \tag{3.7}$$

we have, using the Cauchy-Schwarz inequality,

$$\mathbb{E}\Big[|\zeta_{i+1}^d|^2\Big] \le Ch\left(\int_{t_i}^{t_{i+1}}\mathbb{E}\big[|f(Y_t) - f(Y_{t_i})|^2\big]\,\mathrm{d}t + h\mathbb{E}\big[|f(Y_{t_i}) - f_n(Y_{t_i})|^2\big]\right).$$

Lemma 3.2 then implies that the inequalities $\mathbb{E}\big[|\zeta_{i+1}^d|^2\big] \le Ch(\int_{t_i}^{t_{i+1}}\mathbb{E}\big[|f(Y_t) - f(Y_{t_i})|^2\big]\,\mathrm{d}t + hK_2(n,q,q'))$ and $\frac{1}{h}\sum_{i=1}^{n}\mathbb{E}\big[|\zeta_i^d|^2\big] \le C\big(K_2(n,q,q') + \mathcal{R}_\pi[f(Y)]\big)$ hold. Combining the latter with (3.5) and (3.6) concludes the proof of (3.3). $\qquad\square$

We have kept the above result general, without *a priori* assuming that the drift function belongs to $C^2(D)$. If we consider a constant diffusion and $(\mathbf{H}y2)$, we can recover a better upper bound using (3.4) instead of (3.5) in the first part of the previous proof and prove a first order strong rate of convergence. This will be illustrated in Proposition 3.2 below. We now state the main result of our paper, recalling the projection defined in (2.5).

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**Theorem 3.1.** *The following holds:*

$$\max_{i=0,\dots,n} \mathbb{E}\big[|\delta Y_i|^2\big] \leq C_{q,q'} h^r, \tag{3.8}$$

*with* $r = \min(1 - \frac{2\beta}{q+2}, 1 - \frac{2\alpha}{q'-2}) > 0$ *under* (**H**y1) *by setting* $(k, k') = (1/(q+2), 1/(q'-2))$ *and* $r = \min(1, \frac{q+2}{2\beta} - 1, \frac{q'-2}{2\alpha} - 1) > 0$ *under* (**H**y2) *by setting* $(k, k') = (1/(2\beta), 1/(2\alpha))$.

**Proof.** 1. Assume (**H**y1). Combining Lemma 3.4 (i) with (3.3) yields

$$\max_{i=0,\dots,n} \mathbb{E}\big[|\delta Y_i|^2\big] \leq C(K_2(n, q, q') + L(n)^2 h + h);$$

$$\leq C_{q,q'}(h^{1-2\beta k} + h^{k(q+2)-2\beta k} + h^{1-2\alpha k'} + h^{k'(q'-2)-2\alpha k'} + h).$$

To balance the error terms, set $k = \frac{1}{q+2}$ and $k' = \frac{1}{q'-2}$, observing that under (**H**y1), (**H**p) holds for this choice of parameters. Thus, we obtain $\max_{i=0,\dots,n} \mathbb{E}\big[|\delta Y_i|^2\big] \leq C_{q,q'} h^r$, with $r = \min(1 - \frac{2\beta}{q+2}, 1 - \frac{2\alpha}{q'-2})$, with $r > 0$.

2. We assume that (**H**y2) holds. Combining Lemma 3.4 (ii) with (3.3), we obtain

$$\max_{i=0,\dots,n} \mathbb{E}\big[|\delta Y_i|^2\big] \leq C(K_2(n, q, q') + h).$$

Setting $k = 1/(2\beta)$, $k' = 1/(2\alpha)$ yields $\max_{i=0,\dots,n} \mathbb{E}\big[|\delta Y_i|^2\big] \leq C_{q,q'} h^r$, where $r = \min(1, \frac{q+2}{2\beta} - 1, \frac{q'-2}{2\alpha} - 1)$. Since (**H**y2) $\implies$ (**H**y1), we observe that $r > 0$. $\square$

We now state the convergence results associated to the extensions of the scheme defined in Remark 2.3.

**Corollary 3.1.** *In the setting of Theorem 3.1, we have*

$$\max_{i=0,\dots,n} \mathbb{E}\big[|Y_{t_i} - \bar{Y}_{t_i}|^2\big] + \max_{i=0,\dots,n} \mathbb{E}\big[|Y_{t_i} - \tilde{Y}_{t_i}|^2\big] + \max_{i=0,\dots,n} \mathbb{E}\big[|Y_{t_i} - \breve{Y}_{t_i}|^2\big] \leq C_{q,q'} h^r.$$

*where for* $(\tilde{Y}_{t_i})_{i \leq n}$ *and* $(\breve{Y}_{t_i})_{i \leq n}$, *we set* $\eta = h^{r/q}$ *and* $\zeta = h^{-r/(q'-2)}$, *recalling Remark 2.3.*

**Proof.** 1. For all $i \leq n$, we compute, using the 1-Lipschitz continuity of $p_{\bar{D}}$,

$$\mathbb{E}\big[|Y_{t_i} - \bar{Y}_{t_i}|^2\big] = \mathbb{E}\big[|p_{\bar{D}}(Y_{t_i}) - p_{\bar{D}}(\hat{Y}_{t_i})|^2\big] \leq \mathbb{E}\big[|Y_{t_i} - \hat{Y}_{t_i}|^2\big],$$

and the upper bound in this case follows directly from Theorem 3.1.

2. For $i \leq n$, we compute

$$\mathbb{E}\big[|Y_{t_i} - \tilde{Y}_{t_i}|^2\big] \leq 2 \left( \mathbb{E}\big[|Y_{t_i} - p_{\bar{D}_\eta}(Y_{t_i})|^2\big] + \mathbb{E}\big[|p_{\bar{D}_\eta}(Y_{t_i}) - p_{\bar{D}_\eta}(\hat{Y}_{t_i})|^2\big] \right)$$

$$\leq 2 \left( \mathbb{E}\big[|Y_{t_i} - p_{\bar{D}_\eta}(Y_{t_i})|^2\big] + \mathbb{E}\big[|Y_{t_i} - \hat{Y}_{t_i}|^2\big] \right)$$

$$\leq C_{q,q'} \left( \mathbb{E}\big[|Y_{t_i} - p_{\bar{D}_\eta}(Y_{t_i})|^2\big] + h^r \right), \tag{3.9}$$

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where we applied Theorem 3.1 to get the last inequality. A straightforward adaptation of the proof of Lemma 3.1 yields the inequality $\mathbb{E}\Big[|Y_{t_i} - p_{\bar{D}_\eta}(Y_{t_i})|^2\Big] \leq C_q \eta^q$. Inserting the previous inequality back into (3.9) and setting $\eta = h^{r/q}$ concludes the result.

3. Similarly, for $i \leq n$, the equality $\mathbb{E}[|Y_{t_i} - p_{\breve{D}_\zeta}(Y_{t_i})|^2] = \mathbb{E}[|Y_{t_i} - \zeta|^2 1_{\{Y_{t_i} > \zeta\}}]$ holds, and an application of Hölder's inequality leads to $\mathbb{E}[|Y_{t_i} - p_{\breve{D}_\zeta}(Y_{t_i})|^2] \leq C_{q'} \zeta^{-(q'-2)}$. Choosing $\zeta = h^{-r/(q'-2)}$ concludes the proof. $\qquad\square$

We now show that, as for the classical Euler scheme, our modified scheme may have a first order strong rate of convergence if the diffusion coefficient is constant. This can be observed in practice, see Section 5.1. This also suggests that a similarly modified Milstein scheme, if the diffusion coefficient is not constant, will have a first order strong rate of convergence.

**Proposition 3.2.** *Assume that $\gamma(x) \equiv \gamma > 0$ for all $x \in D$, and $(\mathbf{Hy2})$ holds, with $q > 6\beta - 2$ and $q' > 6\alpha + 2$. Then, $\max_{i=0,\dots,n} \mathbb{E}\big[|\delta Y_i|^2\big] \leq C_{q,q'} h^2$.*

**Proof.** The proof is similar to step 2 in the proof of Proposition 3.1 but it uses the sharper upper bound (3.4). Since the diffusion function is constant, the identity $\sum_{i=1}^{n} \mathbb{E}[|\zeta_i^w|^2] = 0$ holds. Fix $(k, k') = (1/(2\beta), 1/(2\alpha))$ and it follows that

$$\sum_{i=1}^{n} \mathbb{E}\Big[|\zeta_i^d|^2\Big] \leq Ch\left(K_2(n, q, q') + \mathcal{R}_\pi[f(Y)]\right) \leq C_{q,q'} h^2 \, , \qquad (3.10)$$

holds. Consider the term $\mathbb{E}\Big[\big(\mathbb{E}_{t_i}[\zeta_{i+1}^d]\big)^2\Big]$. For $t \in (t_i, t_{i+1}]$, using Itô's Lemma in the first term of (3.7), $\mathbb{E}_{t_i}\Big[\int_{t_i}^{t_{i+1}} f(Y_t) - f(Y_{t_i})\mathrm{d}t\Big]$, we compute

$$\mathbb{E}_{t_i}\left[\int_{t_i}^{t_{i+1}} \left(\int_{t_i}^{t} f'(Y_u)f(Y_u) + \frac{1}{2}f''(Y_u)\gamma^2\mathrm{d}u + \int_{t_i}^{t} f'(Y_u)\gamma\mathrm{d}W_u\right)\mathrm{d}t\right]$$

$$= \mathbb{E}_{t_i}\left[\int_{t_i}^{t_{i+1}} \left(\int_{t_i}^{t} f'(Y_u)f(Y_u) + \frac{1}{2}f''(Y_u)\gamma^2\mathrm{d}u\right)\mathrm{d}t + \int_{t_i}^{t_{i+1}} (t_{i+1} - t)f'(Y_t)\gamma\mathrm{d}W_t\right] \, ,$$

hence taking the expectation of the square and (2.6) yields

$$\mathbb{E}\left[\left(\mathbb{E}_{t_i}\left[\int_{t_i}^{t_{i+1}} f(Y_t) - f(Y_{t_i})\mathrm{d}t\right]\right)^2\right] \leq Ch^4 \, .$$

From the second component of (3.7) it follows that

$$\mathbb{E}\left[\left(\mathbb{E}_{t_i}\left[\int_{t_i}^{t_{i+1}} f(Y_{t_i}) - f_n(Y_{t_i})\mathrm{d}t\right]\right)^2\right] = h^2\mathbb{E}[|f(Y_{t_i}) - f_n(Y_{t_i})|^2] \leq h^2 K_2(n, q, q') \leq C_{q,q'} h^4 \, ,$$

from Lemma 3.2, and the assumptions on $q, q'$. Dividing through by $h$ and combining with (3.10) concludes the proof. $\qquad\square$

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## 3.3  Moments properties of the schemes

For latter use, we show that our approximations have uniformly bounded second moments. This completes the result of Remark 2.4.

**Lemma 3.5.** *Assume that* (**H**$y$1) *holds, then* $\max_{i=0,\dots,n} \mathbb{E}\Big[|\hat{Y}_{t_i}|^2\Big] \leq C_{q,q'}$.

**Proof.** Since $|\hat{Y}_i|^2 \leq 2(|Y_{t_i} - \hat{Y}_{t_i}|^2 + |Y_{t_i}|^2)$, (**H**$y$1) and Theorem 3.1 imply that

$$\mathbb{E}\Big[|\hat{Y}_{t_i}|^2\Big] \leq 2\left(\mathbb{E}\Big[|Y_{t_i} - \hat{Y}_{t_i}|^2\Big] + \mathbb{E}\big[|Y_{t_i}|^2\big]\right) \leq C_{q,q'}(h^r + 1) \leq C_{q,q'}$$

holds for any $i \leq n$, which proves the claim. □

We now consider the modifications $\tilde{Y}$ and $\check{Y}$, and prove some finite moments for them.

**Proposition 3.3.**

- *If* (**H**$y$1) *holds, then* $\max_{i=0,\dots,n} \mathbb{E}(\check{Y}_{t_i}^{p'}) \leq C_{p',q'}$ *for all* $p' \in [1, q'/2]$;

- *if* (**H**$y$1) *holds with* $q \geq 4$, *then* $\max_{i=0,\dots,n} \mathbb{E}(\tilde{Y}_{t_i}^{-p}) \leq C_{p,q}$ *for all* $p \in [1, q/2 - 1]$.

**Proof.** Let $p'$ be some constant such that $p' \in (1, q'/2)$. By the Mean Value Theorem, for some $c \in [\min(Y_{t_i}^{p'}, \check{Y}_{t_i}^{p'}), \max(Y_{t_i}^{p'}, \check{Y}_{t_i}^{p'})]$ it holds that $|Y_{t_i}^{p'} - \check{Y}_{t_i}^{p'}| \leq C_{p'} c^{p'-1} |Y_{t_i} - \check{Y}_{t_i}|$, and an application of the Cauchy-Schwarz inequality yields

$$\mathbb{E}|Y_{t_i}^{p'} - \check{Y}_{t_i}^{p'}| \leq C_{p'} \sqrt{\mathbb{E}|Y_{t_i}^{2(p'-1)}| + \mathbb{E}|\check{Y}_{t_i}^{2(p'-1)}|} \sqrt{\mathbb{E}|Y_{t_i} - \check{Y}_{t_i}|^2} .$$

Since $Y_{t_i}$ has finite moments for the power of $2(p'-1)$, using the result from Corollary 3.1 and setting $\zeta = h^{-r/(q'-2)}$, it follows that

$$\mathbb{E}|Y_{t_i}^{p'} - \check{Y}_{t_i}^{p'}| \leq C_{p',q'}(1 + \zeta^{2(p'-1)})^{1/2} h^{r/2} \leq C_{p',q'}(1 + \zeta^{p'-1}) h^{r/2} \leq C_{p',q'} h^{\frac{r(q'-2p')}{2(q'-2)}} ,$$

which proves that our modification, $\check{Y}_{t_i}$, has finite moments of order $p'$.
Similar proof using $\bar{D}_\eta$, with $\eta = h^{r/q}$ and the modification $\tilde{Y}_{t_i} = p_{\bar{D}_\eta}(\hat{Y}_{t_i})$. □

**Remark 3.1.** *For SDEs defined on* $\mathbb{R}$, *strong convergence rates have been proved using a tamed explicit scheme [20]. The authors assumed that the drift satisfies* (2.2) *and* (2.3) *with locally Lipschitz exponents* $\alpha \in (0, \infty)$ *and* $\beta = 0$, *and that the diffusion is* $K$-*Lipschitz. Under these assumptions,* (2.1) *has a unique strong solution [25]. We can recover rates of convergence using our modified scheme and a slight modification to our projection* $p_n$.

# 4 Applications

We now apply our results to various stochastic differential equations widely used in the literature.

## 4.1 CIR model

We consider the Feller diffusion [9], defined as the unique strong solution to

$$\mathrm{d}X_t = \kappa(\theta - X_t)\mathrm{d}t + \xi\sqrt{X_t}\mathrm{d}W_t, \qquad X_0 = x_0 > 0, \tag{4.1}$$

where $W$ is a Brownian motion, and $\kappa$, $\theta$, $\xi$ are strictly positive constant parameters. This process has been widely used in the mathematical finance literature, both for interest rate modelling [5] and as dynamic for the instantaneous variance of a stock price process [15]. If the Feller condition $\omega =: 2\kappa\theta/\xi^2 > 1$ holds, then $X_t$ remains strictly positive for all $t \geq 0$ almost surely. Itô's Lemma implies that the Lamperti transform $Y = \sqrt{X}$ is the unique strong solution to

$$\mathrm{d}Y_t = f(Y_t)\mathrm{d}t + c\,\mathrm{d}W_t, \qquad Y_0 = \sqrt{x_0} > 0, \tag{4.2}$$

where

$$f(x) \equiv a/x + bx, \qquad a := (4\kappa\theta - \xi^2)/8, \qquad b := -\kappa/2, \qquad c := \xi/2; \tag{4.3}$$

furthermore, $a > 0$ when the Feller condition holds. Since $X = Y^2$, proving a rate of convergence for a discretisation scheme for process $Y$ is sufficient to obtain a rate of convergence for process $X$. In the following corollary, we apply Theorem 3.1 to provide bounds for $\mathbb{E}(|\delta Y_i|^2)$ and $\mathbb{E}[|\delta X_i|]$, where $\delta X_i := X_{t_i} - \hat{X}_{t_i} = Y_{t_i}^2 - \hat{Y}_{t_i}^2$.

**Corollary 4.1.** *Suppose that $\omega > 2$ holds, then $\max_{i=0,\ldots,n} \mathbb{E}[|\delta Y_i|^2] \leq C_q h^r$ and $\max_{i=0,\ldots,n} \mathbb{E}[|\delta X_i|] \leq C_q h^{r/2}$, with $r = 1 - 2/(\omega + 1) > 0$ if $2 < \omega \leq 3$ and $r = 1$ if $3 < \omega$.*

**Proof.** The drift of $Y$ is one-sided Lipschitz continuous and locally Lipschitz continuous with exponents $\alpha = 0$ and $\beta = 2$, and the diffusion is constant, hence Lipschitz continuous. It has been shown that $\sup_{t\in[0,T]} \mathbb{E}(|X_t|^p)$ is finite for all $p > -2\kappa\theta/\xi^2$; therefore $\sup_{t\in[0,T]} \mathbb{E}(|Y_t|^{-q})$ is finite for all $q < 4\kappa\theta/\xi^2 = 2\omega$ [8, page 5].
Let us now prove the first part directly from Theorem 3.1. Assume $2 < \omega \leq 3$. Fix $k = 1/(2\omega + 2)$, such that (**H**$p$) holds (no condition on $k'$ is required, since $\alpha = 0$). Using (**H**$y1$), we require that $q > 2\beta = 4$ (which is satisfied when $\omega > 2$) such that

$\max_{t \in [0,T]} \mathbb{E}(|Y_t|^{-q})$ is finite. From Theorem 3.1 it follows that $r = 1 - 2\beta/(q+2) = 1 - 4/(2\omega + 2) = 1 - 2/(\omega + 1) > 0$.

Let us now prove the second part directly from Theorem 3.1, and assume that $3 < \omega$. The drift function $f$ belongs to $\mathcal{C}^2(D)$ and differentiation yields $\mathbb{E}(|f'(Y_t)|^2) \leq C\mathbb{E}(1 + |Y_t|^{-4})$, which is finite since $\omega > 3$. Similarly, the inequalities $\mathbb{E}(|f(Y_t)f'(Y_t) + \frac{1}{2}c^2 f''(Y_t)|^2) \leq C\mathbb{E}(|Y_t|^2 + |Y_t|^{-6}) \leq C$ hold. Combining the above ensures that $(\mathbf{H}y2)$ holds. For $k = 1/4$, it follows directly that $r = \min(1, (2\omega + 2)/4 - 1) = 1$ from Theorem 3.1. We now prove the corollary for the difference $\delta X_i$. The Cauchy-Schwarz inequality and the result above imply

$$\mathbb{E}[|\delta X_n|] = \mathbb{E}\left[|(Y_{t_n} - \hat{Y}_{t_n})(Y_{t_n} + \hat{Y}_{t_n})|\right] \leq \sqrt{\mathbb{E}(|\delta Y_n|^2)\mathbb{E}\left[|Y_{t_n} + \hat{Y}_{t_n}|^2\right]}$$
$$\leq Ch^{r/2}\sqrt{\mathbb{E}(|Y_{t_n}|^2) + \mathbb{E}(|\hat{Y}_{t_n}|^2)} \leq Ch^{r/2},$$

since $\mathbb{E}(|\hat{Y}_{t_n}|^2)$ and $\mathbb{E}(|Y_{t_n}|^2)$ are finite from [17, Lemma 3.2] and Lemma 3.5. $\qquad\square$

**Remark 4.1.** *We are able to obtain a rate of convergence for a larger set of parameters compared to the results using an implicit Euler scheme in [26]. However, note that their results are stated using $L^p$-norms with $p > 2$, whereas we consider $p = 1$ throughout.*

## 4.2 Locally smooth coefficients

We now consider a stochastic differential equation of the form (2.4), with drift function $\mu(x) \equiv \mu_1(x) - \mu_2(x)x$, where $\mu_1, \mu_2 : D \to \mathbb{R}$, and diffusion function $\sigma(x) \equiv \gamma x^\nu$, with $\gamma > 0$ and $\nu \in [1/2, 1]$. This model encompasses the Feller diffusion (see Section 4.1) and the CEV model [6], both widely used in mathematical finance. For the special case $\nu = 1$, the diffusion function is $K$-Lipschitz and our scheme applies directly to the process $X$ as long as (2.2) and (2.3) hold for the drift function $\mu$.

We now focus on the case $\nu \in [1/2, 1)$. The Lamperti transform reads $F(x) \equiv \int^x \mathrm{d}y/\sigma(y) \equiv \frac{1}{\gamma(1-\nu)}x^{1-\nu}$. Its inverse $F^{-1}(y) \equiv [\gamma(1-\nu)y]^{\frac{1}{1-\nu}}$, is such that $X_t := F^{-1}(Y_t)$. The process $Y$ is the solution to $\mathrm{d}Y_t = f(Y_t)\mathrm{d}t + \mathrm{d}W_t$, with $Y_0 = F(x_0)$ and

$$f(y) \equiv \frac{\mu\left(F^{-1}(y)\right)}{\sigma\left(F^{-1}(y)\right)} - \frac{1}{2}\sigma'\left(F^{-1}(y)\right). \tag{4.4}$$

In order for the functions $\mu$ and $\sigma$ to satisfy the required conditions, we assume:

$(\mathbf{H}s0)$: $\nu \in [1/2, 1)$, and $\mu_1, \mu_2$ are bounded and belong to $\mathcal{C}_b^2(D)$; furthermore $\mu_1$ is non-negative and non-increasing, and $\mu_2$ is non-decreasing.

We distinguish between two cases for parameter $\nu$:

(**H**$s1$): $\nu \in (1/2, 1)$ and $\mu_1(0) > 0$.

(**H**$s2$): $\nu = 1/2$ and there exists $\bar{x} > 0$ such that $2\mu_1(x)/\gamma^2 \geq 1$ for all $0 < x < \bar{x}$.

We now prove a rate of convergence as a corollary of Theorem 3.1.

**Corollary 4.2** (Locally smooth coefficients). *Assume that* (**H**$s0$) *holds.*

1. *If* (**H**$s1$), *then* $\max_{i=0,\ldots,n} \mathbb{E}[|\delta Y_i|^2] \leq Ch$ *and* $\max_{i=0,\ldots,n} \mathbb{E}[|\delta X_i|] \leq Ch^{1/2}$.

2. *If* (**H**$s2$) *and* $2\mu_1(0)/\gamma^2 =: \omega > 3$ *hold, then* $\max_{i=0,\ldots,n} \mathbb{E}[|\delta Y_i|^2] \leq C_q h^r$ *and* $\max_{i=0,\ldots,n} \mathbb{E}[|\delta X_i|] \leq C_q h^{r/2}$, *with* $r = 1 - 2/\omega > 0$ *if* $3 < \omega \leq 4$ *and* $r = 1$ *if* $4 < \omega$.

**Proof.** In [7, Proposition 3.1], the author proves that if (**H**$s0$) holds, then there exists a unique strong solution to (2.4), which stays in $[0, \infty)$ almost surely. In addition, he shows that (**H**$s1$) and (**H**$s2$) further implies that $\mathbb{P}(\tau_0 = \infty) = 1$, where $\tau_0$ is the first time process $X$ reaches zero. We recall that once we perform the Lamperti transformation, the diffusion function is a constant.

We divide the proof in several parts: (i) we show that the drift function $f$ is one-sided Lipschitz continuous; (ii) we show that $f$ is locally Lipschitz continuous, and hence conclude that (2.2) and (2.3) hold.

(i) We first show that the function defined in (4.4) is globally one-sided Lipschitz continuous. From (4.4), it follows that, for all $(x, y) \in D^2$,

$$(x - y)(f(x) - f(y)) = (x - y)\left(\frac{\mu(F^{-1}(x))}{\sigma(F^{-1}(x))} - \frac{1}{2}\sigma'(F^{-1}(x)) - \frac{\mu(F^{-1}(y))}{\sigma(F^{-1}(y))} + \frac{1}{2}\sigma'(F^{-1}(y))\right).$$

Using $\sigma'(F^{-1}(x)) = \nu/[(1-\nu)x]$, we observe that

$$(x - y)\left(\frac{1}{2}\sigma'(F^{-1}(y)) - \frac{1}{2}\sigma'(F^{-1}(x))\right) = \frac{\nu}{2(1-\nu)}(x - y)\left(\frac{1}{y} - \frac{1}{x}\right) \leq 0,$$

since $x, y > 0$ and $\nu/(2-2\nu) > 0$. By direct computation $\sigma(F^{-1}(x)) = \gamma\,[\gamma(1-\nu)x]^{\frac{\nu}{1-\nu}}$ and

$$\mu(F^{-1}(x)) = \mu_1\left([\gamma(1-\nu)x]^{\frac{1}{1-\nu}}\right) - \mu_2\left([\gamma(1-\nu)x]^{\frac{1}{1-\nu}}\right)[\gamma(1-\nu)x]^{\frac{1}{1-\nu}}.$$

Now, consider the remaining terms, namely

$$(x - y)\left(\frac{\mu(F^{-1}(x))}{\sigma(F^{-1}(x))} - \frac{\mu(F^{-1}(y))}{\sigma(F^{-1}(y))}\right).$$

Introduce $\tilde{x} := [\gamma(1-\nu)x]^{\frac{1}{1-\nu}}$ and $\tilde{y} := [\gamma(1-\nu)y]^{\frac{1}{1-\nu}}$. Note that

$$(x - y)\left(\frac{\mu_1(\tilde{x})}{\sigma(F^{-1}(x))} - \frac{\mu_1(\tilde{y})}{\sigma(F^{-1}(y))}\right) =$$
$$(x - y)\mu_1(\tilde{x})\left(\frac{1}{\sigma(F^{-1}(x))} - \frac{1}{\sigma(F^{-1}(y))}\right) + \frac{(x-y)}{\sigma(F^{-1}(y))}[\mu_1(\tilde{x}) - \mu_1(\tilde{y})] \leq 0,$$

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since $\mu_1$ is non-negative and non-increasing, $\nu/(1-\nu) \geq 1$, and using the fact that the map $\sigma \circ F^{-1}$ is increasing. Additionally,

$$(x-y)\left(\frac{\mu_2(\tilde{y})\tilde{y}}{\sigma(F^{-1}(y))} - \frac{\mu_2(\tilde{x})\tilde{x}}{\sigma(F^{-1}(x))}\right)$$
$$= (1-\nu)(x-y)\mu_2(\tilde{y})(y-x) + x(x-y)[\mu_2(\tilde{y}) - \mu_2(\tilde{x})] \leq C(x-y)^2,$$

since $\sigma\left(F^{-1}(x)\right) \equiv \gamma\left[\gamma(1-\nu)x\right]^{\frac{\nu}{1-\nu}}$, and since $\mu_2$ is bounded and non-decreasing. Combining these results shows that the function $f$ is one-sided Lipschitz continuous.
(ii) We now show that $f$ is locally Lipschitz continuous. By differentiation, it is clear that $\sigma\left(F^{-1}(x)\right) = \left(F^{-1}\right)'(x)$, and hence

$$f'(x) = \mu'\left(F^{-1}(x)\right) - \frac{\mu\left(F^{-1}(x)\right)\sigma'\left(F^{-1}(x)\right)}{\sigma\left(F^{-1}(x)\right)} - \frac{1}{2}\left(F^{-1}\right)'(x)\sigma''\left(F^{-1}(x)\right). \quad (4.5)$$

We now prove that the first term on the right-hand side of (4.5) is bounded by $Cx^\alpha$, for some $\alpha$ to be determined. By $(\mathbf{H}s0)$,

$$|\mu'\left(F^{-1}(x)\right)| \leq |\mu_1'\left(F^{-1}(x)\right)| + |\mu_2\left(F^{-1}(x)\right)| + |\mu_2'\left(F^{-1}(x)\right)F^{-1}(x)| \leq C\left(1 + |x|^{1/(1-\nu)}\right),$$

hence the first term on the right-hand side of (4.5) is bounded by $C(1 + x^\alpha)$, where $\alpha = 1/(1-\nu)$.
We now consider the second term on the right-hand side of (4.5). Since $\sigma'\left(F^{-1}(x)\right) = \gamma\nu\left[\gamma(1-\nu)x\right]^{\frac{\nu-1}{1-\nu}} = \frac{\nu}{(1-\nu)x}$, and

$$\mu\left(F^{-1}(x)\right) = \mu_1\left(\left[\gamma(1-\nu)x\right]^{1/(1-\nu)}\right) - \mu_2\left(\left[\gamma(1-\nu)x\right]^{1/(1-\nu)}\right)\left[\gamma(1-\nu)x\right]^{1/(1-\nu)},$$

we see that

$$\left|\frac{\mu\left(F^{-1}(x)\right)\sigma'\left(F^{-1}(x)\right)}{\sigma\left(F^{-1}(x)\right)}\right| \leq \left|C_1\frac{\mu_1\left(C_2 x^{\frac{1}{1-\nu}}\right)}{x^{\frac{1}{1-\nu}}}\right| + \left|C_3\mu_2(C_4 x^{\frac{1}{1-\nu}})\right|, \quad (4.6)$$

where $C_1, C_2, C_3, C_4$ are positive constants. By $(\mathbf{H}s0)$ it follows that (4.6) is bounded by $C\left(1 + x^{-\beta}\right)$, for $\beta = 1/(1-\nu)$.
We finally consider the last term on the right-hand side of (4.5). Observe that

$$\sigma''\left(F^{-1}(x)\right) = \gamma\nu(\nu-1)\left[\gamma(1-\nu)x\right]^{\frac{\nu-2}{1-\nu}} = -Cx^{\frac{\nu-2}{1-\nu}}$$

and $|\frac{1}{2}\left(F^{-1}\right)'(x)\sigma''\left(F^{-1}(x)\right)| \leq C/x^2 \leq Cx^{-\beta}$, since $\nu \in [1/2, 1)$. These three results yield $|f'(x)| \leq C(1 + x^{1/(1-\nu)} + x^{-1/(1-\nu)})$, and hence the drift function is locally Lipschitz continuous, with $\alpha = \beta = 1/(1-\nu)$. Combining this with (i) allows us to conclude that (2.2) and (2.3) hold.

We now prove statements 1 and 2 in the corollary.

1) Assume $(\mathbf{H}s1)$ holds. Since the locally Lipschitz exponents are $\alpha = \beta = 1/(1-\nu)$, fix $k = k' = (1-\nu)/2$, so that $(\mathbf{H}p)$ holds. By [7], $\mathbb{E}(\sup_{t\in[0,T]}|X_t^p|)$ is finite for all $p > 0$, and $\mathbb{E}(\sup_{t\in[0,T]}|X_t|^{-p})$ is finite for all $p > 0$; therefore $\mathbb{E}(\sup_{t\in[0,T]}|Y_t|^{-q})$ is finite for all $q > 0$ [7, Lemma 3.1]. We note that $f$ belongs to the class $\mathcal{C}^2(D)$ and $(\mathbf{H}y2)$ holds, therefore $r = 1$.

2) Assume $(\mathbf{H}s2)$ holds and let $2\mu_1(0)/\gamma^2 =: \omega > 3$. Then, $\max_{t\in[0,T]}\mathbb{E}(|X_t|^{-p})$ is finite for all $p < \omega - 1$ [7, Lemma 3.1]: therefore the term $\max_{t\in[0,T]}\mathbb{E}(|Y_t|^{-q})$ is finite for all $q < 2(\omega - 1)$. Fix $k = 1/(q+2)$, so that $(\mathbf{H}p)$ holds. For some $q > 4$ we have that $\max_{t\in[0,T]}\mathbb{E}(|Y_t|^{-q})$ is finite, so $(\mathbf{H}y1)$ also holds. From Theorem 3.1, $r = 1 - 2\beta/(q+2) = 1 - 2/\omega$ holds.

Further assume that $\omega > 4$. Note that the drift function $f$ belongs to the class $\mathcal{C}^2(D)$. Fix $k = 1/4$, so that $(\mathbf{H}p)$ holds. By the assumptions on the parameters it follows that the term $\max_{t\in[0,T]}\mathbb{E}(|Y_t|^{-6}) = \max_{t\in[0,T]}\mathbb{E}(|X_t|^{-3})$ is finite, therefore $(\mathbf{H}y2)$ holds. From Theorem 3.1, $r = \min(1, (q+2)/4 - 1) = \min(1, (2\omega - 2 + 2)/4 - 1) = 1$.

$\square$

In the CIR model, we obtain $r = 1$ for $3 < \omega$ using the finite inverse moment for the process $Y$ from [8]. For the general case in Corollary 4.2, we assumed that $4 < \omega$. In the next corollary, we impose additional assumptions in order to recover the same parameter constraints as for the Feller diffusion in the previous section:

**Corollary 4.3.** *Assume* $(\mathbf{H}s0)$ *and* $(\mathbf{H}s2)$, *and let* $a^*, b^* > 0$ *be such that* $\mu_1(x) \geq a^*$ *and* $\mu_2(x) \leq b^*$ *for all* $x \in D$. *If* $3 < \omega := 2\mu_1(0)/\gamma^2$, *then* $\max_{i=0,\ldots,n}\mathbb{E}[|\delta Y_i|^2] \leq C_q h$ *and* $\max_{i=0,\ldots,n}\mathbb{E}[|\delta X_i|] \leq C_q h^{1/2}$.

**Proof.** From the assumptions on $\mu_1$ and $\mu_2$, there exists $a^*, b^*$ strictly positive such that the inequality $\mu_1(x) - \mu_2(x)x \geq a^* - b^*x$ holds in the domain $D = (0, \infty)$. We define $Z$ as the process with drift function $a^* - b^*x$ (instead of $\mu_1(x) - \mu_2(x)x$), and diffusion function $\sigma(x) \equiv \gamma x^{1/2}$. Therefore, by the Comparison Theorem (see [22]) the inequality $X_t \geq Z_t$ holds for all $t \in [0,T]$ almost surely, and hence $\mathbb{E}(|X_t|^{-p}) \leq \mathbb{E}(|Z_t|^{-p})$ holds for all $p > 0$. We recognise process $Z$ as the Feller diffusion: from the assumption on $\omega$, it follows that $\max_{t\in[0,T]}\mathbb{E}(|Z_t|^{-3})$ is finite. The result then follows directly from the second part of Corollary 4.1. $\square$

### 4.3 3/2 model

The 3/2 process $X = (X_t)_{t \geq 0}$ [16] is the solution to

$$\mathrm{d}X_t = c_1 X_t(c_2 - X_t)\mathrm{d}t + c_3 X_t^{3/2}\mathrm{d}W_t , \quad X_0 = x_0 > 0 , \quad (4.7)$$

with $c_1, c_2, c_3 > 0$. Introduce the quantity $\omega := 2 + 2c_1/c_3^2$. The Feller diffusion and the 3/2 process are related as follows: using the map $F(y) \equiv y^{-1/2}$ yields the Lamperti transformed CIR process $Y := F(X)$, as in (4.2) and (4.3), with parameters, $a := (4c_1 + 3c_3^2)/8$, $b := -c_1 c_2/2$ and $c := -c_3/2$. Existence and uniqueness can be retrieved from the properties of the Feller diffusion. Furthermore, $\max_{t \in [0,T]} \mathbb{E}(|X_t|^p)$ is finite for all $p < \omega$.

**Corollary 4.4** (3/2 model). *Let* $Y := X^{-1/2}$. *Then,* $\max_{i=0,\dots,n} \mathbb{E}\big[|\delta Y_i|^2\big] \leq Ch^r$, *with* $r = (\omega - 1)/(\omega + 1) > 0$ *if* $\omega \in (2, 3]$, *and* $r = 1$ *if* $\omega > 3$.

**Proof.** In terms of the CIR coefficients, we have $\omega = 2 + 2c_1/c_3^2 = 2\kappa\theta/\xi^2$. We directly apply Corollary 4.1 to note that for $2 < \omega \leq 3$, it follows that $r = (\omega - 1)/(\omega + 1) > 0$. For $r = 1$, we require that $\omega > 3$ and an application of Corollary 4.1. □

We now establish a convergence result for the 3/2 process $X$, using the modification $\tilde{X}$ (recall Remark 2.3).

**Corollary 4.5.** *If* $\omega > 3$, *then* $\max_{i=0,\dots,n} \mathbb{E}(|X_{t_i} - \tilde{X}_{t_i}|) \leq C_q h^{(\omega-3)/(2\omega)}$.

**Proof.** Recall that $\mathbb{E}(|Y_{t_i}|^{-6})$ is finite for $\omega > 3$ and from Corollary 4.4 we recall that $r = 1$. Using a similar approach to Proposition 3.3, $\mathbb{E}(|\tilde{Y}_{t_i}|^{-6}) \leq \eta^{-6}$, and the result from Corollary 3.1 for $\eta := h^{1/q}$ yield

$$\mathbb{E}(|X_{t_i} - \tilde{X}_{t_i}|) \leq C_q(1 + \eta^{-6})^{1/2}h^{1/2} \leq C_q(1 + \eta^{-3})h^{1/2} \leq C_q h^{(\omega-3)/(2\omega)} ,$$

which concludes the result. □

By imposing additional assumptions, one can obtain $\max_{i=0,\dots,n} \mathbb{E}(|X_{t_i} - \tilde{X}_{t_i}|) \leq C_q h^{1/2}$ using Proposition 3.3. In [26, Proposition 3.2] the authors prove strong convergence for the 3/2 process using a drift-implicit scheme when $\omega > 6$ holds. Using our scheme, we obtain strong rates of convergence for $\omega > 3$.

### 4.4 Ait-Sahalia model

In the Ait-Sahalia interest rate model [1], $X$ is the solution to

$$dX_t = \left(\frac{a_{-1}}{X_t} - a_0 + a_1 X_t - a_2 X_t^\varrho\right)\mathrm{d}t + \gamma X_t^\rho \mathrm{d}W_t , \quad X_0 = x_0 > 0 , \quad (4.8)$$

where all constant parameters are non-negative, and $\rho, \varrho > 1$. From [28], we know that there exists a strong solution on $(0, \infty)$. Using the Lamperti transformation $F(x) \equiv x^{1-\rho}$, $x > 0$, we define the process $Y$, which satisfies

$$\mathrm{d}Y_t = f(Y_t)\mathrm{d}t + (1 - \rho)\gamma\mathrm{d}W_t \,, \quad Y_0 = x_0^{1-\rho} > 0 \,, \tag{4.9}$$

where

$$f(x) \equiv (1 - \rho)\left(a_{-1}x^{\frac{-1-\rho}{1-\rho}} - a_0 x^{\frac{-\rho}{1-\rho}} + a_1 x - a_2 x^{\frac{-\rho+\varrho}{1-\rho}} - \frac{\rho\gamma^2}{2}x^{-1}\right) \,.$$

**Corollary 4.6.** *If $\varrho + 1 > 2\rho$, then $\displaystyle\max_{i=0,\dots,n} \mathbb{E}\big[|\delta Y_i|^2\big] \leq Ch$.*

**Proof.** Straightforward differentiation yields

$$f'(x) = -a_{-1}(1 + \rho)x^{\frac{2}{\rho-1}} + a_0\rho x^{\frac{1}{\rho-1}} + a_1(1 - \rho) - a_2(-\rho + \varrho)x^{-\frac{r-1}{\rho-1}} - \frac{\rho\gamma^2}{2}(\rho - 1)x^{-2}.$$

We have $\lim_{x\to 0} f'(x) = \lim_{x\to\infty} f'(x) = -\infty$, hence $\sup_{0<x<\infty} f'(x)$ is finite by continuity and therefore $f$ is one-sided Lipschitz continuous. In addition, $|f'(x)| \leq C(1 + x^{\frac{2}{\rho-1}} + x^{-\frac{\varrho-1}{\rho-1}})$ for $x > 0$, so $f$ is locally Lipschitz continuous with $\alpha = 2/(\rho - 1)$ and $\beta = (\varrho - 1)/(\rho - 1)$. The diffusion is a constant, hence Lipschitz continuous. Using the locally Lipschitz continuous properties of the drift, fix $k = 1/(2\beta)$ and $k' = 1/(2\alpha)$. We recall that if $\varrho + 1 > 2\rho$, then $\max_{t\in[0,T]} \mathbb{E}(|X_t|^p)$ and $\max_{t\in[0,T]} \mathbb{E}(|X_t|^{-p})$ are finite for all $p \geq 2$ [28, Lemma 2.3] so that (**H**$y$1) holds. Differentiation yields

$$f''(x) = \frac{-2a_{-1}(\rho + 1)}{\rho - 1}x^{\frac{3-\rho}{\rho-1}} + \frac{a_0\rho}{\rho - 1}x^{\frac{2-\rho}{\rho-1}} + a_2\frac{(-\rho + \varrho)(\varrho - 1)}{\rho - 1}x^{-\frac{\varrho+\rho-2}{\rho-1}} + \rho\gamma^2(\rho - 1)x^{-3} \,.$$

Since $f$ belongs to $\mathcal{C}^2(D)$ and (2.6) is finite by [28, Lemma 2.3], then (**H**$y$2) holds. The result follows from Theorem 3.1. □

We now compute a strong rate of convergence for the Ait-Sahalia process $X$, and recall the modification $\tilde{X}_{t_i} = p_{\bar{D}_\eta}(\hat{X}_{t_i})$.

**Corollary 4.7.** *Let $\varrho + 1 > 2\rho$, then $\max_{i=0,\dots,n} \mathbb{E}(|X_{t_i} - \tilde{X}_{t_i}|) \leq Ch^{\frac{1}{2} - \frac{\rho}{q(\rho-1)}}$, where $\eta := h^{1/q}$ for some $q > 2\rho/(\rho - 1)$.*

**Proof.** Using a similar approach to Proposition 3.3 yields

$$\mathbb{E}[|X_{t_i} - \tilde{X}_{t_i}|] \leq C\left(\mathbb{E}\left[|Y_{t_i}|^{2\rho/(1-\rho)}\right] + \mathbb{E}\left[|\tilde{Y}_{t_i}|^{2\rho/(1-\rho)}\right]\right)^{1/2}(\mathbb{E}|Y_{t_i} - \tilde{Y}_{t_i}|^2)^{1/2}.$$

Since $\rho > 1$ and $\varrho + 1 > 2\rho$, $\mathbb{E}[|Y_{t_i}|^{2\rho/(1-\rho)}]$ is finite and the result follows. □

# 5    Numerical results

In this section, numerical simulations demonstrate the strong convergence rate of the modified Euler scheme. The CIR model, the one-dimensional stochastic Ginzburg-Landau equation with multiplicative noise, and an example of the Ait-Sahalia model are all considered. For process $X$, denote by $\hat{X}_T^{(j)}$ the modified Euler-Maruyama approximation at time $T$ and $X_T^{(j)}$ the closed-form solution (or reference solution), using the same Brownian motion path (the $j^{\text{th}}$ path). The empirical average absolute error $\mathcal{E}$ for the process $X$ is defined by

$$\mathcal{E} := \frac{1}{M} \sum_{j=1}^{M} |X_T^{(j)} - \hat{X}_T^{(j)}|,$$

over $M$ sample paths. Throughout, fix $M = 10000$. The error quantity $\mathcal{E}$ measures the error at time $T$, as typically that will yield the largest value throughout the path. An equidistant time grid is used, with step sizes $h := T/2^N$, for different values of $N$. The strong error rates are computed by plotting $\mathcal{E}$ against the number of discretisation steps on a log-log scale: the strong rate of convergence $r$ is then retrieved using linear regression.

## 5.1    CIR model

The Lamperti-transformed drift-implicit square-root Euler method (see [8, 26]) has the unique strictly positive solution defined for $i = 0, \ldots, n-1$ by

$$Y_{t_{i+1}} = \frac{Y_{t_i} + c\Delta W_{i+1}}{2(1 - bh_{i+1})} + \sqrt{\frac{(Y_{t_i} + c\Delta W_{i+1})^2}{4(1 - bh_{i+1})^2} + \frac{ah_{i+1}}{1 - bh_{i+1}}}, \qquad Y_0 = \sqrt{x_0} > 0,$$

with $a, b, c$ defined in (4.3). The CIR/Feller diffusion is recovered by setting $X_{t_i} = Y_{t_i}^2$ for $i \leq n$, and we compare the modified explicit Euler scheme with this implicit scheme used as a reference solution (with a large number of time steps).

We compute the strong rates of convergence for the CIR process, where the implicit scheme is used as a reference solution. Set $(\kappa, \theta, \xi, T, x_0) = (0.125\omega, 1, 0.5, 1, 1)$, such that $2\kappa\theta/\xi^2 = \omega$. The cases $\omega = (1, 1.5, 2, 2.5, 3, 3.5, 4)$ are considered. The reference solution is computed using $N = 12$. Figure 1 shows the rates of convergence $r$ achieved for the CIR process, where $k = 1/4$ in the modified scheme, according to Corollary 4.1. In the corollary, we prove a strong rate of convergence of $1/2$ when $\omega > 3$. The coefficient of determination $R^2$ is above 0.998 for all $\omega$.
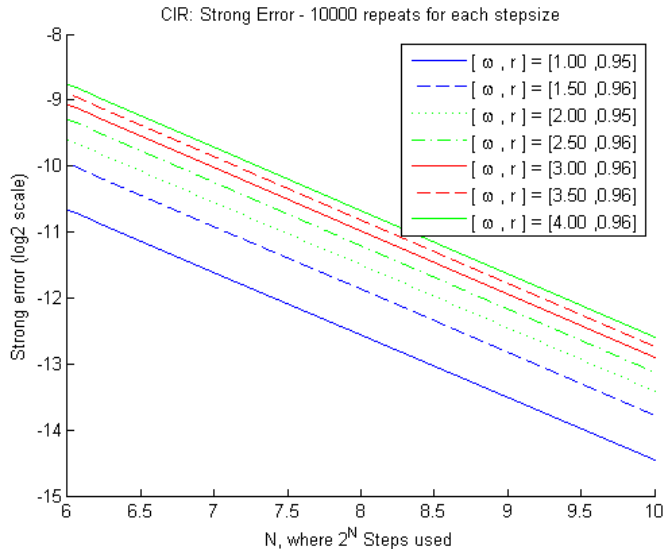
Figure 1: CIR model: $\mathcal{E}$ against number of steps ($\log_2$ scale).

**Remark 5.1.** *The empirical rates of convergence achieved are higher than the predicted rates from Corollary 4.1. In fact, $r$ is approximately 1, due to the constant diffusion function in the transformed Feller diffusion (see Proposition 3.2). The "classical" Euler scheme is a strong order 1 scheme in this case.*

**Remark 5.2.** *The projection introduced in Definition 2.1 can be modified to $\tilde{p}_n(x) := Ln^{-k} \vee x \wedge Un^{k'}$, with $L, U > 0$ suitably chosen constant. This is beneficial if the process has extreme initial conditions or average state, and does not impact the convergence results.*

For small $x_0$, it is intuitive to use the projection in Remark 5.2 to achieve faster convergence (albeit not affect the asymptotic behaviour). Set $(\kappa, \theta, \xi, T) = (0.375, 1, 0.5, 1)$, such that $2\kappa\theta/\xi^2 = 3$. In Figure 2, we let $x_0$ vary between 0.05 and 1.2 in increments of 0.05. We compare the errors achieved for $k = 1/4$, using the projections $p_n$ ($L = U = 1$) and $\tilde{p}_n$ ($L = \sqrt{x_0}$ and $U = 1$). By using projection $\tilde{p}_n$, smaller errors can be achieved for small $x_0$.

## 5.2 Ginzburg-Landau

Consider the one-dimensional stochastic Ginzburg-Landau SDE [24, Chapter 4], where the process $X$ is the unique strong solution to

$$\mathrm{d}X_t = \left[ -X_t^3 + \left( \lambda + \frac{1}{2}\sigma^2 \right) X_t \right] \mathrm{d}t + \sigma X_t \mathrm{d}W_t, \quad X_0 = x_0 > 0 \,,$$
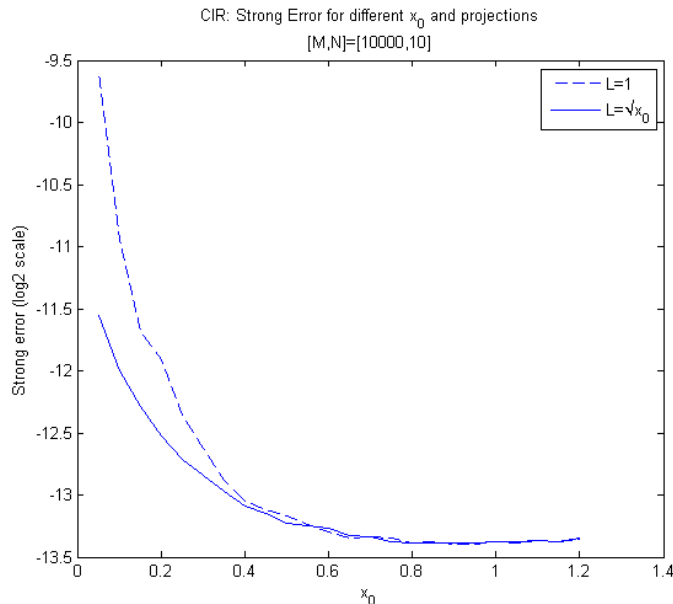
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Figure 2: Absolute error ($\log_2$ scale) for $N = 10$.

for $\lambda, \sigma \geq 0$, which admits the closed-form solution

$$X_t = \frac{x_0 \exp(\lambda t + \sigma W_t)}{\sqrt{1 + x_0^2 \int_0^t \exp(2\lambda s + 2\sigma W_s) \mathrm{d}s}} \ .$$

This SDE is a special case of the Ait-Sahalia process with $(a_{-1}, a_0, a_1, a_2, \varrho, \rho) = (0, 0, \lambda + 1/2\sigma^2, 1, 3, 1)$. For this choice of parameters, $\varrho + 1 > 2\rho$ holds and hence the moments and inverse moments of $X_t$ are finite for all $t \in [0, T]$, and the solution stays in $(0, \infty)$ almost surely. The drift function satisfies (2.2), with $(\alpha, \beta) = (2, 0)$, e.g. set $k' = 1/4$ in the modified scheme. In addition, the drift is one-sided Lipschitz continuous and the diffusion is $K$-Lipschitz. As a result, theoretical convergence for this example can be obtained with rate $r = 1/2$.

### 5.2.1    Ginzburg-Landau strong convergence

For this SDE, the closed-form solution is used in the definition of $\mathcal{E}$ to compute the strong rate of convergence $r$. Figure 3 shows the average absolute error $\mathcal{E}$ using the modified scheme, for parameters $(\sigma, \lambda, T, x_0) = (1, 1/2, 1, 1)$. The empirical rate achieved of 0.54 coincides with the predicted rate of $1/2$.
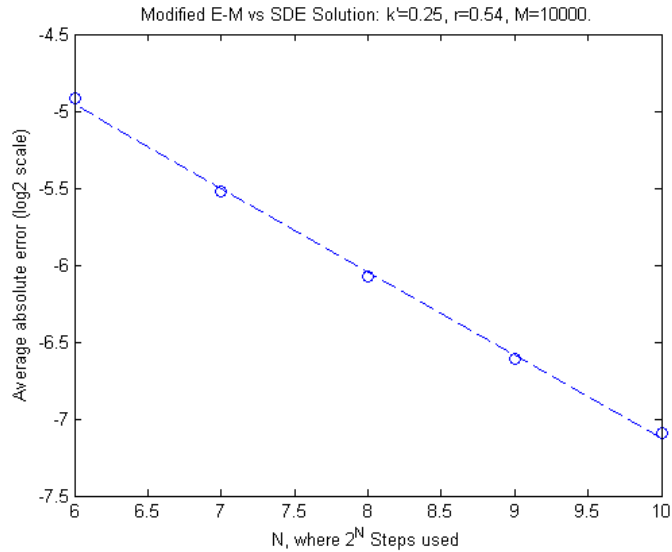
Figure 3: Ginzburg-Landau model: average absolute error $\mathcal{E}$ vs $N$ ($\log_2$ scale).

### 5.2.2 Ginzburg-Landau Euler-Maruyama divergence

We consider an example of the Ginzburg-Landau SDE for which the standard Euler-Maruyama scheme diverges, and compare the results with the modified explicit scheme. Fix parameters $(\sigma, \lambda, T, x_0) = (7, 0, 3, 1)$ as in [19], for which the authors prove moment explosion for the classical Euler-Maruyama scheme, see [19, Table 1]. Figure 4 shows the error $\mathcal{E}$ for the classical and the modified schemes, for different $N$. For the modified scheme, set $k = 1$ and $k' = 1/4$. It can be seen that both schemes eventually converge, with rates $(r_e, r_m) = (0.84, 0.83)$ for the classical and modified Euler schemes. However, for a range of step sizes, the classical Euler scheme explodes, as proven in [19] (N.B. very large and $NaN$ values are set to $2^{10}$ in the figure, to illustrate the explosions for the classical scheme).

### 5.3 Ait-Sahalia model

The strong rate of convergence for the Ait-Sahalia model is computed using a reference solution with a large number of steps. Consider the parameters $(a_{-1}, a_0, a_1, a_2, \gamma, x_0) = (1, 1, 1, 1, 1, 1)$, and $(\varrho, \rho, T) = (2, 3/2, 1)$. From these parameters, note that $\alpha = 4$ and $\beta = 2$. Fix $k$ and $k'$, such that $2\beta k = 1$ and $2\alpha k' = 1$: therefore ($\mathbf{H}y1$) holds. Figure 5 shows $\mathcal{E}$ against the number of steps (log-log plot), where $2^{12}$ steps are used for the reference solution. The Ait-Sahalia strong rate of convergence $r = 1.25$ could be
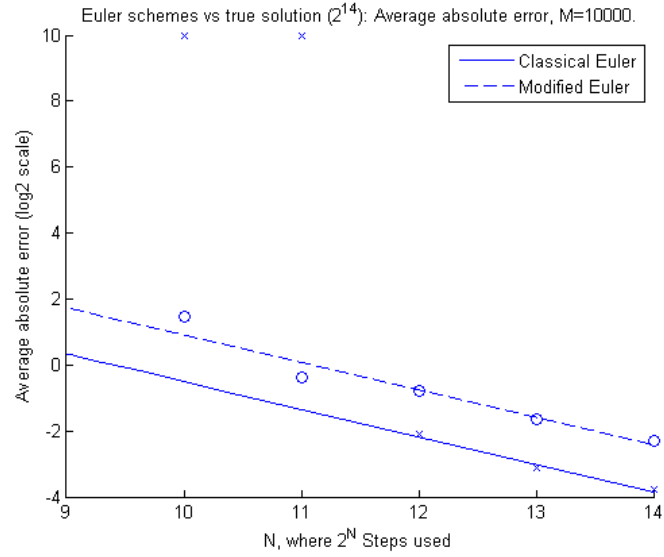
23

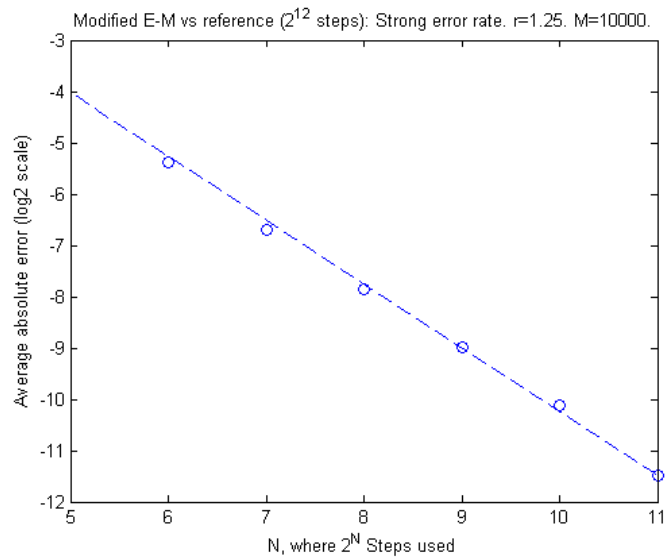Figure 4: Average absolute error $\mathcal{E}$ vs number of steps ($\log_2$ scale).



Figure 5: Ait-Sahalia model: average absolute error vs $N$ ($\log_2$ scale).

justified by recalling Remark 5.1, and since a reference solution is used, as opposed to the true solution.

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# A  Proof of Lemma 2.1

1. Let $r > l > 0$ such that $D_n \subset (l, r)$. Assume that $f$ is $C^1$ on $(l, r)$. From (2.2), we have, for $z, z' \in D_n$, $z > z'$,

$$\frac{f(z) - f(z')}{z - z'} \le K,$$

and letting $z' \to z$, we retrieve that $f'(z) \le K$. This shows that $f = g + \ell$, where $g$ is a non-increasing function and $\ell$ is $K$-Lipschitz continuous, setting e.g. $g(x) \equiv \int_{\frac{l+r}{2}}^{x} f'(u)\mathbf{1}_{\{f'(u) \le 0\}}\mathrm{d}u$ and $\ell(x) \equiv \int_{\frac{l+r}{2}}^{x} f'(u)\mathbf{1}_{\{f'(u)>0\}}\mathrm{d}u$. Since $p_n$ is non-decreasing and 1-Lipschitz on $\mathbb{R}$, we have $f_n = g \circ p_n + \ell \circ p_n$, with $g \circ p_n$ non-increasing and $\ell \circ p_n$ $K$-Lipschitz continuous on $\mathbb{R}$. This shows that $f_n$ satisfies (2.3) as well on $\mathbb{R}$.

2. We now deal with the general case using a smoothing argument. Let $l, r \in D$, $r > l$, such that for all $D_n \subset (l, r)$. We consider a sequence $(\varphi_m)_{m \ge 1}$ of mollifiers whose supports are included in $[-\frac{l}{2}, \frac{l}{2}]$ and define $f^m \equiv \varphi_m \star f \equiv \int_{[-\frac{l}{2}, \frac{l}{2}]} \varphi_m(u) f(x - u)\mathrm{d}u$ as the convolution of $\varphi_m$ and $f$. We observe that, for all $x, y \in (l, r)$,

$$(x - y)(f^m(x) - f^m(y)) = \int_{[-\frac{l}{2}, \frac{l}{2}]} \varphi_m(u)\{(x - y)(f(x - u) - f(y - u))\}\mathrm{d}u$$

$$\le K|x - y|^2 \int_{[-\frac{l}{2}, \frac{l}{2}]} \varphi_m(u)\mathrm{d}u \ \le K|x - y|^2,$$

where we used (2.3) and the fact that $\int_D \varphi_m(u)\mathrm{d}u = 1$. Since $f^m$ is smooth, we can apply Step 1 to obtain, for all $(x, y) \in \mathbb{R}^2$,

$$(x - y)\left(f^m(p_n(x)) - f^m(p_n(y))\right) \le K|x - y|^2.$$

Letting $m$ go to infinity, we then obtain

$$(x - y)\left(f(p_n(x)) - f(p_n(y))\right) \le K|x - y|^2,$$

for all $x, y \in \mathbb{R}$, which concludes the proof.