Modelling electricity forward markets by ambit fields

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Abstract

This paper proposes a new modelling framework for electricity forward markets based on so–called *ambit fields*. The new model can capture many of the stylised facts observed in energy markets and is highly analytically tractable. We give a detailed account on the probabilistic properties of the new type of model, and we discuss martingale conditions, option pricing and change of measure within the new model class. Also, we derive a model for the typically stationary spot price, which is obtained from the forward model through a limiting argument.

Keywords: Electricity markets; forward prices; random fields; ambit fields; Levy basis; Samuelson effect; stochastic volatility.

MSC codes: 60G10, 60G51, 60G55, 60G57, 60G60, 91G99.

1 Introduction

This paper introduces a new type of model for electricity forward prices, which is based on *ambit fields* and *ambit processes*. Ambit stochastics constitutes a general probabilistic framework which is suitable for tempo–spatial modelling. Ambit processes are defined as stochastic integrals with respect to a multivariate random measure, where the integrand is given by a product of a deterministic kernel function and a stochastic volatility field and the integration is carried out over an *ambit set* describing the sphere of influence for the stochastic field.

Due to their very flexible structure, ambit processes have successfully been used for modelling turbulence in physics and cell growth in biology, see Barndorff-Nielsen & Schmiegel (2004, 2007, 2008*a*,*b*,*c*, 2009), Vedel Jensen et al. (2006). The aim of this paper is now to develop a new modelling framework for (electricity) forward markets based on the ambit concept.

Over the past two decades, the markets for power have been liberalised in many areas in the world. The typical electricity market, like for instance the Nordic Nord Pool market or the German EEX market, organises trade in spot, forward/futures contracts and European options on these. Although these assets are parallel to other markets, like traditional commodities or stock markets, electricity

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has its own distinctive features calling for new and more sophisticated stochastic models for risk management purposes, see Benth, Šaltytė Benth & Koekebakker (2008).

The electricity spot cannot be stored directly except via reservoirs for hydro–generated power, or large and expensive batteries. This makes the supply of power very inelastic, and prices may rise by several magnitudes when demand increases, due to temperature drops, say. Since spot prices are determined by supply and demand, some form of mean–reversion or stationarity can be observed. The spot prices have clear deterministic patterns over the year, week and intra–day. The literature has focused on stochastic models for the spot price dynamics, which take some of the various stylised facts into account. Recently, a very general, yet analytically tractable class of models has been proposed in Barndorff-Nielsen et al. (2010), based on Lévy semistationary processes, which are special cases of ambit processes.

One of the fundamental problems in power market modelling is to understand the formation of forward prices. Non–storability of the spot makes the usual buy–and–hold hedging arguments break down, and the notion of convenience yield is not relevant either. There is thus a highly complex relationship between spot and forwards.

A way around this would be to follow the so–called Heath–Jarrow–Morton approach, which has been introduced in the context of modelling interest rates, see Heath et al. (1992), and model the forward price dynamics directly (rather than modelling the spot price and deducing the forward price from the conditional expectation of the spot at delivery). There are many challenging problems connected to this way of modelling forward prices.

Firstly, standard models for the forward dynamics generally depend on the current time and the time to maturity. However, power market trades in contracts which deliver power over a *delivery period*, introducing a new dimension in the modelling. Hence comprehensive forward price models should be functions of both *time to* and *length of* delivery, which calls for random field models in time and space. Furthermore, since the market trades in contracts with overlapping delivery periods, specific no–arbitrage conditions must be satisfied which essentially puts restrictions on the space structure of the field. So far, the literature is not very rich on modelling power forward prices applying the Heath–Jarrow–Morton approach, presumably due to the lack of analytical tractability and empirical knowledge of the price evolution.

Empirical studies, see Frestad et al. (2010), have shown that the logarithmic returns of forward prices are non–normally distributed, with clear signs of (semi-) heavy tails. Also, a principal component analysis by Koekebakker & Ollmar (2005) indicates a high degree of idiosyncratic risk in power forward markets. This strongly points towards random field models which, in addition, allow for stochastic volatility. Moreover, the structure determining the interdependencies between different contracts is by far not properly understood. Some empirical studies, see Andresen et al. (2010), suggest that the correlations between contracts are decreasing with time to maturity, whereas the exact form of this decay is not known. But how to take 'length of delivery' into account in modelling these interdependencies has been an open question. A first approach on how to tackle these problems will be presented later in this paper.

Ambit processes provide a flexible class of random field models, where one has a high degree of flexibility in modelling complex dependencies. These may be probabilistic coming from a driving Levy basis and the stochastic volatility, or functional from a specification of an ambit set or the deterministic kernel function.

Our focus will be on ambit processes which are *stationary* in time. As such, our modelling framework differs from the traditional models, where stationary processes are (if at all) reached by limiting arguments. Modelling directly in stationarity seems in fact to be quite natural in various applications and is e.g. done in physics in the context of modelling turbulence, see e.g. Barndorff-Nielsen & Schmiegel (2007, 2009). Here we show that such an approach has strong potential in finance, too, when we are concerned with modelling commodity markets. In particular, we will argue that energy spot prices are typically well–described by stationary processes, see e.g. Barndorff-Nielsen et al.

(2010) for a detailed discussion on that aspect, and in order to achieve stationarity in the spot price it makes sense to model the corresponding forward price also in stationarity. The precise relation between the spot and the forward price will be established later in the paper.

Due to their general structure, ambit processes easily incorporate leptokurtic behaviour in returns, stochastic volatility and leverage effects and the observed Samuelson effect in the volatility. Note that the *Samuelson effect*, see Samuelson (1965), refers to the finding that, when the time to maturity approaches zero, the volatility of the forward increases and converges to the volatility of the underlying spot price (provided the forward price converges to the spot price).

Although many stylised facts of energy markets can easily be incorporated in an ambit framework, one may question whether ambit processes are not in fact *too* general to be a good building block for financial models. In particular, one property — the martingale property — is often violated by general ambit processes. However, we can and will formulate conditions which ensure that an ambit process is in fact a martingale. So, if we wish to stay within the martingale framework, we can do so by using a restricted subclass of ambit processes. On the other hand, in modelling terms, it is actually not so obvious whether we *should* stay within the martingale framework if our aim is to model electricity forward contracts. Given the illiquidity of electricity markets, it cannot be taken for granted that arbitrage opportunities arising from forward prices outside the martingale framework can be exercised. Also, we know from recent results in the mathematical finance literature, see e.g. Guasoni et al. (2008), Pakkanen (2011), that subclasses of non–(semi)–martingales can be used to model financial assets without necessarily giving rise to arbitrage opportunities in markets which exhibit market frictions, such as e.g. transaction costs.

Next, we will not work with the most general class of ambit processes since we are mainly interested in the time–stationary case as mentioned before.

Last but not least we will show that the ambit framework can shed some light on the connection between electricity spot and forward prices. Understanding the interdependencies between these two assets is crucial in many applications, e.g. in the hedging of exotic derivatives on the spot using forwards. A typical example in electricity markets is so–called user–time contracts, giving the holder the right to buy spot at a given price on a predefined number of hours in a year, say.

The outline for the remaining part of the paper is as follows. Section 2 gives an overview of the standard models used for forward markets. Section 3 reviews basic traits of the theory of ambit fields and processes. In Section 4, we introduce the new modelling framework for electricity forward markets, study its key properties and highlight the most relevant model specifications. In Section 5, we show how some of the traditional models for forward prices relate to ambit processes. Section 6 presents the martingale conditions for our new model and discusses option pricing. Moreover, since we do the modelling under the risk neutral measure, we discuss how a change of measure can be carried out to get back to the physical probability measure, see Section 7. Next we show what kind of spot model is implied by our new model for the forward price, and we discuss that, under certain conditions, the implied spot price process equals in law a Lévy semistationary process, see Section 8. In order to get also a visual impression of the new models for the term structure of forward prices, we present a simulation algorithm for ambit fields in Section 9 and highlight the main theoretical properties of the modelling framework graphically. Section 10 deals with extensions of our new modelling framework: While we mainly focus on arithmetic models for forward prices in this paper, we discuss briefly how geometric models can be constructed. Also, we give an outlook on how ambit field based models can be used to jointly model time and period of delivery. Finally, Section 11 concludes and Appendix A contains the proofs of our main results and some technical results on the correlation structure of the new class of models and extensions to the multivariate framework.

2 Overview on approaches to modelling forward prices

Before introducing ambit fields, let us review the exisiting literature on direct modelling of forward prices in commodity markets, i.e. the approach where one is not starting out with a specification of the underlying spot dynamics.

Although commodity markets have very distinct features, most models for energy forward contracts have been inspired by instantaneous forward rates models in the theory for the term structure of interest rates, see Koekebakker & Ollmar (2005) for an overview on the similarities between electricity forward markets and interest rates.

Hence, in order to get an overview on modelling concepts which have been developed in the context of the term structure of interest rates, but which can also be used in the context of electricity markets, we will now review these examples from the interest rate literature. However, later we will argue that, in order to account for the particular stylised facts of power markets, there is a case for leaving these models behind and focusing instead on ambit fields as a natural class for describing energy forward markets.

Throughout the paper, we denote by $t \in \mathbb{R}$ the current time, by $T \geq 0$ the time *of* maturity of a given forward contract, and by $x = T - t$ the corresponding time *to* maturity. We use $F_t(T)$ to denote the price of a forward contract at time t with time of maturity T. Likewise, we use f for the forward price at time t with time to maturity $x = T - t$, when we work with the Musiela parameterisation, i.e. we define f by

$$f_t(x) = f_t(T - t) = F_t(T).$$

2.1 Multi–factor models

Motivated by the classical Heath et al. (1992) framework, the dynamics of the forward rate under the risk neutral measure can be modelled by

$$df_t(x) = \sum_{i=1}^{n} \sigma_t^{(i)}(x) dW_t^{(i)}, \qquad \text{for } t \geq 0,$$

for $n \in I\!N$ and where $W^{(i)}$ are independent standard Brownian motions and $\sigma^{(i)}(x)$ are independent positive stochastic volatility processes for $i = 1, \ldots, n$. The advantage of using these multi–factor models is that they are to a high degree analytically tractable. Extensions to allow for jumps in such models have also been studied in detail in the literature. However, a principal component analysis by Koekebakker & Ollmar (2005) has indicated that we need in fact many factors (large n) to model electricity forward prices. Hence it is natural to study extensions to infinite factor models which are also called random field models.

2.2 Random field models for the dynamics of forward rates

In order to overcome the shortcomings of the multifactor models, Kennedy (1994) has pioneered the approach of using random field models, in some cases called stochastic string models, for modelling the term structure of interest rates. Random field models have a continuum of state variables (in our case forward prices for all maturities) and, hence, are also called infinite factor models, but they are typically very parsimonious in the sense that they do not require many parameters. Note that finite–factor models can be accommodated by random field models as degenerate cases.

Kennedy (1994) proposed to model the forward rate by a centered, continuous Gaussian random field plus a continuous deterministic drift. Furthermore he specified a certain structure of the covariance function of the random field which ensured that it had independent increments in the time direction t (but not necessarily in the time to maturity direction x). Such models include as special cases the classical Heath et al. (1992) model when both the drift and the volatility functions are deterministic and also two–parameter models, such as models based on Brownian sheets. Kennedy (1994) derived suitable drift conditions which ensure the martingale properties of the corresponding discounted zero coupon bonds.

In a later article, Kennedy (1997) revisited the continuous Gaussian random field models and he showed that the structure of the covariance function of such models can be specified explicitly if one assumes a Markov property. Adding an additional stationarity condition, the correlation structure of such processes is already very limited and Kennedy (1997) proved that, in fact, under a strong Markov and stationarity assumption the Gaussian field is necessarily described by just three parameters.

The Gaussian assumption was relaxed later and Goldstein (2000) presented a term structure model based on non–Gaussian random fields. Such models incorporate in particular conditional volatility models, i.e. models which allow for more flexible (i.e. stochastic) behaviour of the (conditional) volatilities of the innovations to forward rates (in the traditional Kennedy approach such variances were just constant functions of maturity), and, hence, are particularly relevant for empirical applications. Also, Goldstein (2000) points out that one is interested in very *smooth* random field models in the context of modelling the term structure of interest rates. Such a smoothness (e.g. in the time to maturity direction) can be achieved by using *integrated random fields*, e.g. he proposes to integrate over an Ornstein–Uhlenbeck process. Goldstein (2000) derived drift conditions for the absence of arbitrage for such general non–Gaussian random field models.

While such models are quite general and, hence, appealing in practice, Kimmel (2004) points out that the models defined by Goldstein (2000) are generally specified as solutions to a set of stochastic differential equations, where it is difficult to prove the existence and uniqueness of solutions. The Goldstein (2000) models and many other conditional volatility random field models are in fact complex and often infinite dimensional processes, which lack the key property of the Gaussian random field models introduced by Kennedy (1994): that the individual forward rates are low dimensional diffusion processes. The latter property is in fact important for model estimation and derivative pricing. Hence, Kimmel (2004) proposes a new approach to random field models which allows for conditional volatility and which preserves the key property of the Kennedy (1994) class of models: the class of *latent variable term structure models*. He proves that such models ensure that the forward rates and the latent variables (which are modelled as a joint diffusion) follow jointly a finite dimensional diffusion.

A different approach to generalising the Kennedy (1994) framework is proposed by Albeverio et al. (2004). They suggest to replace the Gaussian random field in the Kennedy (1994) model by a (pure jump) Lévy field. Special cases of such models are e.g. the Poisson and the Gamma sheet.

Finally, another approach for modelling forward rates has been proposed by Santa-Clara & Sornette (2001) who build their model on *stochastic string shocks*. We will review that class of models later in more detail since it is related (and under some assumptions even a special case) of the new modelling framework we present in this paper.

2.3 Intuitive description of an ambit field based model for forward prices

After we have reviewed the traditional models for the term structure of interest rates, which are (partially) also used for modelling forward prices of commodities, we wish to give an intuitive description of the new framework we propose in this paper before we present all the mathematical details.

As in the aforementioned models, we also propose to use a random field to account for the two temporal dimensions of current time and time to maturity. However, the main difference of our new modelling framework compared to the traditional ones is that we model the forward price *directly*. This *direct* modelling approach is in fact twofold: First, we model the forward prices directly rather than the spot price, which is in line with the Heath et al. (1992) framework. Second, we do not specify the *dynamics* of the forward price as the solution of an evolution equation, but we specify a random field, an ambit field, which explicitly describes the forward price. In particular, we propose to use random fields given by stochastic integrals of type

$$\int_{A_t(x)} h(\xi, s, x, t)\sigma_s(\xi)L(d\xi, ds), \tag{1}$$

as a building block for modelling $f_t(x)$. A natural choice for L — motivated by the use of Lévy processes in the one–dimensional framework — is the class of *Lévy bases*, which are infinitely divisible random measures as described in more detail below. Here the integrand is given by the product of a deterministic kernel function h and a random field σ describing the stochastic volatility.

We will describe in more detail below, how stochastic integrals of type (1) have to be understood. Note here that we integrate over a set $A_t(x)$, the *ambit set*, which can be chosen in many different ways. We will discuss the choice of such sets later in the paper.

An important motivation for the use of ambit processes is that we wish to work with processes which are *stationary* in time, i.e. in t, rather than formulating a model which *converges* to a stationary process. Hence, we work with stochastic integrals starting from $-\infty$ in the temporal dimension, more precisely, we choose ambit sets of the form $A_t(x) = \{(\xi, s) : -\infty < s \leq t, \xi \in I_t(s, x)\}$, where $I_t(s, x)$ is typically an interval including x, rather than integrating from 0, which is what the traditional models do which are constructed as solutions of stochastic partial differential equations (SPDEs). (In fact, many traditional models coming from SPDEs can be included in an ambit framework when choosing the ambit set $A_t(x) = [0, t] \times \{x\}$, see Barndorff-Nielsen, Benth & Veraart (2011) for more details.)

In order to obtain models which are stationary in the time component t, but not necessarily in the time to maturity component x, we assume that the kernel function depends on t and s only through the difference $t - s$, so having that h is of the form $h(\xi, s, x, t) = k(\xi, t - s, x)$, that σ is stationary in time and that $A_t(x)$ has a certain structure, as described below. Then the specification (1) takes the form

$$\int_{A_t(x)} k(\xi, t - s, x)\sigma_s(\xi)L(d\xi, ds). \tag{2}$$

Note that Hikspoors & Jaimungal (2008), Benth (2011) and Barndorff-Nielsen et al. (2010) provide empirical evidence that spot and forward prices are influenced by a *stochastic volatility* field σ. Here we assume that σ describes the volatility of the forward market as a whole. More precisely, we will assume that the volatility of the forward depends on previous states of the volatility both in time and in space, where the spatial dimension reflects the time to maturity. We will come back to that in Section 4.2.3.

The general structure of ambit fields makes it possible to allow for general *dependencies between forward contracts*. In the electricity market, a forward contract has a close resemblance with its neighbouring contracts, meaning contracts which are close in maturity. Empirics (by principal component analysis) suggest that the electricity markets need many factors, see e.g. Koekebakker & Ollmar (2005), to explain the risk, contrary to interest rate markets where one finds 3–4 sources of noise as relevant. Since electricity is a non–storable commodity, forward looking information plays a crucial role in settling forward prices. Different information at different maturities, such as plant maintenance, weather forecasts, political decisions etc., give rise to a high degree of idiosyncratic risk in the forward market, see Benth & Meyer-Brandis (2009). These empirical and theoretical findings justify a random field model in electricity and also indicate that there is a high degree of dependency around contracts which are close in maturity, but much weaker dependence when maturities are farther apart. The structure of the ambit field and the volatility field which we propose in this paper will allow us to "bundle" contracts together in a flexible fashion.

3 Ambit fields and processes

This section reviews the concept of ambit fields and ambit processes which form the building blocks of our new model for the electricity forward price. For a detailed account on this topic see Barndorff-Nielsen, Benth & Veraart (2011) and Barndorff-Nielsen & Schmiegel (2007). Throughout the paper, we denote by $(\Omega, \mathcal{F}, P^*)$ our probability space. Note that we use the $*$ notation since we will later refer to this probability measure as a risk neutral probability measure.

3.1 Review of the theory of ambit fields and processes

The general framework for defining an ambit process is as follows. Let $Y = \{Y_t(x)\}$ with $Y_t(x) := Y(x,t)$ denote a stochastic field in space–time $\mathcal{X} \times \mathbb{R}$ and let $\tau(\theta) = (x(\theta), t(\theta))$ denote a curve in $\mathcal{X} \times \mathbb{R}$. The values of the field along the curve are then given by $X_\theta = Y_{t(\theta)}(x(\theta))$. Clearly, $X = \{X_\theta\}$ denotes a stochastic process. In most applications, the space $\mathcal{X}$ is chosen to be $\mathbb{R}^d$ for $d = 1, 2$ or 3. Further, the stochastic field is assumed to be generated by innovations in space–time with values $Y_t(x)$ which are supposed to depend only on innovations that occur prior to or at time t and in general only on a restricted set of the corresponding part of space–time. I.e., at each point (x,t), the value of $Y_t(x)$ is only determined by innovations in some subset $A_t(x)$ of $\mathcal{X} \times \mathbb{R}_t$ (where $\mathbb{R}_t = (-\infty, t]$), which we call the *ambit set* associated to (x,t). Furthermore, we refer to Y and X as an *ambit field* and an *ambit process*, respectively.

In order to use such general ambit fields in applications, we have to impose some structural assumptions. More precisely, we will define $Y_t(x)$ as a stochastic integral plus a smooth term, where the integrand in the stochastic integral will consist of a deterministic kernel times a positive random variate which is taken to embody the *volatility* of the field Y. More precisely, we think of ambit fields as being of the form

$$Y_t(x) = \mu + \int_{A_t(x)} h(\xi, s, x, t)\, \sigma_s(\xi)\, L(\mathrm{d}\xi, \mathrm{d}s) + \int_{D_t(x)} q(\xi, s, x, t)\, a_s(\xi)\, \mathrm{d}\xi \mathrm{d}s, \tag{3}$$

where $A_t(x)$, and $D_t(x)$ are ambit sets, h and q are deterministic functions, $\sigma \geq 0$ is a stochastic field referred to as *volatility*, a is also a stochastic field, and L is a *Lévy basis*. Throughout the paper we will assume that the volatility field σ is independent of the Lévy basis L for modelling convenience.

The corresponding ambit process X along the curve τ is then given by

$$X_\theta = \mu + \int_{A(\theta)} h(\xi, s, \tau(\theta))\sigma_s(\xi)L(d\xi, ds) + \int_{D(\theta)} q(\xi, s, \tau(\theta))a_s(\xi)d\xi ds, \tag{4}$$

where $A(\theta) = A_{t(\theta)}(x(\theta))$ and $D(\theta) = D_{t(\theta)}(x(\theta))$.

Of particular interest in many applications are ambit processes that are stationary in time and nonanticipative. More specifically, they may be derived from ambit fields Y of the form

$$Y_t(x) = \mu + \int_{A_t(x)} h(\xi, t-s, x)\, \sigma_s(\xi)\, L(\mathrm{d}\xi, \mathrm{d}s) + \int_{D_t(x)} q(\xi, t-s, x)\, a_s(\xi)\, \mathrm{d}\xi \mathrm{d}s. \tag{5}$$

Here the ambit sets $A_t(x)$ and $D_t(x)$ are taken to be *homogeneous* and *nonanticipative* i.e. $A_t(x)$ is of the form $A_t(x) = A + (x,t)$ where A only involves negative time coordinates, and similarly for $D_t(x)$. We assume further that $h(\xi, u, x) = q(\xi, u, x) = 0$ for $u \leq 0$.

Due to the structural assumptions we made to define ambit fields, we obtain a class of random fields which is highly analytically tractable. In particular, we can derive moments and the correlation structure explicitly, see the Appendix A.4 for detailed results.

In any concrete modelling, one has to specify the various components of the ambit field, and we do that for electricity forward prices in Section 4.1.

3.2 Background on Lévy bases

Let $\mathcal{S}$ denote the δ–ring of subsets of an arbitrary non–empty set S, such that there exists an increasing sequence $\{S_n\}$ of sets in $\mathcal{S}$ with $\cup_n S_n = S$, see Rajput & Rosinski (1989). Recall from e.g. Rajput & Rosinski (1989), Pedersen (2003), Barndorff-Nielsen (2011) that a Lévy basis $L = \{L(B), B \in \mathcal{S}\}$ defined on a probability space $(\Omega, \mathcal{F}, P)$ is an independently scattered random measure with Lévy–Khinchin representation

$$C\{v \ddagger L(B)\} = \log\left(\mathbb{E}(\exp(ivL(B)))\right),$$

given by

$$C\{v \ddagger L(B)\} = iva(B) - \frac{1}{2}v^2 b(B) + \int_{\mathbb{R}} \left(e^{ivr} - 1 - ivr\mathbb{I}_{[-1,1]}(r)\right) l(dr, B), \tag{6}$$

where a is a signed measure on $\mathcal{S}$, b is a measure on $\mathcal{S}$, $l(\cdot,\cdot)$ is the generalised Lévy measure such that $l(dr, B)$ is a Lévy measure on $\mathbb{R}$ for fixed $B \in \mathcal{S}$ and a measure on $\mathcal{S}$ for fixed dr. Without loss of generality we can assume that the generalised Lévy measure factorises as $l(dr, d\eta) = U(dr, \eta)\mu(d\eta)$, where μ is a measure on $\mathcal{S}$. Concretely, we take μ to be the *control measure*, see Rajput & Rosinski (1989), defined by

$$\mu(B) = |a|(B) + b(B) + \int_{\mathbb{R}} \min(1, r^2) l(dr, B), \tag{7}$$

where $|\cdot|$ denotes the total variation. Further, $U(dr, \eta)$ is a Lévy measure for fixed η.

Note that a and b are absolutely continuous with respect to μ and we can write $a(d\eta) = \widetilde{a}(\eta)\mu(d\eta)$, and $b(d\eta) = \widetilde{b}(\eta)\mu(d\eta)$.

For $\eta \in \mathcal{S}$, let $L'(\eta)$ be an infinitely divisible random variable such that

$$C\{v \ddagger L'(\eta)\} = \log\left(\mathbb{E}(\exp(ivL'(\eta)))\right),$$

with

$$C\{v \ddagger L'(\eta)\} = iv\widetilde{a}(\eta) - \frac{1}{2}v^2 \widetilde{b}(\eta) + \int_{\mathbb{R}} \left(e^{ivr} - 1 - ivr\mathbb{I}_{[-1,1]}(r)\right) U(dr, \eta), \tag{8}$$

then we have

$$C\{v \ddagger L(d\eta)\} = C\{v \ddagger L'(\eta)\}\mu(d\eta). \tag{9}$$

In the following, we will (as in Barndorff-Nielsen (2011)) refer to $L'(\eta)$ as the *Lévy seed* of L at η.

If $U(dr, \eta)$ does not depend on η, we call l and L *factorisable*. If L is factorisable, with $\mathcal{S} \subset \mathbb{R}^n$ and if $\widetilde{a}(\eta), \widetilde{b}(\eta)$ do not depend on η and if μ is proportional to the Lebesgue measure, then L is called *homogeneous*. So in the homogeneous case, we have that $\mu(d\eta) = c\, leb(d\eta)$ for a constant c. In order to simplify the exposition we will throughout the paper assume that the constant in the homogeneous case is given by $c = 1$.

3.3 Integration concepts with respect to a Lévy basis

Since ambit processes are defined as stochastic integrals with respect to a Lévy basis, we briefly review in this section in which sense this stochastic integration should be understood. Throughout the rest of the paper, we work with stochastic integration with respect to martingale measures as defined by Walsh (1986), see also Dalang & Quer-Sardanyons (2011) for a review. We will review this theory here briefly and refer to Barndorff-Nielsen, Benth & Veraart (2011) for a detailed overview on integration concepts with respect to Lévy bases. Note that the integration theory due to Walsh can be regarded as Itô integration extended to random fields.

In the following we will present the integration theory on a bounded domain and comment later on how one can extend the theory to the case of an unbounded domain.

Let S denote a bounded Borel set in $\mathcal{X} = \mathbb{R}^d$ for a $d \in \mathbb{N}$ and let $(S, \mathcal{S}, leb)$ denote a measurable space, where $\mathcal{S}$ denotes the Borel σ–algebra on S and leb is the Lebesgue measure.

Let L denote a Lévy basis on $S \times [0, T] \in \mathcal{B}(\mathbb{R}^{d+1})$ for some $T > 0$. Note that $\mathcal{B}(\mathbb{R}^{d+1})$ refers to the Borel sets generated by $\mathbb{R}^{d+1}$ and $\mathcal{B}_b(\cdot)$ refers to the bounded Borel sets generated by S.

For any $A \in \mathcal{B}_b(S)$ and $0 \leq t \leq T$, we define

$$L_t(A) = L(A, t) = L(A \times (0, t]).$$

Here $L_t(\cdot)$ is a measure–valued process, which for a fixed set $A \in \mathcal{B}_b(S)$, $L_t(A)$ is an additive process in law.

In the following, we want to use the $L_t(A)$ as integrators as in Walsh (1986). In order to do that, we work under the square–integrability assumption, i.e.:

Assumption (A1): For each $A \in \mathcal{B}_b(S)$, we have that $L_t(A) \in L^2(\Omega, \mathcal{F}, P^*)$.

Note that, in particular, assumption (A1) excludes α–stable Lévy bases for $\alpha < 2$.

Remark 1. Note that the square integrability assumption is needed for studying certain dynamic properties of ambit fields, such as martingale conditions. Otherwise one could work with the integration concept introduced by Rajput & Rosinski (1989) (provided the stochastic volatility field σ is independent of the Lévy basis L), which would in particular also work for the case when L is a stable Lévy basis.

Next, we define the filtration $\mathcal{F}_t$ by

$$\mathcal{F}_t = \cap_{n=1}^{\infty} \mathcal{F}_{t+1/n}^0, \qquad \text{where} \qquad \mathcal{F}_t^0 = \sigma\{L_s(A) : A \in \mathcal{B}_b(S), 0 < s \leq t\} \vee \mathcal{N}, \tag{10}$$

and where $\mathcal{N}$ denotes the P–null sets of $\mathcal{F}$. Note that $\mathcal{F}_t$ is right–continuous by construction.

In the following, we will unless otherwise stated, work without loss of generality under the zero–mean assumption on L, i.e.

Assumption (A2): For each $A \in \mathcal{B}_b(S)$, we have that $\mathbb{E}(L_t(A)) = 0$.

One can show that under the assumptions (A1) and (A2), $L_t(A)$ is a (square–integrable) *martingale* with respect to the filtration $(\mathcal{F}_t)_{0 \leq t \leq T}$. Note that these two properties together with the fact that $L_0(A) = 0$ a.s. ensure that $(L_t(A))_{t \geq 0, A \in \mathcal{B}(R^d)}$ is a *martingale measure* with respect to $(\mathcal{F}_t)_{0 \leq t \leq T}$ in the sense of Walsh (1986). Furthermore, we have the following orthogonality property: If $A, B \in \mathcal{B}_b(S)$ with $A \cap B = \emptyset$, then $L_t(A)$ and $L_t(B)$ are independent. Martingale measures which satisfy such an orthogonality property are referred to as *orthogonal martingale measures* by Walsh (1986), see also Barndorff-Nielsen, Benth & Veraart (2011) for more details.

For such measures, Walsh (1986) introduces their *covariance measure* Q by

$$Q(A \times [0, t]) = < L(A) >_t, \tag{11}$$

for $A \in \mathcal{B}(R^d)$. Note that Q is a positive measure and is used by Walsh (1986) when defining stochastic integration with respect to L.

Walsh (1986) defines stochastic integration in the following way. Let $\zeta(\xi, s)$ be an *elementary* random field $\zeta(\xi, s)$, i.e. it has the form

$$\zeta(\xi, s, \omega) = X(\omega) \mathbb{I}_{(a,b]}(s) \mathbb{I}_A(\xi)\,, \tag{12}$$

where $0 \leq a < t$, $a \leq b$, X is bounded and $\mathcal{F}_a$–measurable, and $A \in \mathcal{S}$. For such elementary functions, the stochastic integral with respect to L can be defined as

$$\int_0^t \int_B \zeta(\xi, s)\, L(d\xi, ds) := X\left(L_{t\wedge b}(A \cap B) - L_{t\wedge a}(A \cap B)\right), \tag{13}$$

for every $B \in \mathcal{S}$. It turns out that the stochastic integral becomes a martingale measure itself in B (for fixed a, b, A). Clearly, the above integral can easily be generalised to allow for integrands given by *simple* random fields, i.e. finite linear combinations of elementary random fields. Let $\mathcal{T}$ denote the set of simple random fields and let the *predictable* σ–algebra $\mathcal{P}$ be the σ–algebra generated by $\mathcal{T}$. Then we call a random field *predictable* provided it is $\mathcal{P}$–measurable. The aim is now to define stochastic integrals with respect to L where the integrand is given by a predictable random field.

In order to do that Walsh (1986) defines a norm $\|\cdot\|_L$ on the predictable random fields ζ by

$$\|\zeta\|_L^2 := \mathbb{E}\left[\int_{[0,T]\times S} \zeta^2(\xi, s)\, Q(d\xi, ds)\right], \tag{14}$$

which determines the Hilbert space $\mathcal{P}_L := L^2(\Omega \times [0,T] \times S, \mathcal{P}, Q)$, and he shows that $\mathcal{T}$ is dense in $\mathcal{P}_L$. Hence, in order to define the stochastic integral of $\zeta \in \mathcal{P}_L$, one can choose an approximating sequence $\{\zeta_n\}_n \subset \mathcal{T}$ such that $\|\zeta - \zeta_n\|_L \to 0$ as $n \to \infty$. Clearly, for each $A \in \mathcal{S}$, $\int_{[0,t]\times A} \zeta_n(\xi, s)\, L(d\xi, ds)$ is a Cauchy sequence in $L^2(\Omega, \mathcal{F}, P)$, and thus there exists a limit which is defined as the stochastic integral of ζ.

Then, this stochastic integral is again a martingale measure and satisfies the following *Itô–type isometry*:

$$\mathbb{E}\left[\left(\int_{[0,T]\times A} \zeta(\xi, s) L(d\xi, ds)\right)^2\right] = \|\zeta\|_L^2, \tag{15}$$

see (Walsh 1986, Theorem 2.5) for more details.

Remark 2. In order to use Walsh–type integration in the context of ambit fields, we note the following:

- General ambit sets $A_t(x)$ are not necessarily bounded. However, the stochastic integration concept reviewed above can be extended to unbounded ambit sets using standard arguments, cf. Walsh (1986, p. 289).

- For ambit fields with ambit sets $A_t(x) \subset \mathcal{X} \times (-\infty, t]$, we define Walsh–type integrals for integrands of the form

$$\zeta(\xi, s) = \zeta(\xi, s, x, t) = \mathbb{I}_{A_t(x)}(\xi, s) h(\xi, s, x, t) \sigma_s(\xi). \tag{16}$$

- The original Walsh's integration theory covers integrands which do not depend on the time index t. Clearly, the integrand given in (16) generally exhibits t–dependence due to the choice of the ambit set $A_t(x)$ and due to the deterministic kernel function h. In order to allow for time dependence in the integrand, we can define the integrals in the Walsh sense for any *fixed* t. Note that in the case of having t–dependence in the integrand, the resulting stochastic integral is, in general, not a martingale measure any more. We will come back to this issue in Section 6.

In order to ensure that the ambit fields (as defined in (3)) are well–defined (in the Walsh–sense), throughout the rest of the paper, we will work under the following assumption:

Assumption (A3): Let L denote a Lévy basis on $S \times (-\infty, T]$, where S denotes a not necessarily bounded Borel set S in $\mathcal{X} = \mathbb{R}^d$ for some $d \in I\!N$. We extend the definition of the measure Q, see (11), to an unbounded domain and, next, we define a Hilbert space $\mathcal{P}_L$ with norm $||\cdot||_L$ as in (14) (extended to an unbounded domain) and, hence, we have an Itô isometry of type (15) extended to an unbounded domain. We assume that, for fixed x and t,

$$\zeta(\xi, s) = \mathbb{I}_{A_t(x)}(\xi, s) h(\xi, s, x, t)\sigma_s(\xi)$$

satisfies

1. $\zeta \in \mathcal{P}_L$,
2. $||\zeta||_L^2 = \mathbb{E}\left[\int_{\mathbb{R}\times\mathcal{X}} \zeta^2(\xi, s) Q(d\xi, ds)\right] < \infty$.

Note that in our forward price model we will discard the drift term from the general ambit field defined in (3) and hence we do not add an integrability condition for the drift.

With a precise notion of integration established, let us return to the derivation of characteristic exponents, which will become useful later. It holds that (see also Rajput & Rosinski (1989, Proposition 2.6))

$$\begin{aligned} C\left\{v \ddagger \int \overline{f} dL\right\} &= \log\left(\mathbb{E}\left(\exp\left(iv \int \overline{f} dL\right)\right)\right) = \int \log\left(\mathbb{E}(\exp(iv\overline{f}(\eta)L'(\eta)))\right)\mu(d\eta) \\ &= \int C\{v\overline{f}(\eta) \ddagger L'(\eta)\}\mu(d\eta), \end{aligned} \quad (17)$$

for a deterministic function $\overline{f}$ which is integrable with respect to the Lévy basis.

In order to be able to compute moments of integrals with respect to a Lévy basis, we invoke a generalised Lévy–Itô decomposition, see Pedersen (2003). Corresponding to the generalised Lévy–Khintchine formula, (6), the Lévy basis can be written as

$$\begin{aligned} L(B) &= a(B) + \sqrt{b(B)} W(B) + \int_{\{|y|<1\}} y(N(dy, B) - \nu(dy, B)) + \int_{\{|y|\geq 1\}} yN(dy, B) \\ &= a(B) + \sqrt{b(B)} W(B) + \int_{\{|y|<1\}} y(N - \nu)(dy, B) + \int_{\{|y|\geq 1\}} yN(dy, B), \end{aligned}$$

for a Gaussian basis W and a Poisson basis N with intensity ν.

Now we have all the tools at hand which are needed to compute the conditional characteristic function of ambit fields defined in (3) where σ and L are assumed independent and where we condition on the path of σ.

Theorem 1. *Let C^σ denote the conditional cumulant function when we condition on the volatility field σ. The conditional cumulant function of the ambit field defined by (3) is given by*

$$\begin{aligned} &C^\sigma\left\{v \ddagger \int_{A_t(x)} h(\xi, s, x, t)\sigma_s(\xi) L(d\xi, ds)\right\} \\ &= \log\left(\mathbb{E}\left(\exp\left(iv \int_{A_t(x)} h(\xi, s, x, t)\sigma_s(\xi) L(d\xi, ds)\right) \middle| \sigma\right)\right) \\ &= \int_{A_t(x)} C\left\{vh(\xi, s, x, t)\sigma_s(\xi) \ddagger L'(\xi, s)\right\}\mu(d\xi, ds), \end{aligned} \quad (18)$$

where L' denotes the Lévy seed and μ is the control measure associated with the Lévy basis L, cf. (8) and (7).

The proof of the Theorem is straightforward given the previous results and is hence omitted.

Note that in the homogeneous case, equation (18) simplifies to

$$C^{\sigma}\left\{v \ddagger \int_{A_t(x)} h(\xi,s,x,t)\sigma_s(\xi)L(d\xi,ds)\right\} = \int_{A_t(x)} C\left\{vh(\xi,s,x,t)\sigma_s(\xi) \ddagger L'\right\} d\xi ds.$$

3.4 Lévy Semistationary Processes ($\mathcal{LSS}$)

After having reviewed the basic traits of ambit fields, we briefly mention the null–spatial case of semi–stationary ambit fields, i.e. the case when we only have a temporal component and when the kernel function depends on t and s only through the difference $t-s$. This determines the class of Lévy semistationary processes ($\mathcal{LSS}$), see Barndorff-Nielsen et al. (2010). Specifically, let $Z = (Z_t)_{t\in\mathbb{R}}$ denote a general Lévy process on $\mathbb{R}$. Then, we write $Y = \{Y_t\}_{t\in\mathbb{R}}$, where

$$Y_t = \mu + \int_{-\infty}^{t} k(t-s)\omega_{s-}\mathrm{d}Z_s + \int_{-\infty}^{t} q(t-s)a_s\mathrm{d}s, \tag{19}$$

where μ is a constant, k and q are nonnegative deterministic functions on $\mathbb{R}$, with $k(t) = q(t) = 0$ for $t \leq 0$, and ω and a are càdlàg, stationary processes. The reason for here denoting the volatility by ω rather than σ will become apparent later. In abbreviation the above formula is written as

$$Y = \mu + k * \omega \bullet Z + q * a \bullet leb, \tag{20}$$

where leb denotes Lebesgue measure. In the case that Z is a Brownian motion, we call Y a *Brownian semistationary* ($\mathcal{BSS}$) *process*, see Barndorff-Nielsen & Schmiegel (2009).

In the following, we will often, for simplicity, work within the set–up that both $\mu = 0$ and $q \equiv 0$, hence

$$Y_t = \int_{-\infty}^{t} k(t-s)\omega_{s-}dZ_s. \tag{21}$$

For integrability conditions on ω and k, we refer to Barndorff-Nielsen et al. (2010). Note that the stationary dynamics of Y defined in (21) is a special case of a volatility modulated Lévy–driven Volterra process, which has the form

$$Y_t = \int_{-\infty}^{t} h(t,s)\omega_{s-}\,dZ_s\,, \tag{22}$$

where Z is a Lévy process and h is a real–valued measurable function on $\mathbb{R}^2$, such that the integral with respect to Z exists.

4 Modelling the forward price under the risk neutral measure

After having reviewed the basic definitions of ambit fields and the stochastic integration concept due to Walsh (1986), we proceed now by introducing a general model for (deseasonalised) electricity forward prices based on ambit fields.

We consider a probability space $(\Omega, \mathcal{F}, P^*)$, where P^* denotes the risk neutral probability measure.

Remark 3. Since we model directly under the risk neutral measure, we will ignore any drift terms in the following, but work with a zero–mean specification of the ambit field, which we later derive the martingale conditions for.

We set $\mathbb{R}_+ = [0, \infty)$ and define a Lévy basis $L = (L(A, s))_{A \in \mathcal{B}(\mathbb{R}_+), s \in \mathbb{R}}$ and a stochastic volatility field $\sigma = (\sigma_s(A))_{A \in \mathcal{B}(\mathbb{R}_+), s \in \mathbb{R}}$, which is independent of L. Throughout the remaining part of the paper, we define the filtration $\{\mathcal{F}_t\}_{t \in \mathbb{R}}$ by

$$\mathcal{F}_t = \cap_{n=1}^{\infty} \mathcal{F}^0_{t+1/n}, \qquad \text{where} \qquad \mathcal{F}^0_t = \sigma\{L(A, s) : A \in \mathcal{B}(\mathbb{R}_+), s \leq t\} \vee \mathcal{N}, \tag{23}$$

and where $\mathcal{N}$ denotes the P–null sets of $\mathcal{F}$. Note that $\mathcal{F}_t$ is right–continuous by construction. Also, we define the enlarged filtration $\{\overline{\mathcal{F}}_t\}_{t \in \mathbb{R}}$ by

$$\overline{\mathcal{F}}_t = \cap_{n=1}^{\infty} \overline{\mathcal{F}}^0_{t+1/n}, \qquad \text{where} \qquad \overline{\mathcal{F}}^0_t = \sigma\{(L(A, s), \sigma_s(A)) : A \in \mathcal{B}(\mathbb{R}_+), s \leq t\} \vee \mathcal{N}. \tag{24}$$

4.1 The model

Under the risk neutral measure, the new model type for the forward price $f_t(x)$ is defined for fixed $t \in \mathbb{R}$ and for $x \geq 0$ by

$$f_t(x) = \int_{A_t(x)} k(\xi, t-s, x)\sigma_s(\xi) L(d\xi, ds), \tag{25}$$

where

(i) the Lévy basis L is square integrable and has zero mean (this is an extension of assumptions (A1) and (A2) to an unbounded domain);

(ii) the stochastic volatility field σ is assumed to be adapted to $\{\mathcal{F}_t\}_{t \in \mathbb{R}}$ and independent of the Lévy basis L and in order to ensure stationarity in time, we assume that $\sigma_s(\xi)$ is stationary in s;

(iii) the kernel function k is assumed to be non–negative and chosen such that $k(\xi, u, x) = 0$ for $u < 0$;

(iv) the convolution $k \star \sigma$ is integrable w.r.t. L, i.e. it satisfies (A3);

(v) the ambit set is chosen to be

$$A_t(x) = A_t = \{(\xi, s) : \xi \geq 0, \ s \leq t\}, \tag{26}$$

for $t \in \mathbb{R}, \ x \geq 0$, see Figure 1. Note that the ambit set is of the type $A_t(x) = A_0(x) + (0, t)$ for $A_0(x) = \{(\xi, s) : \xi \geq 0, \ s \leq 0\}$. In the following, we will drop the (x) in the notation of the ambit set, i.e. $A_t(x) = A_t$, since the particular choice of the ambit set defined in (26) does not depend on the spatial component x.

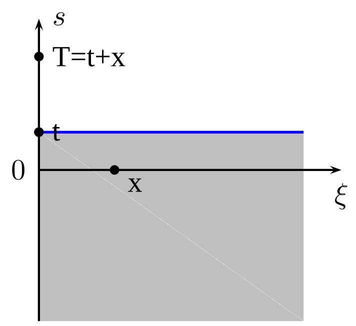

Figure 1: The ambit set $A_t(x) = A_t$.

Note that $f_t(x)$ is a stochastic process in time for each fixed x. Also, it is important to note that for *fixed* x, $f_t(x)$ is stationary in t, more precisely $f_t(\cdot)$ is a stationary field in time. However, as soon as we replace x by a function of t, $x(t)$ say, in our case by $x(t) = T - t$, $f_t(x(t))$ is generally not stationary any more. This is consistent with forward prices derived from stationary spot models (see Barndorff-Nielsen et al. (2010)).

In order to construct a specific model for the forward price, we need to specify the kernel function k, the stochastic volatility field $\sigma_s(\xi)$ and L.

It is important to note that, when working with general ambit processes as defined in (25), in modelling terms we can play around with both the ambit set, the weight function k, the volatility field σ and the Lévy basis in order to achieve a dependence structure we want to have. As such there is generally not a unique choice of the ambit set or the weight function or the volatility field to achieve a particular type of dependence structure and the choice will be based on stylised features, market intuition and considerations of mathematical/statistical tractability.

In order to make the model specification easier in practice, we have decided to work with the encompassing ambit set defined in (26).

Remark 4. We have chosen to model the forward price in (25) as an arithmetic model. One could of course interpret $f_t(x)$ in (25) as the *logarithmic* forward price, and from time to time in the discussion below this is the natural context. However, in the theoretical considerations, we stick to the arithmetic model, and leave the analysis of the geometric case to Section 10.1. We note that Bernhardt et al. (2008), Garcia et al. (2010) proposed and argued statistically for an arithmetic spot price model for Singapore electricity data. An arithmetic spot model will naturally lead to an arithmetic dynamics for the forward price. Benth et al. (2007) proposed an arithmetic model for spot electricity, and derived an arithmetic forward price dynamics. In Benth, Cartea & Kiesel (2008) arithmetic spot and forward price models are used to investigate the risk premium theoretically and empirically for the German EEX market.

Remark 5. Note that the forward price at time 0 implied by the model is given as

$$f_0(x) = \int_{A_0} k(\xi, -s, x)\sigma_s(\xi)L(d\xi, ds). \tag{27}$$

Hence, we view the *observed* forward price as a realisation of the random variable $f_0(x)$ given in (27), contrary to most other models where $f_0(x)$ is considered as deterministic and *put equal to* the observed price.

The ambit field specification we are working with here is highly analytical tractable and its conditional cumulant function is given as follow.

Theorem 2. *Let L be a homogeneous Lévy basis*[1]. *Then*

$$\mathrm{C}^{\sigma}\left\{\zeta \ddagger f_t\left(x\right)\right\} = \int_{-\infty}^{t}\int_{0}^{\infty}\mathrm{C}\left\{\zeta k\left(\xi, t-s, x\right)\sigma_s\left(\xi\right) \ddagger L'\right\}\mathrm{d}\xi\mathrm{d}s, \tag{28}$$

where L' is the Lévy seed associated with L. Further, in the Gaussian case, we have

$$\mathrm{C}\left\{\zeta k\left(\xi, t-s, x\right)\sigma_s\left(\xi\right) \ddagger L'\right\} = -\frac{1}{2}\zeta^2 k^2\left(\xi, t-s, x\right)\sigma_s^2\left(\xi\right).$$

The proof of the theorem is straightforward and hence omitted.

[1]Recall that for every homogeneous Lévy basis the control measure is proportional to the Lebesgue measure. Here we implicitly assume that the proportionality constant is standardised to 1.

4.2 Examples of model specifications

A forward model based on an ambit field has a very general structure and, hence, we would like to point out some concrete model specifications which might be useful in practice. In any particular application, the concrete specification should be determined in a data–driven fashion and we will comment on model estimation and inference in Section 10.2.

Since we have chosen the ambit set to be the encompassing set defined in (26), there are three components of the model which we still need to specify: The Lévy basis L, the kernel function k and the stochastic volatility field σ.

4.2.1 Specification of the Lévy basis

Recall that we have defined our model based on a Lévy basis which is square integrable and has zero mean. Extensions to allow for non–zero mean are straightforward and, hence, omitted.

In principal, we can choose any infinitely divisible distribution satisfying these two assumptions. A very natural choice would be the Gaussian Lévy basis which would result in a smooth random field.

Alternative interesting choices include the Normal Inverse Gaussian (NIG) Lévy basis, see Example 1 below, and a tempered stable Lévy basis.

In an arithmetic modelling set up, if one wants to ensure price positivity, one would need to relax the zero–mean assumption for the Lévy basis and could then e.g. choose a Gamma or Inverse Gaussian Lévy basis.

4.2.2 Specification of the kernel function

Note that the kernel function k plays a key role in our model due to the following three reasons.

1. The kernel function completely determines the tempo–spatial autocorrelation structure of a zero–mean ambit field, see Section A.4.
2. It also characterises the Samuelson effect as we will see in Theorem 7.
3. It determines whether the forward price is indeed a martingale, see Theorem 3 and Corollary 1.

Recall that the kernel k is a function in three variables $\xi, t-s, x$, where $t-s$ refers to the temporal and ξ, x to the spatial dimension.

A rather natural approach for specifying a kernel function is to assume a factorisation.

We will present two different types here, which are important in different contexts as we will see later.

First, we study a factorisation into a temporal and a spatial kernel. In particular, we assume that the kernel function factorises as follows:

Factorisation 1

$$k(\xi, t-s, x) = \phi(\xi, x)\psi(t-s), \tag{29}$$

for a suitable function ψ representing the temporal part and ϕ representing the spatial part.

In a next step, we can study specifications of ϕ and ψ separately.

The choice of the temporal kernel ψ can be motivated by Ornstein–Uhlenbeck processes, which imply an exponential kernel, or more generally by CARMA processes, see Brockwell (2001*a*,*b*).

In empirical work, it will be particularly interesting to focus in more detail on the question of how to model the *spatial* kernel function ϕ, which determines the correlation between various forward contracts. In principal, one could choose similar (or the same) types of functions for the temporal and the spatial dimension. However, we will see in Section 8 that particular choices of ϕ will lead to a rather natural relation between forward and implied spot prices.

Let us briefly study an example which is included in our new modelling framework.

Example 1. *Let L be a homogeneous symmetric normal inverse Gaussian (NIG) Lévy basis, more specifically having Lévy seed L', see Section 3.2, with density*

$$\pi^{-1}\delta|y|^{-1}\gamma K_1(\gamma|y|),$$

where K denotes the modified Bessel function of the second kind and where $\delta, \gamma > 0$, see Barndorff-Nielsen (1998). Then

$$C\{\theta \ddagger L'\} = \delta\gamma - \delta\left(\gamma^2 + \theta^2\right)^{1/2}.$$

If the kernel function k factorises as in (29) and if $\sigma_s(\xi) \equiv 1$, then

$$\begin{aligned}\log(\mathbb{E}(ivf_t(x))) &= \int_{A_t} C\{vk(\xi, t-s, x) \ddagger L'\} d\xi ds \\ &= \int_{A_t} \left[\delta\gamma - \delta\left(\gamma^2 + (\phi(\xi, x)\psi(t-s))^2\right)^{1/2}\right] d\xi ds.\end{aligned}$$

For particular choices of the kernel function, this integral can be computed explicitly. E.g. for $\alpha > 0$, let $\phi(\xi, x) = \exp(-\alpha(\xi + x))$ and $\psi(t-s) = \exp(-\alpha(t-s))$. Then,

$$k(\xi, t-s, x) = \exp(-\alpha(\xi - s))\exp(-\alpha T),$$

for $\alpha > 0$. Then

$$\begin{aligned}\log(\mathbb{E}(ivf_t(x))) &= \int_{A_t} C\{vk(\xi, t-s, x) \ddagger L'\} d\xi ds \\ &= \delta\gamma \int_{-\infty}^{t}\int_0^\infty \left(1 - \sqrt{1 + c^2\exp(-2\alpha(\xi - s))}\right) d\xi ds,\end{aligned}$$

for $c = v\exp(-2\alpha T)/\gamma$. This integral can be expressed in terms of standard functions, see Section A.1 in the Appendix.

An alternative factorisation of the kernel function is given as follows.

Factorisation 2

$$k(\xi, t-s, x) = \Phi(\xi)\Psi(t-s, x), \tag{30}$$

for suitable functions Ψ and Φ.

Although Factorisation 2 does not look very natural at first sight, it is in fact also a very important one since it naturally includes cases where t cancels out in the sense that $\Psi(t-s, x) = \widetilde{\Psi}(t-s+x) = \widetilde{\Psi}(T-s)$ for a suitable function $\widetilde{\Psi}$. This property is crucial when we want to formulate martingale conditions for the forward price, see Section 6. Let us look at some more specific examples for that case in the following.

Example 2. *Motivated by the standard OU models, we choose*

$$\Psi(t-s, x) = \exp(-\alpha(t-s+x)),$$

for some $\alpha > 0$. The choice of Ψ can also be motivated from continuous–time ARMA (CARMA) processes, see Brockwell (2001a,b). Specifically, for $\alpha_i > 0, i = 1, \ldots, p, p \geq 1$, introduce the matrix

$$A = \begin{bmatrix} 0 & I_{p-1} \\ -\alpha_p & -\alpha_{p-1} \cdots - \alpha_1 \end{bmatrix}, \tag{31}$$

where I_n denotes the $n \times n$ identity matrix. For $0 < p < q$, define the p–dimensional vector $\mathbf{b}' = (b_0, b_1, \ldots, b_{p-1})$, where $b_q = 1$ and $b_j = 0$ for $q < j < p$, and introduce

$$\Psi(t-s, x) = \mathbf{b}'\exp(A(t-s+x)\mathbf{e}_p,$$

with $\mathbf{e}_k$ being the kth canonical unit vector in $\mathbb{R}^p$.

Another interesting example which does not belong to the class of linear models is given as follows.

Example 3. *Bjerksund et al. (2010) propose a geometric Brownian motion model for the electricity forward price with kernel function given by $\eta(t,T) = a/(T-t+b)$ for a, b two positive constants. They argue that the Samuelson effect in electricity markets are much steeper than in other commodity markets, defending the choice of a hyperbolic function rather than exponential. The volatility $\eta(t,T)$ motivates the choice (for $x = T-t$),*

$$\Psi(t-s,x) = \frac{a}{t-s+x+b} = \frac{a}{T-s+b}.$$

We will come back to the latter example later on, when we focus in more detail on the Samuelson effect.

4.2.3 Specification of the volatility

The question of how to model the stochastic volatility field $\sigma_t(x)$ in the ambit field specification is a very important and interesting one, and, hence, we will discuss several relevant choices in more detail in the following.

There are essentially two approaches which can be used for constructing a relevant stochastic volatility field: Either one specifies the stochastic volatility field directly as a random field (e.g. as another ambit field), or one starts from a purely temporal stochastic volatility process and then generalises the stochastic process to a random field in a suitable way. In the following, we will present examples for both types of construction.

First, we focus on the modelling approach where we directly specify a random field for the volatility field. A natural starting point for modelling the volatility is given by kernel–smoothing of a Lévy basis – possibly combined with a (nonlinear) transformation to ensure positivity. For instance, let

$$\sigma_t^2(x) = V\left(\int_{A_t^\sigma(x)} j(\xi, t-s, x) L^\sigma(d\xi, ds)\right), \tag{32}$$

where L^σ is a Lévy basis independent of L, j is an integrable kernel function satisfying $j(\xi, u, x) = 0$ for $u < 0$ and $V : \mathbb{R} \to \mathbb{R}_+$ is a continuous, nonnegative function. Further, the ambit set has the structure $A_t^\sigma(x) = A_0^\sigma(x) + (0,t)$ and is therefore homogeneous and nonanticipative. For simplicity, we could choose $A_t^\sigma(x) = A_t(x)$ as defined in (26).

Note that σ^2 defined by (32) with the ambit set defined by (26) is stationary in the temporal dimension.

Let us look at some more concrete examples:

1. A rather simple specification is given by choosing L^σ to be a standard normal Lévy basis and $V(x) = x^2$. Then $\sigma_s^2(\xi)$ would be positive and pointwise χ^2–distributed with one degree of freedom.

2. One could also work with a general Lévy basis, in particular Gaussian, and V given by the exponential function, see e.g. Barndorff-Nielsen & Schmiegel (2004) and Schmiegel et al. (2005).

3. A non–Gaussian example would be to choose L^σ as an inverse Gaussian Lévy basis and V to be the identity function.

Regarding the choice of the kernel function j of the volatility field, we should note that it determines the tempo–spatial autocorrelation structure of the volatility field.

For simplicity, we might want to start off with kernel functions which have no spatial component, e.g. $j(\xi, t-s, x) = \exp(-\lambda(t-s))$ for $\lambda > 0$ mimicking the Ornstein–Uhlenbeck–based stochastic volatility models, see e.g. Barndorff-Nielsen & Schmiegel (2004). In a next step (if necessary in the particular application), we could then add spatial correlation.

Second, we show how to construct a stochastic volatility field by extending a stochastic processes by a spatial dimension. Note that our objective is to construct a stochastic volatility field which is stationary (at least in the temporal direction). Clearly, there are many possibilities on how this can be done and we focus on a particularly relevant one in the following, namely the *Ornstein–Uhlenbeck–type volatility field* (OUTVF). The choice of using an OU process as the stationary base component is motivated by the fact that non–Gaussian OU–based stochastic volatility models, as e.g. studied in Barndorff-Nielsen & Shephard (2001), tend to perform fairly well in practice, at least in the purely temporal case.

Suppose now that $\tilde{Y}$ is a positive OU type process with rate parameter $\lambda > 0$ and generated by a Lévy subordinator Y, i.e.

$$\tilde{Y}_t = \int_{-\infty}^{t} e^{-\lambda(t-s)} \mathrm{d}Y_s,$$

We call a stochastic volatility field $\sigma_t^2(x)$ on $\mathbb{R}_+ \times \mathbb{R}$ an *Ornstein–Uhlenbeck–type volatility field* (OUTVF), if it is defined as follows

$$\tau_t(x) = \sigma_t^2(x) = e^{-\mu x}\tilde{Y}_t + \int_0^x e^{-\mu(x-\xi)} \mathrm{d}Z_{\xi|t}, \tag{33}$$

where $\mu > 0$ is the spatial rate parameter and where $\mathcal{Z} = \left\{Z_{\cdot|t}\right\}_{t\in\mathbb{R}_+}$ is a family of Lévy processes, which we define more precisely in the next but one paragraph.

Note that in the above construction, we start from an OU process in time. In particular, $\tau_t(0)$ is an OU process. The spatial structure is then introduced by two components: First, we we add an exponential weight $e^{-\mu x}$ in the spatial direction, which reaches its maximal for $x = 0$ and decays the further away we get from the purely temporal case. Second, an integral is added which resembles an OU–type process in the spatial variable x. However, note here that the integration starts from 0 rather than from $-\infty$, and hence the resulting component is not stationary in the spatial variable x.

Let us now focus in more detail on how to define the family of Lévy processes $\mathcal{Z}$. Suppose $\tilde{X} = \left\{\tilde{X}_t\right\}_{t\in\mathbb{R}}$ is a stationary, positive and infinitely divisible process on $\mathbb{R}$. Next we define $Z_{|\cdot} = \left\{Z_{x|\cdot}\right\}_{x\in\mathbb{R}_+}$ as the so–called *Lévy supra–process* generated by $\tilde{X}$, that is $\left\{Z_{x|\cdot}\right\}_{x\in\mathbb{R}_+}$ is a family of stationary processes such that $Z_{|\cdot}$ has independent increments, i.e. for any $0 < x_1 < x_2 < \cdots < x_n$ the processes $Z_{x_1|\cdot}, Z_{x_2|\cdot} - Z_{x_1|\cdot}, ..., Z_{x_n|\cdot} - Z_{x_{n-1}|\cdot}$ are mutually independent, and such that for each x the cumulant functional of $Z_{x|\cdot}$ equals x times the cumulant functional of $\tilde{X}$, i.e.

$$\mathrm{C}\{m \ddagger Z_{x|\cdot}\} = x\mathrm{C}\{m \ddagger \tilde{X}\},$$

where

$$\mathrm{C}\{m \ddagger \tilde{X}\} = \log \mathrm{E}\left\{e^{im(\tilde{X})}\right\},$$

with $m\left(\tilde{X}\right) = \int \tilde{X}_s m(\mathrm{d}s)$, m denoting an ‘arbitrary’ signed measure on $\mathbb{R}$. Then at any $t \in \mathbb{R}$ the values $Z_{x|t}$ of $Z_{\cdot|\cdot}$ at time t as x runs through $\mathbb{R}_+$ constitute a Lévy process that we denote by $Z_{\cdot|t}$. This is the Lévy process occurring in the integral in (33).

Note that τ is stationary in τ and that $\tau_t(x) \to \tilde{Y}_t$ as $x \to 0$.

Example 4. *Now suppose, for simplicity, that $\tilde{X}$ is an OU process with rate parameter κ and generated by a Lévy process X. Then*

$$\mathrm{Cov}\{\tau_t(x), \tau_{t'}(x')\} = \frac{1}{2}\Big(\mathrm{Var}\{Y_1\}\lambda^{-1}e^{-\lambda(|t-t'|)-\mu(x+x')} + \mathrm{Var}\{\tilde{X}_0\}\mu^{-1}e^{-\kappa|t-t'|-\mu|x-x'|} \\ -\mathrm{Var}\{\tilde{X}_0\}\mu^{-1}e^{-\kappa|t-t'|-\mu(x+x')}\Big).$$

If, furthermore, $\mathrm{Var}\{Y_1\} = \mathrm{Var}\{\tilde{X}_0\}$ *and* $\kappa = \lambda = \mu$ *then for fixed* x *and* x' *the autocorrelation function of* τ *is*

$$\mathrm{Cor}\{\tau_t(x), \tau_{t'}(x')\} = e^{-\kappa|t-t'|}e^{-\kappa|x-x'|}.$$

This type of construction can of course be generalised in a variety of ways, including dependence between X and Y and also superposition of OU processes.

Note that the process $\tau_t(x)$ is in general not predictable, which is disadvantageous given that we want to construct Walsh–type stochastic integrals. However, if we choose $\widetilde{X}$ to be of OU type, then we obtain a predictable stochastic volatility process.

Let us conclude this section with two further important remarks.

Remark 6. Note that if we work with a non–Gaussian Lévy basis in the volatility specification (32), then we do not obtain a continuous volatility field. If smoothness is a concern, then one could integrate the random field again, see Santa-Clara & Sornette (2001) for a similar approach, to obtain the necessary smoothness. The same argument also holds for smoothness of the forward price model.

Remark 7. So far we have only focused on one method for introducing stochastic volatility in a model based on a kernel–smoothed Lévy basis. An alternative approach would be to use extended subordination as introduced in Barndorff-Nielsen (2010) and Barndorff-Nielsen & Pedersen (2011), which we will study in more detail in future research.

4.3 Autocorrelation and cross–correlation

It is important to note that our new model does not only model one particular forward/futures contract, but it models the entire forward curve at once. Hence it is interesting to study the correlation structure for various forward contracts implied by our new modelling framework. The detailed results are relegated to the Appendix, see Section A.4, but we wish to highlight our main findings here: We see that the correlation structure is determined by three factors: the intersection of the corresponding ambit sets, the kernel function and the autocorrelation structure of the stochastic volatility field. More precisely, for the particular model defined in (25), where we additionally assume that the Lévy basis is homogeneous (satisfying Assumption (H) in the Appendix), we have for $t \in \mathbb{R}$, $h \geq 0$ and for $x, x' \geq 0$ that for the ambit set defined in (26), we have $A_t(x) \cap A_{t+h}(x') = A_t(x) = A_t$ and hence

$$Cor(f_t(x), f_{t+h}(x')) \\ = \frac{\int_{[0,\infty)\times[0,\infty)} k(\xi, u, x)k(\xi, u+h, x')\mathbb{E}\left(\sigma_0^2(\xi)\right) d\xi du,}{\sqrt{\int_{[0,\infty)\times[0,\infty)} k^2(\xi, u, x)\mathbb{E}\left(\sigma_0^2(\xi)\right) d\xi du \int_{[0,\infty)\times[0,\infty)} k^2(\xi, u', x')\mathbb{E}\left(\sigma_0^2(\xi)\right) d\xi du'}}.$$

Furthermore, one could think of modelling various commodity forward or futures contracts, such as electricity and natural gas futures simultaneously. In such a situation it becomes even more clear how flexible the ambit set–up is: We can specify different ambit sets, kernel function, stochastic volatility fields and Lévy bases and obtain a rather flexible correlation structure. The details of these multivariate extensions can be found in the Appendix in Section A.5.

5 Relating traditional model classes to the ambit framework

As already mentioned, the use of ambit fields for constructing models for forward prices is entirely new to the literature and extends the use of correlated random fields to allow for both functional and statistical dependence, as described in more detail below.

In the following, we will describe how some of the traditional models can be related to the ambit framework.

5.1 Heath–Jarrow–Morton model

In the *geometric* Heath et al. (1992) framework, the dynamics of the logarithmic forward price under the risk neutral measure are modelled by

$$d\log(f_t(x)) = \sigma_t(x)dW_t, \qquad \text{for } t \geq 0,$$

where W is a standard Brownian motion and σ is a positive stochastic volatility process. Note that we start at time 0 here. Hence, the explicit formula for the forward price is given by

$$f_t(x) = f_0(x)\exp\left(\int_0^t \sigma_s(x)dW_s - \frac{1}{2}\int_0^t \sigma_s^2(x)ds\right).$$

Clearly, such a model is a special case of an ambit field defined in (3), where $A_t(x) = [0,t] \times \{x\}$, L is a Gaussian Lévy basis and the kernel function h satisfies $h \equiv 1$.

5.2 Random field models

Ambit processes embed the Gaussian and Lévy field models proposed in Albeverio et al. (2004), Kennedy (1994, 1997). To see that note that we can set $\sigma \equiv 1$ and we can choose $A_t(x)$ to be an interval.

If we allow for a non–trivial kernel function h or stochastic volatility field σ we can obtain some of the conditional volatility models proposed in Goldstein (2000), Kimmel (2004).

5.3 Stochastic string shock model

Also, the *stochastic string shock* model by Santa-Clara & Sornette (2001), which was designed to model the term structure of interest rates, is related to the ambit framework. Their modelling framework is given as follows. The dynamics of the forward rate is given by

$$d_t f_t(x) = \alpha_t(x)dt + \sigma_t(x)d_t Z(t,x),$$

for adapted processes a and σ and a stochastic string shock Z. Note here that the notation d_t is taken from Santa-Clara & Sornette (2001) and refers to the fact that we look at the differential operator w.r.t. t. A string shock is defined as a random field $(Z(t,x))_{t,x\geq 0}$ which is continuous in both t and x and is a martingale in t. Furthermore the variance of the t–increments has to equal the time change, i.e. $Var(d_t Z(t,x)) = dt$ for all $x \geq 0$, and the correlation of the t–increments, i.e. $Cor(d_t Z(t,x), d_t Z(t,y))$, does not depend on t. Santa-Clara & Sornette (2001) show that such stochastic strings can be obtained as solutions to second order linear stochastic partial differential equations (SPDEs). It is well–known that such SPDEs have a unique solution (under some boundary conditions), see Morse & Feshbach (1953) and the references in Barndorff-Nielsen, Benth & Veraart (2011), and the solution is representable in terms of an integral, often of convolution type, of a Green function with respect to the random noise. The class of stochastic strings given by solutions to SPDEs is large and includes in particular (rescaled) Brownian sheets and Ornstein–Uhlenbeck sheets. Similarly to the procedure presented in Goldstein (2000), Santa-Clara & Sornette (2001) argue that it might also be useful to smoothen the string shocks further, so that they are particularly smooth in direction of time to delivery x. Again, this can be achieved by integrating a stochastic string shock with respect to its second component. Stochastic string shock models are true generalisations of the Heath et al. (1992) framework which do not lose the parsimonious structure of the original HJM model. Also, due to their general structure, string models can give rise to a variety of different correlation functions and, hence, are very flexible tools for modelling various stylised facts without needing many parameters.

The main element in the stochastic string model, see Santa-Clara & Sornette (2001, p. 159), is the term

$$\int_0^t \sigma_v(T-v)d_vZ(v,T-v) = \int_0^t\int_0^\infty \sigma_s(T-s)G(T-s,z)\eta(s,z)dzds, \tag{34}$$

where Z is a stochastic string shock, η is white noise, σ is an adapted process and G is the corresponding Green function. The derivation by Santa-Clara & Sornette (2001) is partly heuristic. However, rigorous mathematical meaning can be given to the integral in (34) by the Walsh (1986) concept of martingale measures, see Section 3.3.

This may be compared to a special case of our ambit process where the integration is carried out with respect to a Gaussian Lévy basis, i.e. by choosing

$$\int_0^t \sigma_v(T-v)d_vZ(v,T-v) = \int_0^t\int_0^\infty \sigma_s(T-s)G(T-s,z)d_sW_s(dz).$$

So, for a deterministic function σ the product of σ and G is what we can model by the function h in the ambit framework, i.e.

$$h(\xi,s,T) = \sigma_s(T-s)G(T-s,\xi).$$

The main difference between the approach advocated in the present paper and the stochastic string shock approach lies in the fact that the ambit fields focused on here are constructed as stationary processes in time where the integration of the temporal component starts at $-\infty$ and not at 0 and, also, we consider general Lévy bases with a wide range of infinitely divisible distributions and do not restrict ourselves to the continuous Gaussian case. Finally, we provide a mathematically rigorous framework for defining the fields of forward prices.

5.4 Audet et al. (2004) model

Consider the model by Audet et al. (2004) written in the Musiela parameterisation. They study the electricity market on a finite time horizon $[0,T^*]$ and model the dynamics of the forward price $f_t(x)$ by

$$df_t(x) = f_t(x)e^{-\alpha x}\sigma_{x+t}dB_{x+t}(t),$$

for a deterministic, bounded volatility curve $\sigma : [0,T^*] \to \mathbb{R}_+$, a constant $\alpha > 0$ and where B_{x+t} denotes a Brownian motion for the forward price with time of maturity $x+t$. Further, the correlation structure between the Brownian motions is given by

$$\mathrm{corr}(dB_{x'}(t), dB_x(t)) = \exp(-\rho(x-x'))\,dt = \exp(-\rho|T-T'|)\,dt, \quad \text{for all } 0 \le x, x' \le T^*-t, \tag{35}$$

where $x' = T'-t, x = T-t$. Such a model implies that the volatility of the forward price is lower than the volatility of the spot price, an effect which is described by the parameter α. Also, forward contracts which are close in maturity can be modelled to be strongly correlated, an effect which is reflected by the choice of the parameter ρ.

We observe that the above model for the logarithmic forward price is in fact another special case of an ambit process, with deterministic volatility and an ambit set $\mathcal{A}_t(x) = [0,t] \times \{x\}$, and the Lévy basis being a Gaussian random field which is Brownian in time and has a spatial correlation structure in space as specified in (35).

6 Martingale conditions and option pricing

We have introduced the model for the forward price under the risk neutral probability measure. We have already mentioned that a martingale condition for an electricity forward contract is not absolutely necessary. In fact, there are at least two arguments which can be brought forward to support the choice of more general classes of stochastic processes than (semi-) martingales.

First, in the energy context it might not be as crucial that $f_t(T-t)$ is a martingale as it is in the context of modelling interest rates. In fact, as already indicated in the Introduction, one can argue that from a liquidity point of view, it would be possible to use non–martingales for modelling forward prices since in many emerging electricity markets, one may not be able to find any buyer to get rid of a forward, nor a seller when one wants to enter into one. Hence, the illiquidity prevents possible arbitrage opportunities from being exercised.

Second, independently of the particular structure of energy markets, the recent literature in mathematical finance, see e.g. Schachermayer (2004), Guasoni et al. (2008) has highlighted that some classes of non–semimartingales, in particular, stochastic processes with conditional full support, do not necessarily give raise to arbitrage opportunities when more realistic market characteristics, such as the existence of transaction costs, are taken into account. In the null–spatial setting Pakkanen (2011) has shown that $\mathcal{BSS}$ processes have in fact conditional full support. In future research it will hence be interesting to study extensions of this result to the (Gaussian) ambit framework.

However, the question of establishing martingale conditions for ambit fields is nevertheless interesting and will be studied in the following so that we can get a better understanding which classes of ambit fields form a subclass of models suitable for classical modelling of general forward prices (not necessarily restricted to electricity forward contracts).

6.1 Martingale conditions

We need to formulate conditions which ensure that the forward price under the risk–neutral P^*–measure becomes a (local) martingale. In the standard HJM framework in interest rate theory the martingale condition is stated as a drift condition on the dynamics. However, here we have an explicit dynamics, and the (semi-) martingale property is connected to the regularity of the input in the stochastic integral.

First, we will formulate the martingale conditions for more general ambit fields as defined in (3), where the ambit set $A_t(x) = A_t$ is chosen as in (26). Next, we show how such conditions simplify in the new modelling framework described in (25).

Note that all proofs will be given in the appendix.

Theorem 3. *Let $x = T - t$ for some $T > 0$ and for a fixed $t \in \mathbb{R}$ write*

$$Y_t(x) = Y_t(T-t) = \int_{A_t} h(\xi, s, T-t, t)\sigma_s(\xi)L(d\xi, ds), \quad \text{where } A_t = \{(\xi, s) : \xi > 0, s \leq t\},$$

for a deterministic kernel function h, an adapted, non–negative random field σ and a Lévy basis L satisfying both Assumptions (A1) and (A2) on an unbounded domain and (A3), see Section 3.3. Further σ and L are assumed to be independent.

Then $(Y_t(T-t))_{t\in\mathbb{R}}$ is a martingale w.r.t. $\{\mathcal{F}_t\}_{t\in\mathbb{R}}$ if and only if for all $\xi > 0$, $s \leq t \leq T$ we have

$$h(\xi, s, T-t, t) = \tilde{h}(\xi, s, T), \tag{36}$$

for some deterministic kernel function $\tilde{h}$.

Remark 8. If we would like to work with Lévy bases L which do not have zero mean, then the martingale conditions have to be extended by an additional drift condition.

Corollary 1. *In the special case of the new model defined in (25), we get that $(f_t(T-t))_{t\in\mathbb{R}}$ is a martingale w.r.t. $\{\mathcal{F}_t\}_{t\in\mathbb{R}}$ if and only if for all $\xi > 0$, $s \leq t \leq T$ we have*

$$k(\xi, t-s, T-t) = \tilde{k}(\xi, T-s)\,, \tag{37}$$

for a deterministic kernel function $\tilde{k}$. This is a special case of Factorisation 2, see (30).

Remark 9. Note that we have stated the martingale property for all t on the real line (which does not include $-\infty$). We refer to Basse-O'Connor et al. (2010) for a study on martingale properties at $-\infty$.

However, in practical terms, we are mainly interested in the martingale property for $t \geq 0$ since this is when the market is active. Negative times are *only* a modelling device in order to have stationary models.

Clearly, the martingale condition is rather strong and it is hence necessary to check whether there are actually any relevant cases left, which are not excluded by condition (37). Hence, let us study some examples.

First we show that the condition (37) covers the standard Heath et al. (1992) models, that come from stochastic partial differential equations.

Example 5. *The traditional way to model the forward dynamics using the Musiela parameterisation with $x = T-t$, is given by*

$$df_t(x) = \frac{\partial f_t}{\partial x}(x)\,dt + h(x,t)\,dW_t\,,$$

where, for simplicity, we disregard any spatial dependency in the Gaussian field W so that it is indeed a Brownian motion. Under appropriate (weak) conditions, the mild solution of this stochastic partial differential equation (SPDE) is given by

$$f_t(x) = S_t f_0(x) + \int_0^t S_{t-s}h(x,s)\,dW_s\,,$$

where S_t is the right–shift operator, $S_t g(x) = g(x+t)$, see Carmona & Tehranchi (2006), Da Prato & Zabczyk (1992) for more details. Hence,

$$f_t(x) = f_0(x+t) + \int_0^t h(s,(t+x)-s)\,dW_s = f_t(T-t) = f_0(T) + \int_0^t h(s, T-s)\,dW_s.$$

Thus, we see that the martingale condition (37) is satisfied.

Another important example is motivated by the Audet et al. (2004) model.

Example 6. *In our modelling framework defined in (25), we choose k to be of the form*

$$k(\xi, t-s, x) = k(\xi, t-s, T-t) = \exp(-\alpha((\xi+x)+(t-s))) = \exp(-\alpha((\xi+T-s))),$$

for some $\alpha > 0$. Then the martingale condition is clearly satisfied. Note that this choice of the kernel function belongs to both the class of Factorisation 1 and of Factorisation 2.

Further important examples of kernel functions which satisfy the martingale condition can be constructed as follows.

Example 7. *We can focus on kernel functions k which factorise as in (30), i.e.*

$$k(\xi, t-s, x) = \Psi(t-s, x)\Phi(\xi).$$

Clearly, the choice of the function Φ does not have any impact on the question whether the ambit field is a martingale. This is determined by the choice of the function Ψ.

As mentioned in Section 4, every choice of the form $\Psi(t-s,T-t)=\widetilde{\psi}(t-s+T-t)=\widetilde{\psi}(T-s)$ *satisfies the martingale condition. Motivated from the Bjerksund et al. (2010) model, see also Example 3, we could choose*

$$\Psi(t-s,x)=\frac{a}{t-s+x+b}=\frac{a}{T-s+b},$$

for $a,b>0$*.*

Moreover, motivated by the CARMA models discussed in Example 2, the following choice of Ψ *is also interesting:*

$$\Psi(t-s,x)=\mathbf{b}'\exp(A(t-s+x)\mathbf{e}_p\,,$$

for the p*–dimensional vector* $\mathbf{b}'=(b_0,b_1,\ldots,b_{p-1})$*, where* $b_q=1$ *and* $b_j=0$ *for* $q<j<p$*, with* $\mathbf{e}_k$ *being the* k*th canonical unit vector in* $\mathbb{R}^p$ *and where the matrix* A *is defined as in (31).*

So we have seen that it is possible to formulate martingale conditions for ambit fields and we have studied some relevant examples of forward price models which satisfy the martingale condition, which implies that we cannot have t–dependence in the kernel function.

6.2 Option pricing

We review briefly how to price options based on forward contracts with a price dynamics given by an ambit field. To this end, suppose we place ourselves in the risk-neutral context, and assume that the forward price at time $t\geq 0$ of a contract maturing at time $T\geq t$ is

$$f_t(T-t)=\int_{-\infty}^{t}\int_{0}^{\infty}k(\xi,t-s,T-t)\sigma_s(\xi)L(d\xi ds),$$

with the kernel function k satisfying the martingale condition of Thm. 3. Given a measurable function $g:\mathbb{R}\mapsto\mathbb{R}$, consider the problem of pricing a European option which pays $g(F_\tau(T))$ at exercise time $\tau\leq T$. From the arbitrage theory, we find that the price of this option at time $t\leq\tau$ is

$$C(t)=\mathrm{e}^{-r(\tau-t)}\mathbb{E}\left[g(f_\tau(T-\tau))\,|\,\mathcal{F}_t\right]\,. \tag{38}$$

Here, the constant $r>0$ is the risk–free interest rate. For C to be well–defined, we must suppose that $g(F_\tau(T-\tau))$ is integrable.

Since the cumulant function of the ambit field is available (see Thm. 1), the Fourier–based pricing method is an attractive approach (see Carr & Madan (1998)). If $g,\widehat{g}\in L^1(\mathbb{R})$, with $\widehat{g}$ being the Fourier transform of g, we can express the price of the option as

$$C(t)=\frac{1}{2\pi}\int_{\mathbb{R}}\widehat{g}(z)\mathbb{E}\left[\mathrm{e}^{\mathrm{i}zf_\tau(T-\tau)}\,|\,\mathcal{F}_t\right]\,dz. \tag{39}$$

Here, we make use of the integral representation of the inverse Fourier transform, see Folland (1984). Thus, to find $C(t)$, we must compute the conditional cumulant function of Y.

First, we split the ambit field to get

$$f_\tau(T-\tau)=\int_{-\infty}^{t}\int_{0}^{\infty}k(\xi,\tau-s,T-\tau)\sigma_s(\xi)L(d\xi ds)+\int_{t}^{\tau}\int_{0}^{\infty}k(\xi,\tau-s,T-\tau)\sigma_s(\xi)L(d\xi ds).$$

The first integral on the right hand side is $\mathcal{F}_t$-measurable. Hence,

$$\begin{aligned}\mathbb{E}\left[\mathrm{e}^{\mathrm{i}zf_\tau(T-\tau)}\,|\,\mathcal{F}_t\right]&=\exp\left(\mathrm{i}z\int_{-\infty}^{t}\int_{0}^{\infty}k(\xi,\tau-s,T-\tau)\sigma_s(\xi)L(d\xi ds)\right)\\&\quad\times\mathbb{E}\left[\exp\left(\mathrm{i}z\int_{t}^{\tau}\int_{0}^{\infty}k(\xi,\tau-s,T-\tau)\sigma_s(\xi)L(d\xi ds)\right)\,|\,\mathcal{F}_t\right].\end{aligned}$$

The conditional expectation can be expressed analogously as in Thm. 1. Note that the option price will not depend explicitly on $f_t(T-t)$.

Many relevant payoff functions g will not be in $L^1(\mathbb{R})$. For example, the payoff of a call option $g(x) = \max(x-K, 0)$ will fail to satisfy this condition. In such circumstances, one can dampen the payoff function by some exponential, and use the same procedure as above (see Carr & Madan (1998) for more details, including examples). In case of geometric forward price models, we apply the machinery above to the payoff function $h(x) = g(\exp(x))$.

7 Change of measure

If our forward price model is formulated under a risk–neutral pricing measure, it is of interest to understand how to get back to the physical measure in order to have a model for the observed prices. We will introduce an Esscher transform to accommodate this.

Throughout this section we will assume that the Lévy basis is *homogeneous* to simplify the notation.

In order to define the change of measure we work on a market with finite time horizon $T^* > 0$, hence we define our model on $\mathbb{R}_{T^*} = (-\infty, T^*]$ rather than on $\mathbb{R}$.

Theorem 4. *Define the process*

$$M_t^\theta = \exp\left(\int_{A_t} \theta(s,\xi)\, L(d\xi, ds) - \int_{A_t} C\{-i\theta(s,\xi) \ddagger L'\}\, d\xi\, ds\right), \tag{40}$$

where $C\{\cdot \ddagger L'\}$ is the characteristic exponent of the seed of L (and related to $C\{\cdot \ddagger L\}$ through equation (9)). The deterministic function $\theta : A_t \mapsto \mathbb{R}_{T^*}$ is supposed to be integrable with respect to the Lévy basis L in the sense of Walsh. Assume that

$$\mathbb{E}\left(\exp\left(\int_{A_t} C\{-i\theta(s,\xi) \ddagger L'\}\, d\xi\, ds\right)\right) < \infty, \text{ for all } t \in \mathbb{R}_{T^*}. \tag{41}$$

Then M_t^θ is a martingale with respect to $\mathcal{F}_t$ with $\mathbb{E}[M_0^\theta] = 1$.

The proof of the previous theorem is straightforward and, hence, omitted. We use that result now in order to define an equivalent probability P by

$$\left.\frac{dP}{dP^*}\right|_{\mathcal{F}_t} = M_t^\theta, \tag{42}$$

for $t \geq 0$. Hence, we have a change of measure from the risk neutral probability P^* under which the forward price is defined to a real world probability P. In effect, the function θ is an additional parameter to be modelled and estimated, and it will play the role as the *market price of risk*, as it models the difference between the risk–neutral and objective price dynamics.

We compute the characteristic exponent of an integral of L under P.

Theorem 5. *For any $v \in \mathbb{R}$, and Walsh–integrable function f with respect to L, it holds that*

$$\begin{aligned}
&C_{\mathbb{P}}\left\{v \ddagger \int_{A_t} f(s,\xi) L(d\xi, ds)\right\} = \log \mathbb{E}_P\left[\exp(iv \int_{A_t} f(s,\xi) L(d\xi, ds)\right] \\
&= \log \mathbb{E}\left[\exp\left(\int_{A_t} (ivf(s,\xi) + \theta(s,\xi))\, L(d\xi, ds)\right)\right] \exp\left(-\int_{A_t} C\{-i\theta(s,\xi)) \ddagger L'\}\, d\xi\, ds\right) \\
&= \int_{A_t} \left(C\{vf(s,\xi) - i\theta(s,\xi) \ddagger L'\} - C\{-i\theta(s,\xi) \ddagger L'\}\right)\, d\xi\, ds\,.
\end{aligned}$$

Note that the transform above is a simple generalization of the Esscher transform of Lévy processes, see Shiryaev (1999), Benth, Šaltytė Benth & Koekebakker (2008) and Barndorff-Nielsen & Shiryaev (2010) for more details on this aspect.

8 Constructing the spot model from the forward model

After having studied the new model for the forward price, we investigate in detail the nature of the spot price model implied by our new modelling framework for the forward price.

Note that this study should be understood as a theoretical exercise for now, since we typically do *not* observe convergence of the electricity forward price to the electricity spot price. However, Section 8.4 will explain how the results in this Section can be adapted to the corresponding empirical findings.

By the no–arbitrage assumption, the forward price for a contract which matures in zero time, $x = 0$, has to be equal to the spot price, that is, $f_t(0) = S_t$. Thus,

$$S_t = \int_{-\infty}^{t} \int_{0}^{\infty} k(\xi, t-s, 0)\sigma_s(\xi)L(d\xi, ds). \tag{43}$$

We have the following Lemma:

Lemma 1. *Suppose that*

$$\lim_{x \downarrow 0} \int_{-\infty}^{t} \int_{0}^{\infty} (k(\xi, t-s, x) - k(\xi, t-s, 0))^2 \mathbb{E}[\sigma_s^2(\xi)]\, d\xi ds = 0,$$

then $f_t(x) \to S_t$ in $L^2(P^*)$ as time to maturity x tends to zero.

Proof. This follows readily by appealing to the Itô-type isometry for Walsh integrals, see (15). ☐

The Lemma gives us that the forward price will tend continuously in variance to the spot price as time to maturity decreases to zero. Note that whenever $\sigma_s(\xi)$ is a stationary field, the condition in the Lemma is translated to a convergence of $k(\cdot, \cdot, x)$ to $k(\cdot, \cdot, 0)$ in $L^2(\mathbb{R}_+^2)$.

8.1 The general case

Note that the spot price process implied by our ambit field–based forward price model is driven by a tempo–spatial Lévy base, more precisely by a two–parameter random field and not just by a Brownian motion or a Lévy process. In fact, S_t is a superposition of $\mathcal{LSS}$ spot models, in the same sense as superposition of OU processes. Furthermore, S_t is a stationary process if $\sigma_s(\xi)$ is stationary in the temporal dimension s.

Similarly to the result for the forward price, see Theorem 2, we can derive the conditional cumulant function for the implied spot price:

Theorem 6. *Let L be a homogeneous Lévy basis. Then, for S_t as defined by (43),*

$$\mathrm{C}^{\sigma} \{\zeta \ddagger S_t\} = \int_{-\infty}^{t} \int_{0}^{\infty} \mathrm{C}\left\{\zeta k\left(\xi, t-s, 0\right)\sigma_s\left(\xi\right) \ddagger L'\right\} \mathrm{d}\xi \mathrm{d}s, \tag{44}$$

where L' is the Lévy seed associated with L.

8.2 The Gaussian case

A case of some special interest is the situation where the driving Lévy basis of the ambit field is a homogeneous Gaussian Levy basis. Then we get the following result.

Corollary 2. *In the Gaussian case, where $C\{\zeta \ddagger L'\} = -\frac{1}{2}\zeta^2$.*

$$\mathrm{C}^{\sigma} \{\zeta \ddagger S_t\} = -\frac{1}{2}\zeta^2 \int_{-\infty}^{t} \int_{0}^{\infty} k^2\left(\xi, t-s, 0\right)\sigma_s^2\left(\xi\right) d\xi ds.$$

If k factorises as in (29), then

$$\mathrm{C}^{\sigma}\left\{\zeta \ddagger S_t\right\} = -\frac{1}{2}\zeta^2 \int_{-\infty}^{t} \psi^2\left(t-s\right)\omega_s^2 ds, \qquad \text{where} \qquad \omega_s^2 = \int_0^{\infty} \phi^2(\xi, 0)\sigma_s^2(\xi) d\xi.$$

This implies that

$$S_t \stackrel{law}{=} \int_{-\infty}^{t} \psi^2(t-s)\omega_{s-} dW_s, \tag{45}$$

where W is a Brownian motion. The latter is indeed an $\mathcal{LSS}$ process, or more precisely, a Brownian semistationary ($\mathcal{BSS}$) process. Such processes have been used as a model for energy spot prices in Barndorff-Nielsen et al. (2010).

8.2.1 A concrete example

Let us assume that the kernel function factorises as in Factorisation 1, see (29). Then, in particular, we have

$$k^2(\xi, t-s, x) = \phi^2(\xi, x)\psi^2(t-s). \tag{46}$$

Now, let W in (25) be a standard Brownian motion and assume that σ is continuous at $\xi = 0$. In the case that $\phi^2(\xi, x)dx$ converges weakly to the delta measure at 0, we expect to have

$$\mathrm{C}^{\sigma}\left\{\zeta \ddagger S_t\right\} = -\frac{1}{2}\zeta^2 \int_{-\infty}^{t} \psi^2\left(t-s\right)\sigma_s^2(0) ds,$$

and hence

$$S_t \stackrel{law}{=} \int_{-\infty}^{t} \psi^2(t-s)\sigma_{s-}(0) dW_s. \tag{47}$$

That is the forward price implies an $\mathcal{BSS}$–based model for the spot price. Here we should recall that if e.g. the stochastic volatility field in the forward price is given by an an OUTVF defined in (33), where $\sigma_s(\xi) \to \sigma_s(0)$ with $\sigma_s^2(0)$ being an Ornstein–Uhlenbeck process, then the stochastic volatility of the spot price process would be given by an Ornstein–Uhlenbeck process.

As a concrete example, suppose that

$$\psi(t-s) = \alpha e^{-\alpha(t-s)}, \qquad\qquad \phi(\xi, x) = p(\xi; x, \gamma),$$

where

$$p(\xi; x, \gamma) = \frac{\gamma x}{\sqrt{2\pi}} e^{\gamma^2 x} \xi^{-3/2} e^{-\frac{1}{2}\gamma^2(x^2\xi^{-1}+\xi)}$$

i.e. the inverse Gaussian density with mean x and variance $x\gamma^{-2}$

Then, we get the forward–spot relation described above and, further, we obtain an explicit formula for the correlation between forward contracts with different times to delivery x and $\tilde{x}$:

$$\begin{aligned}
Cor(f_t(x), f_t(\tilde{x})) &= \int_0^{\infty} \left[p(\xi; x, \gamma)p(\xi; \tilde{x}, \gamma)\right]^{1l2} \mathrm{d}\xi \\
&= \frac{\gamma\sqrt{x\tilde{x}}}{\sqrt{2\pi}} e^{\frac{1}{2}\gamma^2(x+\tilde{x})} \int_0^{\infty} \xi^{-3/2} e^{-\frac{1}{4}\gamma^2((x^2+\tilde{x}^2)\xi^{-1}+2\xi)} \mathrm{d}\xi \\
&= \frac{\gamma\sqrt{x\tilde{x}}}{\sqrt{2\pi}} e^{\frac{1}{2}\gamma^2(x+\tilde{x})} \int_0^{\infty} \xi^{-3/2} e^{-\frac{1}{2}\gamma^2(\frac{x^2+\tilde{x}^2}{2}\xi^{-1}+\xi)} \mathrm{d}\xi \\
&= \frac{\gamma\sqrt{x\tilde{x}}}{\sqrt{2\pi}} e^{\frac{1}{2}\gamma^2(x+\tilde{x})} \frac{\sqrt{2\pi}}{\gamma\sqrt{\frac{x^2+\tilde{x}^2}{2}}} e^{-\gamma^2\sqrt{\frac{x^2+\tilde{x}^2}{2}}} \\
&= \frac{\sqrt{x\tilde{x}}}{\sqrt{\frac{x^2+\tilde{x}^2}{2}}} e^{-\frac{1}{2}\gamma^2(2\sqrt{\frac{x^2+\tilde{x}^2}{2}}-(x+\tilde{x}))}.
\end{aligned}$$

Note that for any fixed x this expression tends to 0 both for $\tilde{x} \to \infty$ and for $\tilde{x} \to 0$. Hence, the forward price $f_t(x)$ is uncorrelated with the forward price in the long end of the curve, $f_t(\infty)$, as well as with the spot (corresponding to $\tilde{x} = 0$). As is well–known in electricity markets, there exists no spot–forward relationship derived from a buy–and–hold strategy due to non–storability. This model may serve as an extreme case of such a situation, where forward and spot are perfectly uncorrelated statistically.

8.3 Relation to the Samuelson effect

Recall that the Samuelson effect describes the empirical fact that the volatility of the forward price increases and converges to the volatility of the spot price when the time to maturity approaches zero.

This finding is in fact naturally included in our modelling framework which we will show in the following.

Theorem 7. *Assume that the function $x \mapsto k(\xi, u, x)$ is monotonically non–decreasing in $x \geq 0$ for every $(\xi, u) \in \mathbb{R}_+^2$. Then the conditional variance of the forward price $f_t(x)$, given by*

$$v_t(x) := c \int_{-\infty}^{t} \int_{0}^{\infty} k^2(\xi, t-s, x) \sigma_s^2(\xi) d\xi ds,$$

is monotonically non–decreasing in x, for $t \geq 0$. Here, c is a suitable constant $c = b + \kappa_2$ defined in the Appendix.

The proof is given in the Appendix. Note that the conditional variance of the spot is given by $v_t(0)$, and it follows from the Theorem above that $v_t(x) \leq v_t(0)$. As a monotonically increasing sequence being bounded by $v_t(0)$, there exist a limit $\lim_{x \downarrow 0} v_t(x) \leq v_t(0)$. Under the condition in Lemma 1, this limit will be $v_t(0)$, the spot price conditional variance. That is, we have a Samuelson effect. Note that we have this effect also for non–stationary models, since we do not require $\sigma_s(\xi)$ to be stationary, for example.

8.4 Accounting for the fact that the electricity forward typically does not converge to the spot

We have previously discussed how a spot model can be constructed from our general forward model. However, it is well–known that there is no convergence of electricity forward prices to the spot as time to *start of delivery* approaches. That is, if the delivery period is $[T_1, T_2], T_1 < T_2$, then the forward price $F_t(T_1, T_2)$ at time t does not converge to the spot price as $t \to T_1$. One could mimic such a behaviour with the model class we study here, by choosing the 'delivery time' T as the mid–point, say, in the delivery interval $[T_1, T_2]$, $T = (T_1 + T_2)/2$. Then we can still associate a spot price to the forward dynamics $f_t(x)$, but we will never actually observe the convergence in the market since at start of delivery we have $x = (T_2 - T_1)/2$. On the other hand, we will get a model where there is an explicit connection between the forward at "maturity" $t = T_1$ and the spot Y_{T_1}. This opens for modelling spot and forward jointly, taking into account their dependency structure.

9 Simulation study

In this section, we will discuss how to simulate an ambit field. Recall that the (deseasonalised) forward price is modelled as

$$Y_t(x) := f_t(x) = \int_{-\infty}^{t} \int_{0}^{\infty} k(\xi, t-s, x) \sigma_s(\xi) L(d\xi, ds),$$

where

$$\sigma_t^2(x) = V\left(\int_{-\infty}^{t}\int_0^{\infty} j(\xi, t-s, x)L^{\sigma}(d\xi, ds)\right),$$

where all quantities are as defined above.

In the following, we will describe how such an ambit field can be simulated.

Note that we describe the simulation for a fixed t and fixed x. An extension to the case, where one simulates the entire forward curve for various values of t and x is then straightforward.

9.1 Simulation algorithm

We note that we integrate over the ambit set $A_t(x) = (-\infty, t] \times [0, \infty)$, which is unbounded. Hence, in a first step we need to truncate the ambit set, and we define the corresponding ambit field by

$$Y_t^{trunc}(x) = \int_{M_1}^{t}\int_0^{M_2} k(\xi, t-s, x)\sigma_s(\xi)L(d\xi, ds),$$

for constants $M_1 < t$ and $M_2 > 0$. Note that letting M_1 tend to $-\infty$ and M_2 to $+\infty$, $Y_t^{trunc}(x)$ will converge to $Y_t(x)$.

Next, we construct a grid for the interval $[M_1, t] \times [0, M_2]$ by dividing the temporal dimension $[M_1, t]$ into n equidistant intervals of length $(t - M_1)/n$, where we write $t = t_1 > t_2 > \cdots > t_n = M_1$, and by dividing the spatial dimension $[0, M_2]$ into m equidistant intervals of length M_2/m, where we write $0 = x_1 < x_2 < \cdots < x_m = M_2$, for $n, m \in I\!N$.

1. Simulate the stochastic volatility field on the grid points (t_i, x_j) for $i = 1, \ldots, n$ and $j = 1, \ldots, m$. We obtain the values $\sigma_{t_i}(x_j)$. In the absence of stochastic volatility, we set $\sigma_{t_i}(x_j) = 1$ for all i, j.

2. Simulate $n \cdot m$ i.i.d. random variables $Z_{i+j} \stackrel{d}{=} L(\Delta)$, where $i = 1, \ldots, n$, $j = 1, \ldots, m$ and

$$\Delta := \Delta(n, m, M_1, M_2) = \frac{(t - M_1)}{n}\frac{M_2}{m}.$$

3. We approximate $Y_t^{trunc}(x)$ by

$$\widehat{Y}_t^{trunc}(x) := \sum_{i=1}^{n-1}\sum_{j=1}^{m-1} k(x_j, t - t_i, x)\sigma_{t_i}(x_j)Z_{i+j}.$$

The last step, Step 3, of the simulation algorithm makes use of the definition of the stochastic integral in the sense of Walsh for simple processes. This allows us to represent the stochastic integral of $Y_t^{trunc}(x)$ as $\widehat{Y}_t^{trunc}(x)$ in a discretized form, ensuring convergence when Δ goes to zero.

Remark 10. Note that Δ is the area of each rectangular $[t_i, t_{i+1}] \times [x_j, x_{j+1}]$ for $i = 1, \ldots, n-1,\ j = 1, \ldots, m-1$ on the grid. (Hence, we implicitly work with the Lebesgue measure in the specification of the ambit field. In the case that $L = W$ is a Gaussian Lévy basis, with characteristic triplet $(\mu\, leb(), c^2\, leb(), 0)$, we simulate $Z_{i+j} \sim i.i.d.\ N(\mu\Delta, c^2\Delta)$. In the case that L is an NIG Lévy basis we simulate $Z_{i+j} \sim i.i.d.\ NIG(\alpha, \beta, \mu\Delta, \delta\Delta)$.

In the presence of stochastic volatility, we need to use the same procedure as described above for simulating the stochastic volatility field first. I.e. for each grid point (t_i, x_j) we define a truncated interval for the ambit set of the volatility field by $[M_1(i), t_i] \times [0, M_2(j)]$ for constants $M_1(i) < t_i,\ 0 < M_2(j)$. Next, we divide the temporal dimension $[M_1(i), t(i)]$ into $n(i)$ equidistant intervals of length $(t(i) - M_1(i))/n(i)$, where we write $t_i = \tau_1(i) > \cdots > \tau_{n(i)}(i) = M_1(i)$, and by dividing the spatial dimension $[0, M_2(j)]$ into $m(j)$ equidistant intervals of length $M_2(j)/m(j)$, where we write $0 = \xi_1(j) < \cdots < \xi_{m(j)}(j) = M_2(j)$, for $n(i), m(j) \in I\!N$.

Then, we approximate

$$\widehat{\sigma}^2_{t_i}(x_j) = V\left(\sum_{\nu=1}^{n(i)-1}\sum_{\nu'=1}^{m(j)-1} j(\xi_{\nu'}(j), t - \tau_\nu(i), x_j) Z^\sigma_{\nu+\nu'}\right),$$

where $Z_{\nu+\nu'} \stackrel{d}{=} L^\sigma(\Delta)$.

With these general simulation algorithms at hand, let us consider two specific examples: First, suppose that the kernel function k is a weighted sum of two exponential functions, i.e.

$$k(\xi, t-s, x) = k(\xi + t - s + x) = w\exp(-\lambda_1(\xi + t - s + x)) + (1-w)\exp(-\lambda_2(\xi + t - s + x)),$$

for $w \in [0,1]$ and $\lambda_1, \lambda_2 > 0$. Note that k satisfies the martingale condition. This choice of kernel function is motivated by the empirical studies in Barndorff-Nielsen et al. (2010), where such a specification in an $\mathcal{LSS}$–model was shown to fit spot price data collected from the German power exchange EEX. In the same study, one found that a NIG Lévy process was a successful choice, and inspired by this we suppose that L is a NIG Lévy basis. Finally, suppose that V is equal to one, which means that we do not have any volatility process in our ambit field specification.

Due to the specification of the kernel function, we can split the ambit field into two parts,

$$Y_t(x) = wY^1_t(x) + (1-w)Y^2_t(x),$$

where

$$Y^i_t(x) = \int_{-\infty}^t\int_0^\infty \mathrm{e}^{-\lambda_i(\xi+t-s+x)}\,L(d\xi ds) = \mathrm{e}^{-\lambda_i(t+x)}\int_{-\infty}^t\int_0^\infty \mathrm{e}^{-\lambda_i(\xi-s)}\,L(d\xi ds),$$

for $i = 1, 2$. We immediately see that, for $\Delta_t > 0$,

$$Y^i_{t+\Delta_t}(x) = \mathrm{e}^{-\lambda_i\Delta_t}Y^i_t(x) + \mathrm{e}^{-\lambda_i\Delta_t}\mathrm{e}^{-\lambda_i(t+x)}\int_t^{t+\Delta_t}\int_0^\infty \mathrm{e}^{-\lambda_i(\xi-s)}\,L(d\xi ds).$$

For small Δ_t, the last integral can be approximated by

$$\int_t^{t+\Delta_t}\int_0^\infty \mathrm{e}^{-\lambda_i(\xi-s)}\,L(d\xi ds) \approx \mathrm{e}^{\lambda_i t}\int_0^\infty \mathrm{e}^{-\lambda_i\xi}\,L(d\xi\times\Delta_t),$$

to get the iterative Euler–like time–stepping scheme

$$Y^i_{t+\Delta_t}(x) \approx \mathrm{e}^{-\lambda_i\Delta_t}Y^i_t(x) + \mathrm{e}^{-\lambda_i(x+\Delta_t)}\int_0^\infty \mathrm{e}^{-\lambda_i\xi}\,L(d\xi\times\Delta_t). \tag{48}$$

The integral over ξ can be computed numerically by a Riemann–like approximation as in the general case above. We note that we can iterate numerically over space as well, since for Δ_x we have the equality

$$Y^i_t(x+\Delta_x) = \mathrm{e}^{-\lambda_i\Delta_x}Y^i_t(x). \tag{49}$$

We make use of (48) and (49) to implement efficient numerical schemes for the simulation of the whole field $Y_t(x)$.

In an example, let $\lambda_1 = 0.012$, $\lambda_2 = 0.226$, and $w = 0.07$ in the kernel function specification. The NIG Lévy basis has parameters $\delta = 0.7$, $\alpha = 0.0556$, $\mu = \beta = 0$ (using the standard notation for the parameters in the NIG distribution). These figures are taken from the estimates of the $\mathcal{LSS}$ spot

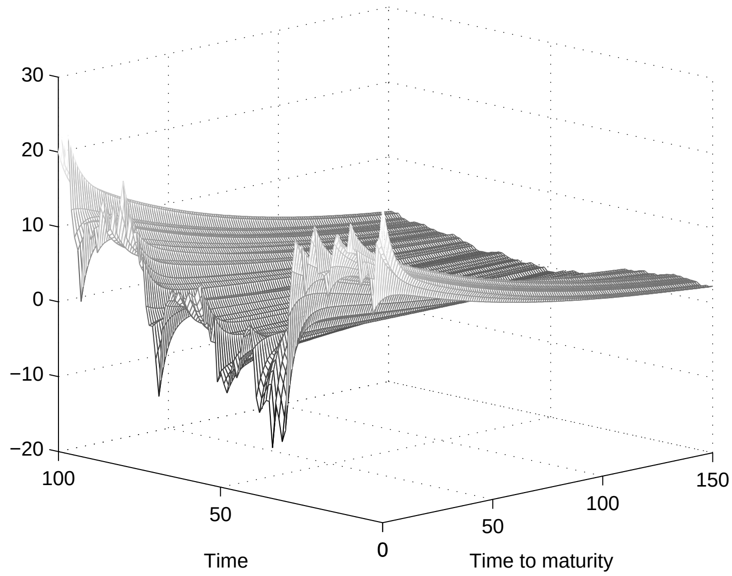

Figure 2: Simulated ambit field based on an NIG Lévy basis without stochastic volatility.

price model for EEX data found in Barndorff-Nielsen et al. (2010). We initialize $Y_0^i(0)$ by simulating backwards in time sufficiently long in order to reach stationarity. In Fig. 2 we show a simulated realization of the ambit field. Along the time axis for $x = 0$ one identifies a sample path of the implied spot dynamics from the model, whereas in the *time–to–maturity* axis x we see that the field is very smooth.

In our second example, we include a stochastic volatility in the ambit field specification of $Y_t(x)$, and exchange the NIG Lévy basis with a Gaussian basis, $L = W$. More specifically, we suppose that L^σ is an IG Lévy basis, V being the identity function and j being the exponential function, i.e. $j(\xi, t-s, x) = \exp(-\lambda^\sigma(\xi + t - s + x))$ for $\lambda^\sigma > 0$. Since the kernel function j is of exponential type, we can use the same considerations as above to obtain a time and space iterative scheme for simulation of $\sigma_t(x)$. A realization of the ambit field is shown in Fig 3, where we have used $\gamma = \alpha$, and the decay rate $\lambda^\sigma = 0.5$ as an example. Note that this stochastic volatility process is a spatial generalization of a so–called BNS model for the volatility process σ_t^2, defined as the stationary solution of an IG–Lévy process driven OU dynamics. The choice of speed of mean reversion equal to 0.5 will in that context yield a half–life of $\ln 2/0.5 \approx 1.4$, meaning that the volatility is fast mean reverting. This is in accordance with the empirical results found in Barndorff-Nielsen et al. (2010).

As we observe, the stochastic volatility case generates much higher variation in prices. This is due to the integration over the volatility in space, that creates a higher variation than the NIG ambit field model.

10 Extensions

We consider various extensions of our model, in particular, a geometric forward model and the question of how to model forwards with delivery period.

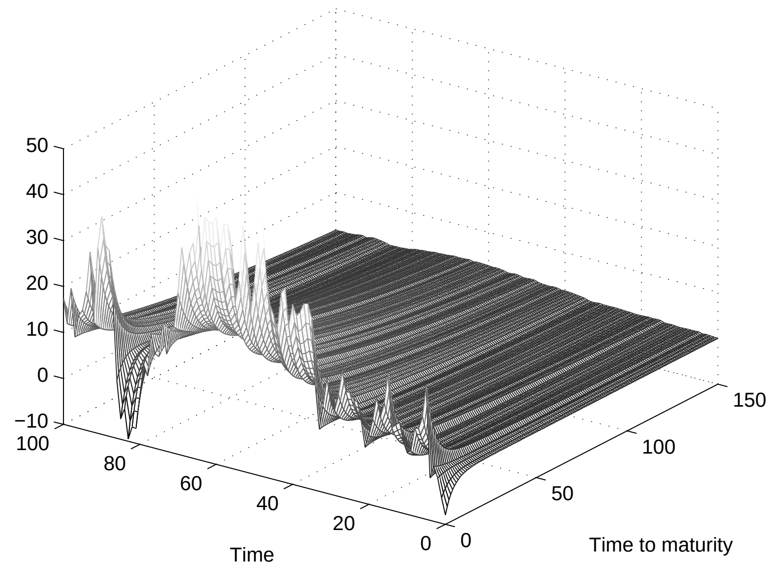

Figure 3: Simulated ambit field based on a Gaussian Lévy basis with a stochastic volatility component based on an IG Lévy basis.

10.1 Geometric modelling framework

So far, we have worked with an arithmetic model for the forward price since this is a very natural model choice and is in line with the traditional random field based models where the forward rate is directly modelled by e.g. a Gaussian random field. However, standard critical arguments include that such models can in principal produce negative prices and hence might not be realistic in practice. One way to overcome that problem would be to work with positive Lévy bases (recall that the kernel function and the stochastic volatility component in the ambit field are by definition positive). Clearly, in such a set–up we would have to relax the zero–mean assumption. But this is straightforward to do. An alternative and more traditional approach would be to work with geometric models, i.e. we model the forward price as the exponential of an ambit processes. Most of the results we derived before can be directly carried over to the geometric set–up. E.g. when we study the link between the forward price and the spot price, this has to be interpreted as the link between the logarithmic forward price and the logarithmic spot price. Likewise, when looking at probabilistic properties such as the moments and cumulants of the processes, they can be regarded as the moments/cumulants of the logarithmic forward price.

The only result, which indeed needs some adjustment, is in fact the martingale property. The condition on the kernel function h stays the same as in Theorem 3 when we go to the geometric model framework, but on top of that there will be an additional drift condition. In order to keep the exposition as simple as possible, we will focus on *homogeneous* Lévy bases, see Section 3.2, in this section.

Before we formulate the martingale condition, we specify an additional integrability assumption.

Assumption (A4) Let Y be defined as in (3), where we assume that L is a *homogeneous* Lévy basis and h satisfies the condition of Theorem 3. We assume that

$$\mathbb{E}\left(\exp\left(\int_{A_t} C\{-i\widetilde{h}(\xi,s,T)\sigma_s(\xi)\ddagger L'\}d\xi ds\right)\right) < \infty, \text{ for all } t \in \mathbb{R}.$$

Now we can formulate the martingale conditions for the geometric forward price model.

Theorem 8. *Let $A_t = \{(\xi, s) : s \leq t; x \geq 0\}$. Further, we assume that the integrability condition (A4) is satisfied. Then, the forward price at time t with delivery at time $t \leq T$, $f_t(T) = (f_t(T))_{t \leq T}$ with*

$$f_t(T) = \exp\left(\int_{A_t} \widetilde{h}(\xi, s, T)\sigma_s(\xi)L(d\xi, ds) - \int_{A_t} C\{-i\widetilde{h}(\xi, s, T)\sigma_s(\xi) \ddagger L'\}d\xi ds\right),$$

is a martingale with respect to $\{\mathcal{F}_t\}_{t\in\mathbb{R}}$.

Consider the example of a Gaussian Lévy basis:

Example 8. *In the special case that $L = W$ is a standardised, homogeneous Gaussian Lévy basis and that (A4) is satisfied, we have that*

$$f_t(T) = \exp\left(\int_{A_t} \widetilde{h}(\xi, s, T)\sigma_s(\xi)W(d\xi, ds) - \frac{1}{2}\int_{A_t} \widetilde{h}^2(\xi, s, T)\sigma_s^2(\xi)d\xi ds\right),$$

is a martingale with respect to $\{\mathcal{F}_t\}_{t\in\mathbb{R}}$.

10.2 Inference

In this section, we will sketch how to estimate the new model for the forward price based on an ambit field. A more detailed analysis is relegated to future research.

10.2.1 The case of constant volatility

In the absence of stochastic volatility, the estimation procedure for an ambit field is rather straightforward and can be carried out in two steps, as described in Jónsdóttir et al. (2011, Section 6).

First, we use the fact that the autocorrelation function of an ambit field defined in (25), but without stochastic volatility, is completely determined by the kernel function k, see Section A.4 in the Appendix for more details. Hence, given a concrete parametric specification for the kernel function k, one can estimate the corresponding parameters from the variogram of the observed random field, see Cressie (1993) for more details.

After having estimated the parameters of the kernel function, one can then proceed and estimate the parameters of a parametric specification of the Lévy basis, see Jónsdóttir et al. (2011) for more details.

Remark 11. Note that the estimation method described above works for a fully specified parametric model. However, one might also be interested in non–parametric estimation techniques – of the kernel function in particular. Brockwell et al. (2011*a*,*b*) have developed such a method for estimating the kernel function for continuous–time moving average processes. In future research, it will be interesting to study how such techniques can be extended to the tempo–spatial framework of ambit fields.

10.2.2 The case of stochastic volatility

As soon as we have a truly *stochastic* volatility field σ in the ambit field specification, inference becomes significantly more involved and the detailed estimation theory is beyond the scope of this paper. However, we still wish to state the main points which have to be addressed.

Note that in order to ensure the identifiability of the model we need to formulate restrictions for k, σ and L.

As before, we can use the variogram to estimate the autocorrelation function. However, in the general case that σ is stochastic, both the second moment of σ and its autocorrelation function enter in the autocorrelation function of the ambit field, see Theorem 9 in the Appendix. This makes it more difficult to identify the kernel function k. Additional assumptions (such as zero mean of the Lévy basis L and a simple parametric form of the second moment of σ) will help to solve that problem.

In a next step, one needs to estimate the stochastic volatility and the Lévy basis. A natural approach to tackle this problem would be to construct a non–parametric estimator of the stochastic volatility field, similar to realised quadratic variation for semimartingales in the null–spatial case, see e.g. Barndorff-Nielsen & Shephard (2002) and Barndorff-Nielsen, Corcuera & Podolskij (2011*a*,*b*). Then one can estimate the volatility parameters separately based on a non–parametric proxy, as it was done in the one–dimensional case in Todorov (2009), Veraart (2011). In order to follow this approach, it might be helpful to focus on the class of ambit fields which are indeed martingales first so that standard theory on quadratic variation is applicable. In a next step, extensions to the ambit fields which are not semimartingales can be studied.

10.3 Outlook on how to include period of delivery into the modelling framework

So far, we have focused on forward prices with fixed delivery time, i.e. on $f_t(x) = f_t(T-t)$. However, in energy markets, there is not just a time of delivery T, but typically a *delivery period*, i.e. at time of delivery $T = T_1$ a certain amount of electricity, say, gets delivered until time T_2 for some $T_2 \geq T_1$, see e.g. Benth, Šaltytė Benth & Koekebakker (2008, Chapter 6) and Barth & Benth (2010). The forward price $F_t(T_1, T_2)$ at time t with delivery period $[T_1, T_2]$ is defined by (see e.g. Benth, Šaltytė Benth & Koekebakker (2008))

$$F_t(T_1, T_2) = \frac{1}{T_2 - T_1} \int_{T_1}^{T_2} f_t(u-t)du.$$

Hence, given an ambit model of $f_t(x)$, we simply average it over the delivery period in order to have the forward price for a contract with delivery period.

Alternatively, we could think of modelling $F_t(T_1, T_2)$ directly – by an ambit field. The main idea here is to include the length of the delivery period $\tau := T_2 - T_1$ as an additional spatial component. E.g. we could think of using

$$\int_{A_t(x,\tau)} k(\xi, \chi, t-s, \tau, x)\sigma_s(\xi, \chi) L(d\xi, d\chi, ds),$$

as a building block for $F_t(T_1, T_2)$. The main obstacle in building such models is the no–arbitrage condition between contracts with overlapping delivery periods. In fact, any model for $F_t(T_1, T_2)$ must satisfy (see Benth, Šaltytė Benth & Koekebakker (2008))

$$F_t(T_1, T_2) = \frac{1}{T_2 - T_1} \int_{T_1}^{T_2} F_t(\tau, \tau)\, d\tau\,,$$

which puts serious restrictions on the degrees of freedom in modelling.

It will be interesting to study the analytical properties of such models in more detail in future research.

11 Conclusion

This paper presents a new modelling framework for electricity forward prices. We propose to use ambit fields which are special types of tempo–spatial random fields as the building block for the new modelling class. Ambit fields are constructed by stochastic integration with respect to Lévy bases and we have argued in favour of the integration concept of Walsh (1986) in the context of financial applications since it enables us to derive martingale conditions for the forward prices. Furthermore, we have shown that forward and spot prices can be linked to each other within the ambit field framework. Also, we have discussed relevant examples of model specifications within the new modelling framework and have related them to the traditional modelling concepts. In addition, we have discussed how a change of measure between the risk–neutral and physical probability measure can be carried out, so that our model can be used both for option pricing purposes and for statistical studies under the physical measure.

A natural next step to take is to test our new model empirically and to study statistical aspects related to our ambit–field models, such as model estimation and model specification tests etc.. We plan to address these issues in detail in future research.

Another interesting aspect, which we leave for future research, is to adapt our modelling framework for applications to the term structure of interest rates.

A Proofs and some further results

A.1 Explicit results for Example 1

Note that

$$\int_0^\infty \left(1-\sqrt{1+c^2\exp(-2\alpha(\xi-s))}\right)d\xi = -\frac{1}{2\alpha}\left[\left(2\sqrt{1+c^2\exp(2\alpha s)}-2\right)\right.$$
$$\left.+\left(2\log(2)-\log\left(\frac{\left(\sqrt{1+c^2\exp(2\alpha s)}+1\right)c^2\exp(2\alpha s)}{\sqrt{1+c^2\exp(2\alpha s)}-1}\right)\right)\right].$$

Hence, we get

$$\begin{aligned}
&\frac{-8\alpha^2}{\delta\gamma}\log(\mathbb{E}(ivf_t(x))) = -8+8\,\ln(2)-4\,\ln\left(c^2\right)-2\,(\ln(2))^2+4\,\ln(2)\ln\left(c^2\right)\\
&+8\sqrt{1+c^2e^{2\,t\alpha}}+4\,\ln\left(-1+\sqrt{1+c^2e^{2\,t\alpha}}\right)-4\,\ln\left(1+\sqrt{1+c^2e^{2\,t\alpha}}\right)\\
&+\left(\ln\left(-1+\sqrt{1+c^2e^{2\,t\alpha}}\right)\right)^2+4\,\mathit{dilog}\left(1/2+1/2\sqrt{1+c^2e^{2\,t\alpha}}\right)\\
&-4\,\ln\left(-1+\sqrt{1+c^2e^{2\,t\alpha}}\right)\ln(2)+2\,\ln\left(-1+\sqrt{1+c^2e^{2\,t\alpha}}\right)\ln\left(1+\sqrt{1+c^2e^{2\,t\alpha}}\right)\\
&-\left(\ln\left(1+\sqrt{1+c^2e^{2\,t\alpha}}\right)\right)^2-8\,t\alpha+8\,\ln(2)\,t\alpha-\left(\ln\left(c^2e^{2\,t\alpha}\right)\right)^2,
\end{aligned}$$

where the dilogarithm function is defined by $\mathrm{dilog}(t)=\int_1^t\frac{\log(x)}{1-x}dx$ for $t>1$.

A.2 Presence of the Samuelson effect

Proof of Theorem 7: For $x\geq 0$, we have

$$\upsilon_t(x)=c\int_{-\infty}^t\int_0^\infty k^2(\xi,t-s,x)\sigma_s^2(\xi)d\xi ds,$$

where $c:=(b+\kappa_2)$ with b,κ_2 defined as in Section A.4. Now let $0\leq x\leq x'$, then

$$\upsilon_t(x')-\upsilon_t(x)=\int_{-\infty}^t\int_0^\infty\left(k^2(\xi,t-s,x')-k^2(\xi,t-s,x)\right)\sigma_s^2 d\xi ds\geq 0,$$

due to the fact that $k(\xi,t-s,x)$ is non–decreasing in x. □

A.3 Proof of the martingale condition

Proof of Theorem 3. In addition to measurability and integrability (which is straightforward here), we have to show that

$$\mathbb{E}[Y_t(T-t)\,|\,\mathcal{F}_{\widetilde{t}}] = Y_{\widetilde{t}}(T-\widetilde{t})\,, \qquad \text{for all } \widetilde{t} \leq t\,.$$

Note that for $\widetilde{t} \leq t$, we have that $A_{\widetilde{t}} \subseteq A_t$. Using the independence property of σ and L and the fact that L is a zero mean process, we find

$$\begin{aligned}
&\mathbb{E}[Y_t(T-t)\,|\,\mathcal{F}_{\widetilde{t}}] \\
&= \mathbb{E}\left[\int_{A_{\widetilde{t}}} h(\xi,s,T-t,t)\sigma_s(\xi)L(d\xi,ds) + \int_{A_t\setminus A_{\widetilde{t}}} h(\xi,s,T-t,t)\sigma_s(\xi)L(d\xi,ds)\,\middle|\,\mathcal{F}_{\widetilde{t}}\right] \\
&= \int_{A_{\widetilde{t}}} h(\xi,s,t,T-t)\sigma_s(\xi)L(d\xi,ds) = Y_{\widetilde{t}}(T-\widetilde{t}) + I_{\widetilde{t}}(T-\widetilde{t}),
\end{aligned}$$

where

$$I_{\widetilde{t}}(T-\widetilde{t}) = \int_{A_{\widetilde{t}}} \left\{h(\xi,s,T-t,t) - h(\xi,s,T-\widetilde{t},\widetilde{t})\right\}\sigma_s(\xi)L(d\xi,ds)\,.$$

Without loss of generality we assume that $Var(L)=1$. Since L is a Lévy basis with zero mean, we know that $\mathbb{E}(I_{\widetilde{t}}(T-\widetilde{t}))=0$, and from the Itô isometry we therefore get that

$$Var(I_{\widetilde{t}}(T-\widetilde{t})) = \int_{A_{\widetilde{t}}} \left\{h(\xi,s,T-t,t) - h(\xi,s,T-\widetilde{t},\widetilde{t})\right\}^2 \mathbb{E}(\sigma_s^2(\xi))Q(d\xi,ds).$$

Thus, in order to obtain $I_{\widetilde{t}}(T-\widetilde{t})=0$, we need that for all $0\leq \xi$, $s\leq \widetilde{t}\leq t\leq T$

$$h(\xi,s,t,T-t) = h(\xi,s,\widetilde{t},T-\widetilde{t})\,. \tag{50}$$

When we look at condition (50) more closely, then we observe that there is in fact only one class of functions, which satisfy such a condition, i.e. functions of the form

$$h(\xi,s,T-t,t) = \tilde{h}(\xi,s,T),$$

for all $\xi\geq 0$, $s\leq\widetilde{t}\leq t\leq T$ for some deterministic kernel function $\tilde{h}$.

□

Proof of Theorem 8. We show that $M=(M_t)_{t\in\mathbb{R}}$ with $M_t=\exp(Y_t(T-t)-d_t)$ is a martingale with respect to $\{\mathcal{F}_t\}_{t\in\mathbb{R}}$ where

$$Y_t(T-t) = \int_{A_t} \widetilde{h}(\xi,s,T)\sigma_s(\xi)L(ds,d\xi), \qquad d_t = \int_{A_t} C\{-i\widetilde{h}(\xi,s,T)\sigma_s(\xi)\ddagger L'\}d\xi ds,$$

where $A_t=\{(\xi,s):s\leq t; x\geq 0\}$. Clearly, M is measurable with respect to $\{\mathcal{F}_t\}_{t\in\mathbb{R}}$ and also integrable due to the integrability assumption (A4). Further, for all $\widetilde{t}\leq t$, we have that

$$\begin{aligned}
&\mathbb{E}(M_t|\mathcal{F}_{\widetilde{t}}) = \mathbb{E}\left(\exp(Y_t(T-t)-d_t)|\,\mathcal{F}_{\widetilde{t}}\right) \\
&= \mathbb{E}\left(\exp\left(\int_{A_{\widetilde{t}}}\widetilde{h}(\xi,s,T)\sigma_s(\xi)L(ds,d\xi) + \int_{A_t\setminus A_{\widetilde{t}}}\widetilde{h}(\xi,s,T)\sigma_s(\xi)L(ds,d\xi) - d_{\widetilde{t}} + d_{\widetilde{t}} - d_t\right)\middle|\,\mathcal{F}_{\widetilde{t}}\right) \\
&= M_{\widetilde{t}}\,\mathbb{E}\left(\exp\left(\int_{A_t\setminus A_{\widetilde{t}}}\widetilde{h}(\xi,s,T)\sigma_s(\xi)L(ds,d\xi) - (d_t - d_{\widetilde{t}})\right)\middle|\,\mathcal{F}_{\widetilde{t}}\right).
\end{aligned}$$

Using the formula for the characteristic functions of integrals with respect to Lévy bases, see Rajput & Rosinski (1989) and Section 3.2, we get

$$\begin{aligned}
&\mathbb{E}\left(\exp\left(\int_{A_t \setminus A_{\widetilde{t}}} \widetilde{h}(\xi, s, T)\sigma_s(\xi) L(ds, d\xi)\right) \Big| \mathcal{F}_{\widetilde{t}}\right) \\
&= \mathbb{E}\left(\mathbb{E}\left(\exp\left(\int_{A_t \setminus A_{\widetilde{t}}} \widetilde{h}(\xi, s, T)\sigma_s(\xi) L(d\xi, ds)\right) \Big| \overline{\mathcal{F}}_{\widetilde{t}}\right) \Big| \mathcal{F}_{\widetilde{t}}\right) \\
&= \mathbb{E}\left(\exp\left(\int_{A_t \setminus A_{\widetilde{t}}} C\{\widetilde{h}(\xi, s, T)\sigma_s(\xi) \ddagger L'\} d\xi ds\right) \Big| \mathcal{F}_{\widetilde{t}}\right) = \mathbb{E}\left(\exp\left(d_t - d_{\widetilde{t}}\right) \big| \mathcal{F}_{\widetilde{t}}\right).
\end{aligned}$$

Hence the result follows.

In the special case that L is a standardised, homogeneous Gaussian Lévy basis, the drift is given by

$$d_t = \frac{1}{2}\int_{A_t} \widetilde{h}^2(\xi, s, T)\sigma_s^2(\xi) d\xi ds.$$

□

A.4 Second order structure of ambit fields

We provide some results on the probabilistic properties of the ambit fields which are useful in modelling.

For ease of exposition, we will focus on ambit fields based on *homogeneous* Lévy bases.

A.4.1 Results

Now we study the second order properties of a general ambit field given by

$$Y_t(x) = \int_{A_t(x)} h(s, \xi, t, x)\sigma_s(\xi) L(ds, d\xi), \tag{51}$$

for a homogeneous Lévy basis L (not necessarily with zero mean), a homogeneous ambit set $A_t(x)$ (as defined above) and a process σ which is independent of L and where h denotes a damping function (ensuring that the integral exists). In order to compute various moments of the ambit field, we work with the Lévy–Itô decomposition:

$$\begin{aligned}
Y_t(x) = &\int_{A_t(x)} h(s, \xi, t, x)\sigma_s(\xi)\sqrt{b} W(d\xi, ds) + \int_{A_t(x)} \int_{\{|y| \leq 1\}} y h(s, \xi, t, x)\sigma_s(\xi)(N - \nu)(dy, ds, d\xi) \\
&+ \int_{A_t(x)} \int_{\{|y| \geq 1\}} y h(s, \xi, t, x)\sigma_s(\xi) N(dy, ds, d\xi),
\end{aligned}$$

where $b \geq 0$ and N is a Poisson random measure with compensator ν. Hence, $N(A) \sim Poisson(\nu(A))$ and, in particular,

$$\mathbb{E}(N(A)) = \nu(A) = Var(N(A)), \qquad \mathbb{E}((N(A)^2) = \nu(A) + \nu(A)^2.$$

Furthermore, we know that

$$\mathbb{E}\left(N(A) - \nu(A)\right) = 0, \qquad Var(N(A) - \nu(A)) = \mathbb{E}(N(A) - \nu(A))^2 = \nu(A).$$

Assumption (H) In the following, we work under the assumptions that

- The generalised Lévy measure ν is factorisable, i.e. $\nu(dy, d\eta) = U(dy)\mu(d\eta)$, for $\eta = (\xi, s)$,
- the measure μ is homogeneous, i.e. $\mu(d\eta) = cd\eta$, for a constant $c \in \mathbb{R}$. For ease of exposition, we choose $c = 1$. Hence, we have $\nu(dy, ds, d\xi) = U(dy)dsd\xi$.

Furthermore, we use the following notation. Let $\kappa_1 = \int_{\{|y|\geq 1\}} yU(dy)$ and $\kappa_2 = \int_{\mathbb{R}} y^2 U(dy)$ (assuming they exist!) and define a function $\rho : \mathbb{R}^4 \to \mathbb{R}$ by

$$\rho(s, \tilde{s}, \xi, \tilde{\xi}) = \mathbb{E}\left(\sigma_s(\xi)\sigma_{\tilde{s}}(\tilde{\xi})\right) - \mathbb{E}\left(\sigma_s(\xi)\right)\mathbb{E}\left(\sigma_{\tilde{s}}(\tilde{\xi})\right), \tag{52}$$

for $s, \tilde{s}, \xi, \tilde{\xi} \geq 0$.

Theorem 9. *Let $t, \tilde{t}, x, \tilde{x} \geq 0$ and let $Y_t(x)$ be an ambit field as defined in (51) and assume that assumption (H) holds. The second order structure is then as follows. The means are given by*

$$\mathbb{E}\left(Y_t(x)|\,\sigma\right) = \kappa_1 \int_{A_t(x)} h(s, \xi, t, x)\sigma_s(\xi)dsd\xi,$$
$$\mathbb{E}\left(Y_t(x)\right) = \kappa_1 \int_{A_t(x)} h(s, \xi, t, x)\mathbb{E}\left(\sigma_s(\xi)\right)dsd\xi.$$

The variances are given by

$$Var\left(Y_t(x)|\,\sigma\right) = (b + \kappa_2)\int_{A_t(x)} h^2(s, \xi, t, x)\sigma_s^2(\xi)dsd\xi,$$
$$\begin{aligned} Var\left(Y_t(x)\right) &= (b + \kappa_2)\int_{A_t(x)} h^2(s, \xi, t, x)\mathbb{E}\left(\sigma_s^2(\xi)\right)dsd\xi \\ &\quad + \kappa_1^2 \int_{A_t(x)}\int_{A_t(x)} h(s, \xi, t, x)h(\tilde{s}, \tilde{\xi}, t, x)\rho(s, \tilde{s}, \xi, \tilde{\xi})dsd\xi d\tilde{s}d\tilde{\xi}. \end{aligned}$$

The covariances are given by

$$Cov\left(Y_t(x), Y_{\tilde{t}}(\tilde{x})|\,\sigma\right) = (b + \kappa_2)\int_{A_t(x)\cap A_{\tilde{t}}(\tilde{x})} h(s, \xi, t, x)h(s, \xi, \tilde{t}, \tilde{x})\sigma_s^2(\xi)d\xi ds,$$
$$\begin{aligned} Cov\left(Y_t(x), Y_{\tilde{t}}(\tilde{x})\right) &= (b + \kappa_2)\int_{A_t(x)\cap A_{\tilde{t}}(\tilde{x})} h(s, \xi, t, x)h(s, \xi, \tilde{t}, \tilde{x})\mathbb{E}\left(\sigma_s^2(\xi)\right)d\xi ds \\ &\quad + \kappa_1^2 \int_{A_t(x)}\int_{A_{\tilde{t}}(\tilde{x})} h(s, \xi, t, x)h(\tilde{s}, \tilde{\xi}, \tilde{t}, \tilde{x})\rho(s, \tilde{s}, \xi, \tilde{\xi})d\tilde{s}d\tilde{\xi}dsd\xi. \end{aligned}$$

Corollary 3. *The conditional correlation is given by*

$$Cor\left(Y_t(x), Y_{\tilde{t}}(\tilde{x})|\,\sigma\right) = \frac{\int_{A_t(x)\cap A_{\tilde{t}}(\tilde{x})} h(s, \xi, t, x)h(s, \xi, \tilde{t}, \tilde{x})\sigma_s^2(\xi)d\xi ds}{\sqrt{\int_{A_t(x)} h^2(s, \xi, t, x)\sigma_s^2(\xi)d\xi ds \int_{A_{\tilde{t}}(\tilde{x})} h^2(\tilde{s}, \tilde{\xi}, \tilde{t}, \tilde{x})\sigma_{\tilde{s}}^2(\tilde{\xi})d\tilde{\xi}d\tilde{s}}}.$$

For $\kappa_1 = 0$, the unconditional correlation is given by

$$Cor\left(Y_t(x), Y_{\tilde{t}}(\tilde{x})\right) = \frac{\int_{A_t(x)\cap A_{\tilde{t}}(\tilde{x})} h(s, \xi, t, x)h(s, \xi, \tilde{t}, \tilde{x})\mathbb{E}\left(\sigma_s^2(\xi)\right)d\xi ds}{\sqrt{\int_{A_t(x)} h^2(s, \xi, t, x)\mathbb{E}\left(\sigma_s^2(\xi)\right)d\xi ds \int_{A_{\tilde{t}}(\tilde{x})} h^2(\tilde{s}, \tilde{\xi}, \tilde{t}, \tilde{x})\mathbb{E}\left(\sigma_{\tilde{s}}^2(\tilde{\xi})\right)d\tilde{\xi}d\tilde{s}}}.$$

A.4.2 Proofs of the second order properties

Proof of Theorem 9. Recall that $\kappa_1 = \int_{|y|\geq 1} yU(dy)$ and $\kappa_2 = \int_{\mathbb{R}} y^2 U(dy)$. Then

$$\begin{aligned}\mathbb{E}\left(Y_t(x)|\,\sigma\right) &= \int_{A_t(x)}\int_{\{|y|\geq 1\}} yh(s,\xi,t,x)\sigma_s(\xi)U(dy)dsd\xi = \kappa_1\int_{A_t(x)} h(s,\xi,t,x)\sigma_s(\xi)dsd\xi,\\ \mathbb{E}\left(Y_t(x)\right) &= \kappa_1\int_{A_t(x)} h(s,\xi,t,x)\mathbb{E}\left(\sigma_s(\xi)\right)dsd\xi.\end{aligned}$$

For the second moment, we get

$$\begin{aligned}\mathbb{E}\left(Y_t(x)^2\big|\,\sigma\right) &= (b+\kappa_2)\int_{A_t(x)} h^2(s,\xi,t,x)\sigma_s^2(\xi)dsd\xi + \kappa_1^2\left(\int_{A_t(x)} h(s,\xi,t,x)\sigma_s(\xi)dsd\xi\right)^2,\\ \mathbb{E}\left(Y_t(x)^2\right) &= (b+\kappa_2)\int_{A_t(x)} h^2(s,\xi,t,x)\mathbb{E}\left(\sigma_s^2(\xi)\right)dsd\xi\\ &\quad + \kappa_1^2\int_{A_t(x)}\int_{A_t(x)} h(s,\xi,t,x)h(\tilde{s},\tilde{\xi},t,x)\mathbb{E}\left(\sigma_s(\xi)\sigma_{\tilde{s}}(\tilde{\xi})\right)dsd\xi d\tilde{s}d\tilde{\xi}.\end{aligned}$$

The conditional and unconditional variance is then given by

$$\begin{aligned}Var\left(Y_t(x)|\,\sigma\right) &= (b+\kappa_2)\int_{A_t(x)} h^2(s,\xi,t,x)\sigma_s^2(\xi)dsd\xi,\\ Var\left(Y_t(x)\right) &= \mathbb{E}\left(Var\left(Y_t(x)|\,\sigma\right)\right) + Var\left(\mathbb{E}\left(Y_t(x)|\,\sigma\right)\right)\\ &= (b+\kappa_2)\int_{A_t(x)} h^2(s,\xi,t,x)\mathbb{E}\left(\sigma_s^2(\xi)\right)dsd\xi\\ &\quad + \kappa_1^2\int_{A_t(x)}\int_{A_t(x)} h(s,\xi,t,x)h(\tilde{s},\tilde{\xi},t,x)\rho(s,\tilde{s},\xi,\tilde{\xi})dsd\xi d\tilde{s}d\tilde{\xi}.\end{aligned}$$

Next, we compute the covariance. In order to do that, we use throughout that for $y,\tilde{y}\in\mathbb{R}$ and $(s,\xi),(\tilde{s},\tilde{\xi})\in A_t(x)\cap A_{\tilde{t}}(\tilde{x})$:

$$\mathbb{E}\left(N(dy,ds,d\xi)N(d\tilde{y},d\tilde{s},d\tilde{\xi})\right) = \nu(dy,ds,d\xi)\nu(d\tilde{y},d\tilde{s},d\tilde{\xi}) + \nu(d\min(y,\tilde{y}),d\min(s,\tilde{s}),d\min(\xi,\tilde{\xi})),$$

and

$$\mathbb{E}\left((N-\nu)(dy,ds,d\xi)(N-\nu)(d\tilde{y},d\tilde{s},d\tilde{\xi})\right) = \nu(d\min(y,\tilde{y}),d\min(s,\tilde{s}),d\min(\xi,\tilde{\xi})).$$

For the product, we get

$$\begin{aligned}\mathbb{E}\left(Y_t(x)Y_{\tilde{t}}(\tilde{x})|\,\sigma\right) &= (b+\kappa_2)\int_{A_t(x)\cap A_{\tilde{t}}(\tilde{x})} h(s,\xi,t,x)h(s,\xi,\tilde{t},\tilde{x})\sigma_s^2(\xi)d\xi ds\\ &\quad + \kappa_1^2\int_{A_t(x)}\int_{A_{\tilde{t}}(\tilde{x})} h(s,\xi,t,x)h(\tilde{s},\tilde{\xi},\tilde{t},\tilde{x})\sigma_s(\xi)\sigma_{\tilde{s}}(\tilde{\xi})d\tilde{s}d\tilde{\xi}dsd\xi,\\ \mathbb{E}\left(Y_t(x)Y_{\tilde{t}}(\tilde{x})\right) &= (b+\kappa_2)\int_{A_t(x)\cap A_{\tilde{t}}(\tilde{x})} h(s,\xi,t,x)h(s,\xi,\tilde{t},\tilde{x})\mathbb{E}\left(\sigma_s^2(\xi)\right)d\xi ds\\ &\quad + \kappa_1^2\int_{A_t(x)}\int_{A_{\tilde{t}}(\tilde{x})} h(s,\xi,t,x)h(\tilde{s},\tilde{\xi},\tilde{t},\tilde{x})\mathbb{E}\left(\sigma_s(\xi)\sigma_{\tilde{s}}(\tilde{\xi})\right)d\tilde{s}d\tilde{\xi}dsd\xi.\end{aligned}$$

Therefore, the covariance is given by

$$
\begin{aligned}
Cov\,(Y_t(x), Y_{\tilde{t}}(\tilde{x})|\,\sigma) &= (b+\kappa_2)\int_{A_t(x)\cap A_{\tilde{t}}(\tilde{x})} h(s,\xi,t,x)h(s,\xi,\tilde{t},\tilde{x})\sigma_s^2(\xi)d\xi ds,\\
Cov\,(Y_t(x), Y_{\tilde{t}}(\tilde{x})) &= (b+\kappa_2)\int_{A_t(x)\cap A_{\tilde{t}}(\tilde{x})} h(s,\xi,t,x)h(s,\xi,\tilde{t},\tilde{x})\mathbb{E}\left(\sigma_s^2(\xi)\right)d\xi ds\\
&\quad + \kappa_1^2\int_{A_t(x)}\int_{A_{\tilde{t}}(\tilde{x})} h(s,\xi,t,x)h(\tilde{s},\tilde{\xi},\tilde{t},\tilde{x})\rho(s,\tilde{s},\xi,\tilde{\xi})d\tilde{s}d\tilde{\xi}dsd\xi.
\end{aligned}
$$

□

A.5 Multivariate extension and cross correlation

For practical applications it is often necessary to extend the current modelling framework to a multivariate set–up. E.g. one could think of modelling various commodity forwards or futures simultaneously. Such a task can be tackled by using the ambit concept.

In order to simplify the notation, we will focus on the *bivariate* case in the following. Extensions to the n–dimensional case for $n \in I\!N$ are then straightforward.

Let us assume we have a pair of ambit fields, i.e.

$$
Y_t^{(i)}(x) = \int_{A_t^{(i)}(x)} h^{(i)}(s,\xi,t,x)\sigma_s^{(i)}(\xi)L^{(i)}(ds,d\xi),
$$

for $i = 1,2$, where $h^{(i)}$, $\sigma^{(i)}$ and $L^{(i)}$ are defined as above. The corresponding Lévy–Itô decomposition is then given by

$$
\begin{aligned}
Y_t^{(i)}(x) = &\int_{A_t^{(i)}(x)} h^{(i)}(s,\xi,t,x)\sigma_s^{(i)}(\xi)\sqrt{b^{(i)}}W^{(i)}(d\xi,ds)\\
&+\int_{A_t^{(i)}(x)}\int_{\{|y|\leq 1\}} yh^{(i)}(s,\xi,t,x)\sigma_s^{(i)}(\xi)(N^{(i)}-\nu^{(i)})(dy,ds,d\xi)\\
&+\int_{A_t(x)^{(i)}}\int_{\{|y|\geq 1\}} yh^{(i)}(s,\xi,t,x)\sigma_s^{(i)}(\xi)N^{(i)}(dy,ds,d\xi),
\end{aligned}
$$

where $b^{I)} > 0$ and $N^{(i)}$ is a Poisson random measure with compensator $\nu^{(i)}$.

The key issue is now how this two ambit fields are related to each other. A natural way of doing a multivariate modelling is to assume that $L := (L^{(1)}, L^{(2)})$ is a vector–valued homogeneous Lévy basis, where the Gaussian part satisfies

$$
Cov\left(W^{(1)}(d\xi,ds), W^{(2)}(d\xi,ds)\right) = \rho d\xi ds,
$$

for $-1 \leq \rho \leq 1$ and the generalised Lévy measure is given by

$$
\nu(y_1,y_2,s_1,s_2,\xi_1,\xi_2) = U(y_1,y_2)\mu(s_1,s_2,\xi_1,\xi_2).
$$

Since we only consider homogeneous Lévy bases, we get

$$
\nu(dy_1,dy_2,ds_1,ds_2,d\xi_1,d\xi_2) = U(dy_1,dy_2)ds_1ds_2d\xi_1d\xi_2,
$$

where we set the proportionality constant to 1.

So we see that correlations between the two Lévy bases can be incorporated either through the Gaussian part or the jump part or a combination of both.

In order to shorten the exposition slightly, we focus on the Gaussian and pure–jump cases separately.

Theorem 10 (Gaussian Lévy base)**.** *Let $L^{(i)}$ be a Gaussian Lévy base for $i = 1, 2$ and let ρ denote the corresponding correlation coefficient, i.e. $\rho d\xi ds = Cov\left(W^{(1)}(d\xi, ds), W^{(2)}(d\xi, ds)\right)$. Under the assumptions above, we get the following covariation functions:*

$$Cov\left(Y_t^{(1)}(x), Y_{\tilde{t}}^{(2)}(\tilde{x})\middle|\, \sigma^{(1)}, \sigma^{(2)}\right)$$
$$= \rho\sqrt{b^{(1)}b^{(2)}}\int_{A_t^{(1)}(x)\cap A_{\tilde{t}}^{(2)}(\tilde{x})} h^{(1)}(s,\xi,t,x)h^{(2)}(s,\xi,\tilde{t},\tilde{x})\sigma_s^{(1)}(\xi)\sigma_s^{(2)}(\xi)dsd\xi.$$

The unconditional covariation is given by

$$Cov\left(Y_t^{(1)}(x), Y_{\tilde{t}}^{(2)}(\tilde{x})\right) = \rho\sqrt{b^{(1)}b^{(2)}}\int_{A_t^{(1)}(x)\cap A_{\tilde{t}}^{(2)}(\tilde{x})} h^{(1)}(s,\xi,t,x)h^{(2)}(s,\xi,\tilde{t},\tilde{x})\Upsilon(s,\xi)dsd\xi,$$

where

$$\Upsilon(s,\xi) = \mathbb{E}\left(\sigma_s^{(1)}(\xi)\sigma_s^{(2)}(\xi)\right) - \mathbb{E}\left(\sigma_s^{(1)}(\xi)\right)\mathbb{E}\left(\sigma_s^{(2)}(\xi)\right).$$

Likewise, we get the following result in the pure jump case.

Theorem 11 (The pure jump case)**.** *Let $L^{(i)}$ be a pure jump Lévy base for $i = 1, 2$ and let $\kappa_{1,1} = \int_{|y|\geq 1}\int_{|y'|\geq 1} yy'U(dy, dy')$. Then (under the assumptions above), we get the following covariation functions:*

$$Cov\left(Y_t^{(1)}(x), Y_{\tilde{t}}^{(2)}(\tilde{x})\middle|\, \sigma^{(1)}, \sigma^{(2)}\right)$$
$$= \left(\kappa_{1,1} - \kappa_1^{(1)}\kappa_1^{(2)}\right)\int_{A_t^{(1)}(x)\cap A_{\tilde{t}}^{(2)}(\tilde{x})} h^{(1)}(s,\xi,t,x)h^{(2)}(s,\xi,\tilde{t},\tilde{x})\sigma_s^{(1)}(\xi)\sigma_s^{(2)}(\xi)dsd\xi.$$

The unconditional covariance is then given by

$$\begin{aligned}Cov\left(Y_t^{(1)}(x), Y_{\tilde{t}}^{(2)}(\tilde{x})\right) = &\int_{A_t^{(1)}(x)\cap A_{\tilde{t}}^{(2)}(\tilde{x})} h^{(1)}(s,\xi,t,x)h^{(2)}(s,\xi,\tilde{t},\tilde{x})\widetilde{\Upsilon}(s,s,\xi,\xi)dsd\xi \\ &+ \int_{A_t^{(1)}(x)}\int_{A_t^{(2)}(x)} h^{(1)}(s,\xi,t,x)h^{(2)}(\tilde{s},\tilde{\xi},\tilde{t},\tilde{x})\widetilde{\Upsilon}(s,\tilde{s},\xi,\tilde{\xi})d\tilde{s}d\tilde{\xi}dsd\xi,\end{aligned}$$

where

$$\widetilde{\Upsilon}(s,\tilde{s},\xi,\tilde{\xi}) = \kappa_{1,1}\mathbb{E}\left(\sigma_s^{(1)}(\xi)\sigma_{\tilde{s}}^{(2)}(\tilde{\xi})\right) - \kappa_1^{(1)}\kappa_1^{(2)}\mathbb{E}\left(\sigma_s^{(1)}(\xi)\right)\mathbb{E}\left(\sigma_{\tilde{s}}^{(2)}(\tilde{\xi})\right).$$

Note that from a modelling point of view there are many possibilities in modelling the joint Lévy measure U. E.g. one could work with classical multivariate Lévy measures. Another possibility would be to apply Lévy copulas, see e.g. Cont & Tankov (2004), to model the dependence structure.

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