Who Should Sell Stocks?

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Imperial-ETH Workshop on Mathematical Finance 2015

Merton's Problem (1969)

- Frictionless market consisting of one safe and one risky asset
- Constant investment opportunities and CRRA for the investor
- Maximize the expected utility of final wealth
- **Solution**: risky weight $\pi_t \equiv \pi_*$

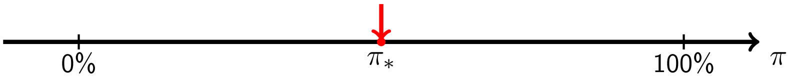
Merton's Problem with Proportional Transaction Costs

Magill and Constantinides (1976)/ Constantinides (1986)/ Davis and Norman (1990) / Shreve and Soner (1994)...

- No trading, if the risky weight is inside a certain no-trade region
- Minimal trading (of local-time type), if the boundaries of the no-trade region are breached

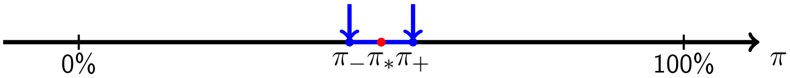
Merton's Problem with Transaction Costs and Continous Dividends

Merton's Problem



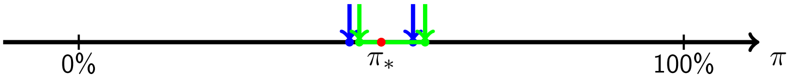
Merton's Problem with Transaction Costs and Continous Dividends

Merton's Problem with $\varepsilon = 1\%$



Merton's Problem with Transaction Costs and Continous Dividends

Merton's Problem and with $\varepsilon = 1\%$ and $\delta = 3\%$



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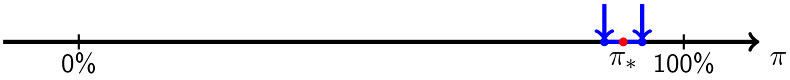

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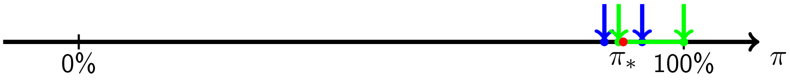

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- For long-term investment problem common advice is to buy-and-hold a stock portfolio: cf. **Siegel (1998)**, **Malkiel (1999)**

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- Buy-and-hold is only optimal for very particular preferences
- **Jang 2007**: numerical approach, but no new effect

This paper

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- Merton's problem with prop. transaction costs and continuous dividends: dynamic Buy-and-Hold can be optimal for a range of realistic parameters
- Dividends are relevant for the portfolio choice problem in contrast to capital structure (M&M theorem)
- More complicated model might lead to simpler optimal solutions
- Closed form optimal strategies even with capital gains tax

Model

Standing Assumptions:

- Black-Scholes dynamics with continuous dividends:

$$dS_t/S_t = (r + \mu - \delta)dt + \sigma dW_t$$

- Proportional Transaction Costs: buy at the ask price $(1 + \varepsilon)S$, sell at the bid price $(1 - \varepsilon)S$
- Constant Relative Risk Aversion $0 < \gamma \neq 1$
- Infinite planning horizon
- Frictionless solution: $0 < \pi_* = \mu/\gamma\sigma^2 < 1$, i.e, no short or levered positions

Long-run Optimality

Goal: maximize the equivalent safe rate ESR among all admissible strategies:

$$\max \left(\liminf_{T\to\infty} \frac{1}{T} \log \mathbb{E} \left[(\Xi_T)^{1-\gamma} \right]^{\frac{1}{1-\gamma}} \right)$$

- Ξ_t = liquidation value at time t
- admissible "=" self financing and $\Xi_t \geq 0$

Main Results: Parameter assumption

Set

$$\pi_{\pm}^{\dagger}(\lambda) = \frac{\mu \pm \varepsilon\delta/(1 \mp \varepsilon) \pm \sqrt{\lambda^2 \pm 2\mu\varepsilon\delta/(1 \mp \varepsilon) + (\varepsilon\delta/(1 \mp \varepsilon))^2}}{\gamma\sigma^2}$$

$$\pi_{-}(\lambda) = \pi_{-}^{\dagger}(\lambda), \quad \pi_{+}(\lambda) = \min\left(\pi_{+}^{\dagger}, 1\right).$$

Suppose one of the following condition is satisfied:

(a) there exists $\lambda > 0$ such that $\pi_{+}(\lambda) < 1$ and the solution $w(\cdot, \lambda)$ of terminal value problem also satisfies a certain initial condition.

(b) there exists $\lambda > 0$ such that $\pi_{+}(\lambda) = 1$ and the solution $w(\cdot, \lambda)$ of a Riccati ODE with a limit condition at infinity also satisfies a certain initial condition.

Main Results: Optimal Policy

Theorem

In the presence of proportional transaction costs $\varepsilon > 0$ and a continuous yield $\delta > 0$ an investor trades to maximizes the equivalent safe rate. Then, under the previous assumption we have:

- *It is optimal to keep the risky weight within the buying and selling boundaries $[\pi_-, \pi_+]$*
- *The optimal equivalent safe rate $\beta = r + (\mu^2 - \lambda^2)/2\gamma\sigma^2$*
- *In case of $\pi_+ < 1$ it holds*

$$\pi_\pm = \pi_* \pm \left(\frac{3}{2\gamma}\pi_*^2 (1-\pi_*)^2\right)^{1/3} \varepsilon^{1/3} + \frac{\delta}{\gamma\sigma^2}\left(\frac{2\gamma\pi_*}{3(1-\pi_*)^2}\right)^{1/3} \varepsilon^{2/3} + \mathcal{O}(\varepsilon) \qquad \textit{as} \qquad \varepsilon \downarrow 0$$

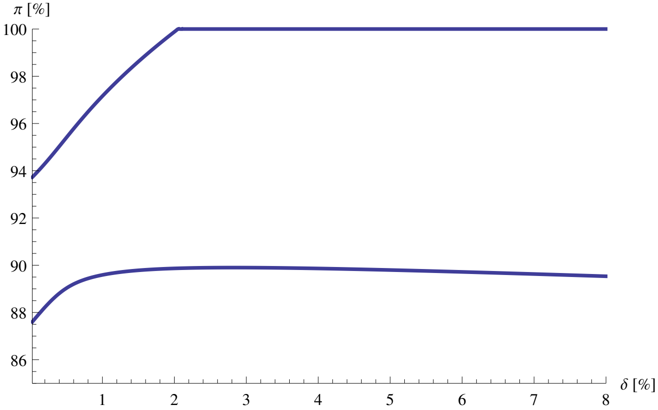

Figure: The no-trade region (vertical axis) plotted against the dividend yield δ (horizontal axis) for $\gamma = 3.45$ ($\pi_* = 90.6\%$), $\mu = 8\%$, $\sigma = 16\%$ and $\varepsilon = 1\%$.

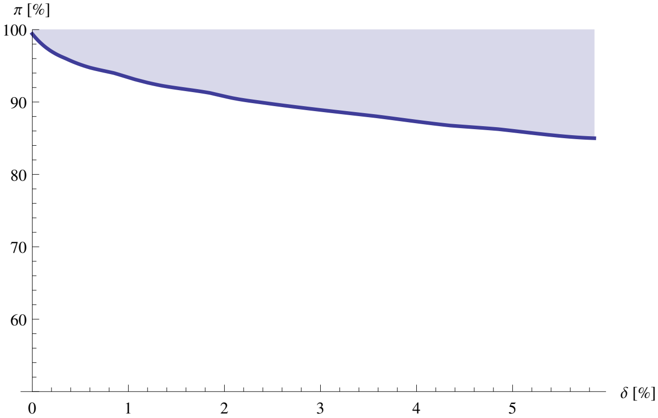

Figure: The never-sell region (shaded) for pairs of dividend yield δ (horizontal axis) and frictionless portfolio weight π_* (vertical axis). Parameters are $\mu = 8\%$, $\sigma = 16\%$ and $\varepsilon = 1\%$.

Robustness

π_*	optimal	never sell	buy & hold
50%	1.67%	2.00%	4.67%
70%	1.58%	1.58%	4.21%
90%	1.52%	1.52%	3.70%

Table: Relative equivalent safe rate loss of the optimal ($[\pi_-, \pi_+]$), never sell ($[\pi_-, 1]$) and buy-and-hold ($[0, 1]$). These numbers are computed using Monte Carlo simulation with $T = 20$, time step $dt = 1/250$ and sample size $N = 2 \cdot 10^7$, $\mu = 8\%$, $\sigma = 16\%$, $r = 1\%$, $\delta = 2\%$, and $\varepsilon = 1\%$.

Robustness with respect to Taxes

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- Choices for B: Share Specification Method/Weighted Average Cost Method cf. **Dammon, Spatt and Zhang (2001)**, **Tahar, Soner and Touzi (2010)**

Taxes

π_*	$[\pi_-, \pi_+]_{ave}$	$[\pi_-, \pi_+]_{ss}$	never sell	buy & hold
50%	2.41%	2.41%	2.07%	4.48%
70%	1.91%	1.91%	1.64%	3.55%
90%	1.36%	1.36%	1.36%	2.94%

Table: Relative equivalent safe rate loss of the capital gains tax adjusted optimal ($[\pi_-, \pi_+]$), never sell ($[\pi_-, 1]$) and buy-and-hold ($[0, 1]$). These numbers are computed using Monte Carlo simulation with $T = 20$, time step $dt = 1/250$ and sample size $N = 2 \cdot 10^7$, $\mu = 8\%$, $\sigma = 16\%$, $\alpha = 20\%$, $\tau = 20\%$, $r = 1\%$, $\delta = 2\%$ and $\varepsilon = 1\%$.

Consumption

- Objective function cf. **Janecek and Shreve (2004)**, **Shreve and Soner (1994)**

$$\max\left(\frac{1}{1-\gamma}\mathbb{E}\left[\int_0^\infty e^{-\rho t}C_t^{1-\gamma}dt\right]\right)$$

- For $\varepsilon = 0$ we have

$$\frac{C_t^*}{X_t+Y_t} = \frac{\rho}{\gamma} + \left(1-\frac{1}{\gamma}\right)\left(r+\frac{\mu^2}{2\gamma\sigma^2}\right)$$

- This consumption policy is approximately optimal even with small proportional transaction costs (**Kallsen and Muhle-Karbe 2013**)

Consumption

π_*	$[\pi_-^{js}, \pi_+^{js}]$	never sell	buy & hold
50%	1.00%	1.67%	2.00%
70%	0.53%	1.05%	1.05%
90%	0.22%	0.65%	0.65%

Table: Relative equivalent safe rate loss of the asymptotically optimal ($[\pi_-^{js}, \pi_+^{js}]$), never sell ($[\pi_-, 1]$) and simple buy-and-hold ($[0, 1]$) strategies with $\pi_\pm^{js}$ as defined in [Janecek and Shreve, Theorem 2]. These numbers are computed using Monte Carlo simulation with $T = 50$, time step $dt = 1/250$, sample size $N = 2 \times 10^7$, $\mu = 8\%$, $\sigma = 16\%$, $\rho = 2\%$, $r = 1\%$, $\delta = 3\%$, $\tau = 0\%$ and $\varepsilon = 1\%$.

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π_*	$[\pi_-^{js}, \pi_+^{js}]$	never sell	buy & hold
50%	1.00%	1.33%	2.33%
70%	0.53%	0.79%	1.05%
90%	0.22%	0.22%	0.22%

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Suggestions and Limitations

- Retirement planning: investors with moderate risk aversions should avoid selling
- After the retirement: gradually liquidate stocks to finance the required consumption or invest in high dividend funds
- Dynamic Buy-and-Hold might be suboptimal for
    - small transaction costs
    - low dividend yields
    - large risk aversions
    - high consumption rates

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- The boundary conditions yield the characterization of the gap parameter λ

Construction of Shadow Market $(S^0, \tilde{S})$

Shadow Price Process $\tilde{S}$*:*

- Lies within the bid-ask spread $[(1-\varepsilon)S, (1+\varepsilon)S]$ a.s.
- Existence of a long-run optimal strategy, i.e.,
  - Finite variation strategy
  - Self-financing strategy and solvent w.r.t. $\tilde{S}$
  - Maximizes the equivalent safe rate w.r.t. $\tilde{S}$
  - Same dividend payments $\tilde{\delta}\tilde{S} = \delta S$
  - Entails buying only when $\tilde{S}_t = (1+\varepsilon)S_t$
  - Entails selling only when $\tilde{S}_t = (1-\varepsilon)S_t$

Verification

- Optimality of the candidate strategy in shadow market (cf. **Guasoni and Robertson 2012**)
  - (super-) Martingale measure $\Rightarrow$ upper bound of the finite horizon ESR
  - Candidate strategy $\Rightarrow$ lower bound of the finite horizon ESR
  - Upper bound $=$ lower bound as $T \to \infty$
- Optimality of the candidate strategy in original market
  - Property of the shadow market

Thank You!