High Frequency Dynamics of Limit Order Markets

Stochastic modeling and Asymptotic Analysis

Rama Cont

3rd Imperial-ETH Workshop in Mathematical Finance (2015)

References :

- Rama Cont, Sasha Stoikov and Rishi Talreja (2010) A stochastic model for order book dynamics, **Operations Research**, Volume 58, No. 3, 549-563.
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- Rama Cont and Adrien de Larrard (2013) Price dynamics in a Markovian limit order market, SIAM Journal on Financial Mathematics, Vol 4, 1–25.
- Rama Cont and Adrien de Larrard (2011) Order book dynamics in liquid markets: limit theorems and diffusion approximations, http://ssrn.com/abstract=1757861, Stochastic Systems, to appear.
- Rama Cont and Adrien de Larrard (2012) Price dynamics in limit order markets: linking volatility with order flow, Working Paper.

Outline

1. At the core of liquidity: the Limit order book
2. The separation of time scales
3. High frequency order book dynamics: empirical properties
4. A tractable framework for order book dynamics
5. Order book dynamics in liquid markets : diffusion limit
6. Analytical results
   - Probability of a price increase at next price change
   - Distribution of duration to next price change
   - Intraday price dynamics retrieved: autocorrelation, volatility and skewness
   - Expression of the volatility of the price
7. Linking volatility with order flow: analytical results and empirical tests

Electronic trading in financial markets

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- Trading in stocks and other financial instruments increasingly takes place through electronic trading platforms.
- Quote-driven markets where prices were set by a market-maker are being increasingly replaced by electronic order driven markets where buy and sell orders are centralized in a **limit order book** and executed against the best available orders on the opposite side.

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- Limit orders may be canceled before execution.

The limit order book

The **limit order book** represents all outstanding limit orders at time t. It is updated at each **order book event**: limit order, market order or cancelation.

Order Book

	SHARES	PRICE
ASKS	13	69900
	17	69800
	22	69700
	25	69600
	2	69500
BIDS	28	69300
	31	69200
	29	69100
	27	69000
	25	68900

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Figure: A limit buy order: Buy 2 at 69200.

A market order

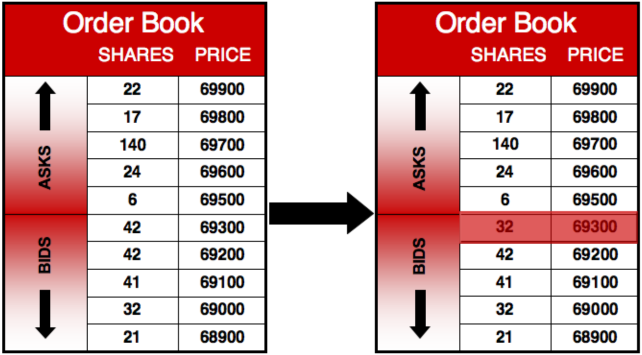

Figure: A market sell order of 10.

A cancellation

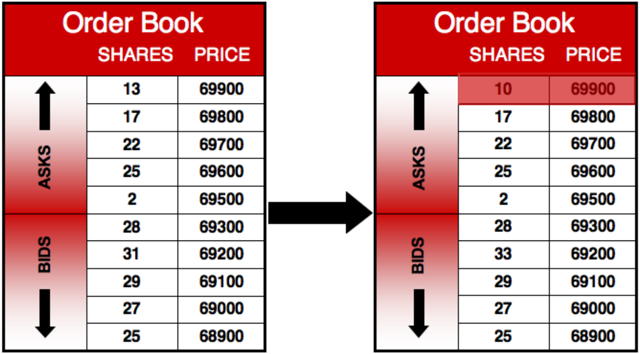

Figure: Cancellation of 3 sell orders at 69900.

Stochastic models of order book dynamics

Stochastic models for order book dynamics aim at

- incorporating the information in
  1. the current state of the order book
  2. statistics on the order flow (arrival rates of market, limit orders and cancellation)

  in view of
  1. estimation of intraday risk (volatility, loss distribution)
  2. short-term ($<$ second) prediction of order flow and price movements for trading strategies
  3. optimal order execution

These applications requires *analytical* tractability and computability.

Limit order books as queueing systems

A limit order book may be viewed as a system of *queues* subject to order book events modeled as a multidimensional **point process**. A variety of stochastic models for dynamics of order book events and/or trade durations at high frequency:

- Independent Poisson processes for each order type (Cont Stoikov Talreja 2010)
- Self exciting and mutually exciting Hawkes processes (Cont, Jafteson & Vinkovskaya 2010, Bacry et al 2010)
- Autoregressive Conditional Duration (ACD) model (Engle & Russell 1997, Engle & Lunde 2003, ..)

Aim: intraday prediction, trade execution, intraday risk management.
In general: price is not Markovian, increments neither independent nor stationary and depend on the state of the order book.
Common approach: model separately order flow dynamics and price dynamics through ad-hoc price impact relations/assumptions.

Stochastic models of limit order markets

General setting: build a stochastic model for the state of the limit order book by modeling

- arrivals of different order types (market, limit, cancel; buy/sell) through *arrival intensities* which may depend on order book configuration, distance to best price level, etc.
- execution of market orders through (deterministic) execution priority rules

Then one tries to deduce from these ingredients the dynamics of the limit order book and, consequently, the price dynamics in an endogenous manner.

Two approaches

- Stochastic models of the extended limit order book: models limit orders at all price levels simultaneously.
- Reduced-form models: focus on the consolidated 'Level-I' order book (best price quotes and corresponding queue lengths).
- More recently: multi-exchange models

A Markovian model for the limit order book

Cont, Stoikov, Talreja (Operations Research, 2010)

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- The above events are mutually independent.

$$
\begin{array}{llccc}
x \to x^{i-1} & \text{with rate} & \lambda(s_a(t)-i) & \text{for} & i < s_a(t), \\
x \to x^{i+1} & \text{with rate} & \lambda(i-s_b(t)) & \text{for} & i > s_b(t), \\
x \to x^{s_b(t)+1} & \text{with rate} & \mu & & \\
x \to x^{s_a(t)-1} & \text{with rate} & \mu & & \\
x \to x^{i+1} & \text{with rate} & \theta(s_a(t)-i)|x^i| & \text{for} & i < s_a(t), \\
x \to x^{i-1} & \text{with rate} & \theta(i-s_b(t))|x^i| & \text{for} & i > s_a(t),
\end{array}
$$

$$
\text{where} \qquad x^{i\pm 1} \equiv x \pm (0, \ldots, \underbrace{1}_{i}, \ldots, 0),
$$

Proposition (Ergodicity)

If $\theta \equiv \min_{1 \le i \le n} \theta_i < \infty$, then X is an ergodic Markov process and admits a unique stationary distribution.

Observations of the order book can then be viewed as a sample from the stationary distribution.

- The dynamics of the order book is then described by a continuous-time Markov chain $X_t \equiv (X_t^1, \ldots, X_t^n)$, where $|X_t^i|$ is the number of limit orders in the book at price i

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- If $X_t^i < 0$ then there are $-X_t^i$ bid orders at price i; if $X_t^i > 0$ then there are X_t^i ask orders at price i.
- Ask price (best offer)

$$s_a(t) = \inf\{i,\ X_t^i > 0\}, \quad t \geq 0.$$

Bid price (best bid)

$$s_b(t) = \sup\{i,\ X_t^i < 0\}, \quad t \geq 0.$$

The limit order book as a measure-valued process

The state of limit order book may also be viewed as a signed measure μ on $\mathbb{R}$ with Hahn-Jordan decomposition

$$\mu = \mu_+ - \mu_- \qquad a(\mu) = \inf(\mathrm{supp}(\mu_-)) \quad \geq b(\mu) = \sup(\mathrm{supp}(\mu_+)),$$

where $\mu_+(B) =$ vol of limit buy orders with prices in B,
$\mu_-(B) =$ vol. of limit sell orders with prices in B

$$\mathrm{supp}(\mu_+) \subset (-\infty, b(\mu)] \qquad \mathrm{supp}(\mu_-) \subset [a(\mu), \infty)$$

We denote $\mathbb{L}$ the set of signed measures whose Hahn-Jordan decomposition is of the form above.
Thus, the limit order book may be viewed in terms of a pair of Radon measures $(\mu_+, \mu-) \in \mathcal{M}(\mathbb{R})^2$.
In the above example, this leads to a measure-valued Markov process with values in $\mathcal{M}(\mathbb{R})^2$.

One step ahead prediction of price moves

- The model allows to compute the probability of a queue going up, when there are m orders in the queue, for $1 \leq d \leq 5$, conditional on the best quotes not changing.

$$p_{up}^{d}(m) = \frac{\hat{\lambda}(d)}{\hat{\theta}(d)m + \hat{\lambda}(d) + \hat{\mu}}$$

for $d = 1$

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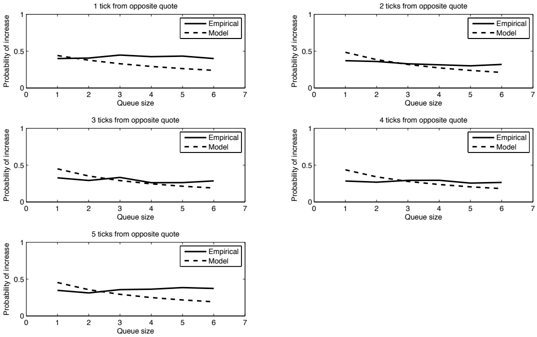
- Empirical test: compare these probabilities to the corresponding empirical frequencies

Empirical performance of one step-ahead prediction



Conditional probabilities of interest

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- The probability that the midprice goes up before it goes down (spread>1)
- The probability that an order at the bid executes before the ask queue disappears (spread=1)

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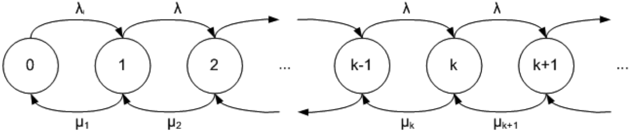
- The probability that the midprice goes up before it goes down (spread>1)
- The probability that an order at the bid executes before the ask queue disappears (spread=1)
- The probability that both a buy and a sell limit order execute before the best quotes move (spread=1)

A birth death process

- At the best quotes, the number of orders is a birth death process with birth rate λ and death rate $\mu_k = \mu + k\theta$.



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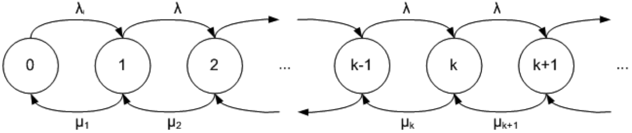

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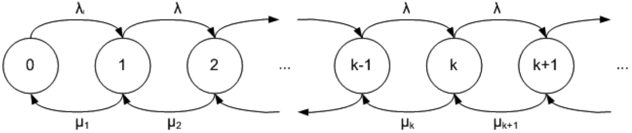

- $\sigma_{i,i-1}$ - first time that the BD process goes from i to $i-1$
- The Laplace transform of the first passage time

$$\hat{f}_{i,i-1}(s) = E[e^{-s\sigma_{i,i-1}}]$$

satisfies a recurrence relation :

$$\hat{f}_{i,i-1}(s) = \frac{\mu_i}{\mu_i + \lambda + s} + \frac{\lambda}{\mu_i + \lambda + s}\hat{f}_{i+1,i}(s)\hat{f}_{i,i-1}(s)$$

First passage time of a birth death process

- The recurrence relation allows us to express the Laplace transform of the first passage time as a continued fraction

$$\hat{f}_{i,i-1}(s) = -\frac{1}{\lambda}\Phi_{k=i}^{\infty}\frac{-\lambda\mu_k}{\lambda+\mu_k+s}$$

where $\Phi_{k=1}^{\infty}\frac{a_k}{b_k} = \frac{a_1}{b_1+\frac{a_2}{b_2+\cdots}}$ is a continued fraction.

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- Let σ_b denote the first-passage time to 0 of a BD process starting at b
- The Laplace transform of σ_b

$$\hat{f}_b(s) = \left(-\frac{1}{\lambda}\right)^b \left(\Pi_{i=1}^{b}\Phi_{k=i}^{\infty}\frac{-\lambda\mu_k}{\lambda+\mu_k+s}\right).$$

Probability of the mid price moving up: spread=1

σ_b is the random time when a bid queue with b orders disappears. σ_a is the random time when an ask queue with a orders disappears.

Theorem

$$P_{a,b} \equiv \mathbb{P}[\sigma_a < \sigma_b]$$

is given by the inverse Laplace transform of

$$\hat{F}_{a,b}(s) = \frac{1}{s}\hat{f}_b(s)\hat{f}_a(-s),$$

evaluated at $t = 0$, where

$$\hat{f}_b(s) = \left(-\frac{1}{\lambda}\right)^b \left(\Pi_{i=1}^b \Phi_{k=i}^{\infty} \frac{-\lambda(\mu + k\theta)}{\lambda + (\mu + k\theta) + s}\right).$$

Proposition (Probability of order execution before mid-price moves)

Let $\Lambda_S \equiv \sum_{i=1}^{S-1} \lambda(i)$ and

$$\hat{f}_j^S(s) = \left(-\frac{1}{\lambda(S)}\right)^j \left(\prod_{i=1}^{b} \Phi_{k=i}^{\infty} \frac{-\lambda(S)\,(\mu + k\theta(S))}{\lambda(S) + \mu + k\theta(S) + s}\right) \tag{1}$$

$$\hat{g}_j^S(s) = \prod_{i=1}^{j} \frac{\mu + \theta(S)(i-1)}{\mu + \theta(S)(i-1) + s}, \tag{2}$$

Then the probability of order execution before the price moves is given by the inverse Laplace transform of

$$\hat{F}_{a,b}^S(s) = \frac{1}{s}\hat{g}_b^S(s)\left(\hat{f}_b^S(2\Lambda_S - s) + \frac{2\Lambda_S}{2\Lambda_S - s}(1 - \hat{f}_b^S(2\Lambda_S - s))\right), \tag{3}$$

evaluated at 0. When $S = 1$, (3) reduces to

$$\hat{F}_{a,b}^1(s) = \frac{1}{s}\hat{g}_b^1(s)\hat{f}_a^1(-s). \tag{4}$$

Probability of increase in mid price

	a				
b	1	2	3	4	5
1	.512	.304	.263	.242	.226
2	.691	.502	.444	.376	.359
3	.757	.601	.533	.472	.409
4	.806	.672	.580	.529	.484
5	.822	.731	.640	.714	.606

	a				
b	1	2	3	4	5
1	.500	.336	.259	.216	.188
2	.664	.500	.407	.348	.307
3	.741	.593	.500	.437	.391
4	.784	.652	.563	.500	.452
5	.812	.693	.609	.548	.500

Table: Empirical frequencies (top) and Laplace transform results (bottom).

Probability of making the spread

- I have one limit order that is b-th order at the bid.
- I have one limit order that is a-th order at the ask.
- The probability that both are executed before the mid price moves:

	a				
b	1	2	3	4	5
1	.266	.308	.309	.300	.288
2	.308	.386	.406	.406	.400
3	.309	.406	.441	.452	.452
4	.300	.406	.452	.471	.479
5	.288	.400	.452	.479	.491

Time scales

Regime	Time scale	Issues
Ultra-high frequency (UHF)	$\sim 10^{-3} - 1$ s	Microstructure, Latency
High Frequency (HF)	$\sim 10 - 10^2$ s	Optimal execution
"Daily"	$\sim 10^3 - 10^4$ s	Trading strategies, Option hedging

Table: A hierarchy of time scales.

Idea: start from a description of the limit order book at the finest scale and use **asymptotics**/ limit theorems to derive quantities at larger time scales.
Analogous to hydrodynamic limits of interacting particle systems.

Moving across time scales: fluid and diffusion limits

Idea: study limit of **rescaled** limit order book as

- tick size $\to 0$
- frequency of order arrivals $\to \infty$
- order size $\to 0$

All these quantities are usually parameterized / scaled as a power of a large parameter $n \to \infty$, which one can think of as number of market participants or frequency of orders.

The limit order book having a natural representation as a (pair of) measures, vague convergence in $D([0,\infty), \mathcal{M}(\mathbb{R})^2)$ is the natural notion of convergence to be considered.

Various combination of scaling assumptions are possible, which may lead to very different limits.

Moving across time scales: fluid and diffusion limits

Various combination of scaling assumptions are possible for the same process, which lead to very different limits.
When scaling assumptions are such that variance vanishes asymptotically, the limit process is deterministic and often described by a PDE or ODE: this is the functional equivalent of a Law of Large Numbers, known as the 'fluid' (or 'hydrodynamic' limit).
Ex: N_i^n Poisson process with intensity λ_n^i.

$$\lambda_n^i \sim n\lambda^i \qquad \left(\frac{N_1^n - N_2^n}{n}, t \geq 0\right) \overset{n\to\infty}{\Rightarrow} ((\lambda^1 - \lambda^2)t, t \geq 0)$$

Other scaling assumptions for the same process may lead to a random limit ("diffusion limit"). Example:

$$\lambda_n^i \sim n\lambda, \quad \lambda_n^1 - \lambda_n^2 = \sigma^2\sqrt{n}, \qquad \frac{N_1^n - N_2^n}{\sqrt{n}} \overset{n\to\infty}{\Rightarrow} \sigma W$$

A reduced-form model for the limit order book

- Idea: if one is primarily interested in price dynamics, the 'action' takes place at the best bid/ask levels
- Empirical data show that the bulk of orders flow to the queues at the best bid/ask (e.g. Biais, Hillion & Spatt 1995)
- Ask price: best selling price: $s^a = (s_t^a, t \geq 0)$
- Bid price: best buying price $s^b = (s_t^b, t \geq 0)$.
- Reduced modeling framework: state variables= number of orders at the ask:

$$(q_t^a, t \geq 0).$$

and number of orders at the bid:

$$(q_t^b, t \geq 0)$$

- State variable: $(s_t^b, q_t^b, s_t^a, q_t^a)_{t \geq 0}$

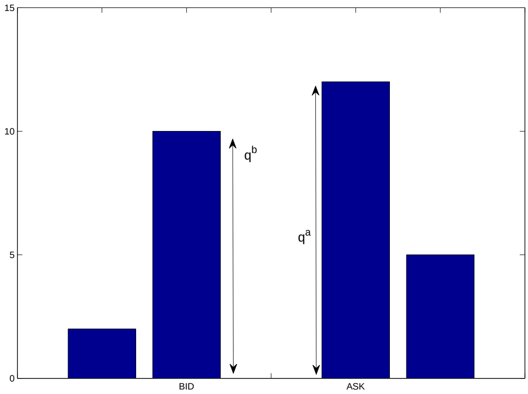


Figure: Reduced-form representation of a limit order book

Limit order book as reservoir of liquidity

Once the bid (resp. the ask) queue is depleted, the price moves to the queue at the next level, which we assume to be one tick below (resp. above).
The new queue size then corresponds to what was previously the number of orders sitting at the price immediately below (resp. above) the best bid (resp. ask).
Instead of keeping track of these queues (and the corresponding order flow) at *all* price levels we treat the new queue sizes as independent variables drawn from a certain distribution f where $f(x, y)$ represents the probability of observing $(q_t^b, q_t^a) = (x, y)$ right after a price increase. Similarly, after a price decrease (q_t^b, q_t^a) is drawn from a distribution $\tilde{f}(\neq f)$ in general.

- if $q_{t-}^a = 0$ then (q_t^b, q_t^a) is a random variable with distribution f, independent from $\mathcal{F}_{t-}$.
- if $q_{t-}^b = 0$ then (q_t^b, q_t^a) is a random variable with distribution $\tilde{f}$, independent from $\mathcal{F}_{t-}$.

Distribution of queue sizes after a price move

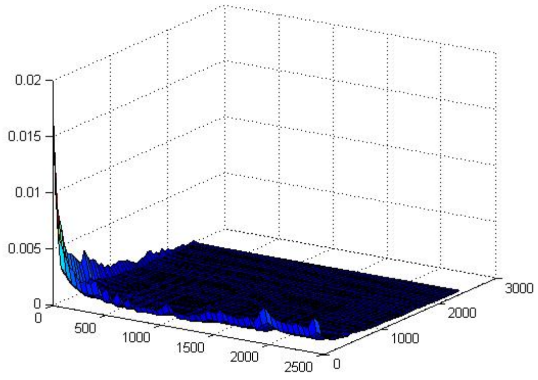


Figure: Joint density of bid and ask queues after a price move.

Distribution of queue sizes after a price move

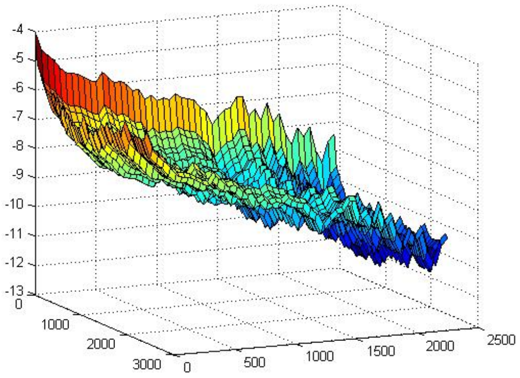


Figure: Joint density of bid and ask queues after a price move: log-scale

Distribution of queue sizes after a price move

We can parameterize this distribution F through

- a radial component $R = \sqrt{|Q^b|^2 + |Q^a|^2}$, which measures the depth of the order book, and
- an angular component $\Theta = \arctan(Q^a/Q^b) \in [0, \pi/2]$ which measures the *imbalance* between outstanding buy and sell orders.

A flexible model which allows for analytical tractability is to assume

$$F(x, y) = H(\sqrt{x^2 + y^2})G\left(\arctan(\frac{y}{x})\right) \quad (5)$$

where H is a probability distributions on $\mathbb{R}_+$ and G a probability distributions on $[0, \pi/2]$.

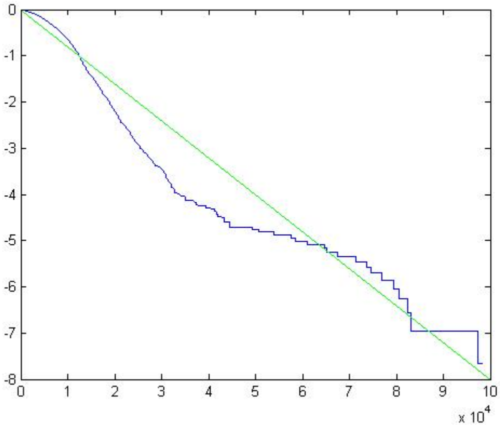


Figure: Radial component $H(.)$ of the empirical distribution function of order book depth: CitiGroup, June 26th, 2008. Green: exponential fit.

Bid-ask spread	1 tick	2 tick	$\geq$ 3 tick
Citigroup	98.82	1.18	0
General Electric	98.80	1.18	0.02
General Motors	98.71	1.15	0.14

Table: % of observations with a given bid-ask spread (June 26th, 2008).

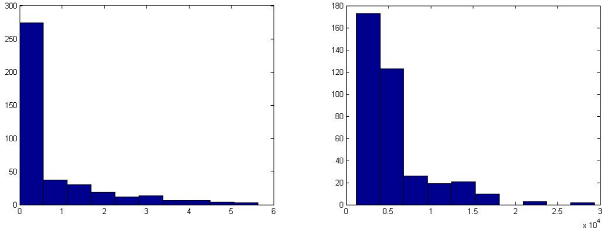


Figure: Distribution of lifetime (in ms) of a spread larger than one tick (left), equal to one tick (right).

GIven these observations, we assume for simplicity that the spread is constant, equal to one tick:

$$\forall t \geq 0, s_t^a = s_t^b + \delta.$$

This assumption of constant spread is justified at a time scale beyond 10 milliseconds, since for many liquid stocks, the lifetime of a spread > 1 tick is $\sim$ a few milliseconds, while the lifetime of a 1-tick spread is $\sim$ seconds. This assumptions allows to deduce *price dynamics* from the dynamics of the order book:

- Price decreases by δ when bid queue is depleted: $q_{t-}^b = 0 \Rightarrow s_t = s_{t-} - \delta$
- Price increases by δ when ask queue is depleted: $q_{t-}^a = 0 \Rightarrow s_t = s_{t-} + \delta$

Dynamics of the Level-I order book

The dynamics of the reduced order book may be described in terms of

- $T_i^a = t_{i+1}^a - t_i^a$: durations between events at the ask
- V_i^a size of the ith event at the ask. If the ith event is a market order or a cancelation, $V_i^a < 0$; if it is a limit order $V_i^a \geq 0$.
- $T_i^b = t_{i+1}^b - t_i^b$ durations between events at the bid
- V_i^b the size of the ith event at the bid

- For general (non-IID) sequences $(T_i^a, V_i^a)_{i\geq 0}$ and $(T_i^b, V_i^b)_{i\geq 0}$, the order book $q = (q^a, q^b)$ is not a Markov process.
- Price changes occur at exit times of $q = (q^a, q^b)$ from $\mathbb{N}^* \times \mathbb{N}^*$.

Dynamics of bid / ask queues and price

The process $X_t = (s_t^b, q_t^b, q_t^a)$ is thus a continuous-time process with piecewise constant sample paths whose transitions correspond to the order book events at the ask $\{t_i^a, i \geq 1\}$ or the bid $\{t_i^b, i \geq 1\}$ with (random) sizes $(V_i^a)_{i \geq 1}$ and $(V_i^b)_{i \geq 1}$.

- Order or cancelation arrives on the ask side $t \in \{t_i^a, i \geq 1\}$:
  If $q_{t-}^a + V_i^a > 0$: $q_t^a = q_{t-}^a + V_i^a$, no price move.
  If $q_{t-}^a + V_i^a \leq 0$: price increases $S_t = S_{t-} + \delta$, queues are regenerated $(q_t^b, q_t^a) = (R_i^b, R_i^a)$ where $(R_i^a, R_i^b)_{i \geq 1}$ are IID variables with (joint) distribution f
- Order or cancelation arrives on the bid side $t \in \{t_i^b, i \geq 1\}$:
  If $q_{t-}^b + V_i^b > 0$: $q_t^a = q_{t-}^a + V_i^a$, no price move.
  If $q_{t-}^a + V_i^a \leq 0$: price decreases $S_t = S_{t-} - \delta$, queues are regenerated $(q_t^b, q_t^a) = (\tilde{R}_i^b, \tilde{R}_i^a)$ where $(\tilde{R}_i)_{i \geq 1} = (\tilde{R}_i^a, \tilde{R}_i^b)_{i \geq 1}$ is a sequence of IID variables with (joint) distribution $\tilde{f}$

Example: a Markovian reduced limit order book

Cont & de Larrard (2010) Price dynamics in a Markovian limit order market.

- Market buy (resp. sell) orders arrive at independent, exponential times with rate μ,
- Limit buy (resp. sell) orders arrive at independent, exponential times with rate λ,
- Cancellations orders arrive at independent, exponential times with rate θ.
- The above events are mutually independent.
- All orders sizes are constant.

$\rightarrow$ Poisson point process $\Rightarrow$ explicit computations possible

- Between price changes, (q_t^a, q_t^b) are independent birth and death process with birth rate λ and death rate $\mu + \theta$.
- Let σ^a (resp. σ^b be the first time the size of the ask (resp bid) queue reaches zero. Duration until next price move: $\tau = \sigma^a \wedge \sigma^b$
- These are hitting times of a birth and death process so conditional Laplace transform of σ^a solves:

$$\mathcal{L}(s,x) = \mathbb{E}[e^{-s\sigma^a}|q_0^a = x] = \frac{\lambda \mathcal{L}(s,x+1) + (\mu+\theta)\mathcal{L}(s,x-1)}{\lambda+\mu+\theta+s},$$

- We obtain the following expression for the (conditional) Laplace transform of σ_a:

$$\mathcal{L}(s,x) = (\frac{(\lambda+\mu+\theta+s) - \sqrt{((\lambda+\mu+\theta+s))^2 - 4\lambda(\mu+\theta)}}{2\lambda})^x.$$

Duration until the next price change

- The duration τ until the next price change is given by:

$$\tau = \sigma^a \wedge \sigma^b.$$

- The distribution of τ conditional on the current queue sizes is

$$\mathbb{P}[\tau > t | q_0^a = x, q_0^b = y] = \mathbb{P}[\sigma^a > t | q_0^a = x] \mathbb{P}[\sigma^b > t | q_0^b = y].$$

- Inverting the Laplace transforms of σ^a, σ^b we obtain

$$\mathbb{P}[\tau > t | q_0^a = x, q_0^b = y] = \int_t^\infty \hat{\mathcal{L}}(u, x) du \int_t^\infty \hat{\mathcal{L}}(u, y) du,$$

where

$$\hat{\mathcal{L}}(t, x) = \sqrt{(\frac{\mu + \theta}{\lambda})^x} \frac{x}{t} I_x(2\sqrt{\lambda(\theta + \mu)} t) e^{-t(\lambda + \theta + \mu)}.$$

Duration until next price move

- Littlewood & Karamata's Tauberian theorems links the tail behavior of τ to the behavior of the conditional Laplace transforms of σ^a and σ^b at zero.
- When $\lambda < \theta + \mu$
  - $\mathbb{P}[\sigma^a > t | q_0^a = x] \sim_{t\to\infty} \dfrac{x(\lambda+\mu+\theta)}{2\lambda(\mu+\theta-\lambda)} \dfrac{1}{t}$
  - $\mathbb{P}[\tau > t | q_0^a = x, q_0^b = y] \sim_{t\to\infty} \dfrac{xy(\lambda+\mu+\theta)^2}{\lambda^2(\mu+\theta-\lambda)^2} \dfrac{1}{4t^2}.$
  - Tail index of order 2
- When $\lambda = \theta + \mu$
  - $\mathbb{P}[\sigma^a > t | q_0^a = x] \sim_{t\to\infty} \dfrac{x}{\sqrt{\pi\lambda}} \dfrac{1}{\sqrt{t}}$
  - $\mathbb{P}[\tau > t | q_0^a = x, q_0^b = y] \sim_{t\to\infty} \dfrac{x}{\sqrt{\pi\lambda}} \dfrac{1}{\sqrt{t}}$
  - Tail index of order 1: the mean between two consecutive moves of the price is infinite.

Forecasting price moves from the Level-I order book

Intuitively, *bid-ask imbalance* gives an indication of the direction of short term price moves. This intuition can be quantified in this model:

Proposition

When $\lambda = \theta + \mu$, the probability $\phi(n, p)$ that the next price move is an increase, conditioned on having the n orders on the bid side and p orders on the ask side is:

$$\phi(n,p) = \frac{1}{\pi}\int_0^{\pi} \left(2 - \cos(t) - \sqrt{(2-\cos(t))^2 - 1}\right)^p \frac{\sin(nt)\cos(t/2)}{\sin(t/2)} dt.$$

Interestingly: this quantity does not depend on the arrival rates λ, θ, μ as long as $\lambda = \theta + \mu$!

Diffusion limit of the price

At a *tick* time scale the price is a piecewise constant, discrete process.
But over larger time scales, prices have "diffusive" dynamics and modeled as such.
Consider a time scale over which the average number of order book events is of order n, i.e.

$$\frac{T_1 + \ldots + T_n}{t_n} = O(1)$$

We will then show that

$$(s_t^n := \frac{s_{t_n}}{\sqrt{n}}, t \geq 0)_{n \geq 1}$$

behaves as a diffusion as for n large and compute its volatility in terms of order flow statistics i.e. a **functional central limit theorem** for $(s_t^n)_{n \geq 1}$.
Diffusion limits of queues have been widely studied (Harrison, Reiman, Williams, Iglehart & Whitt,..) but the *price* process has no analogue in queueing theory.

Diffusion limit of the price: balanced order flow

Balanced order book (C & De Larrard (2010)

If $\lambda = \theta + \mu$ then

$$\left(\frac{s(n \log n \quad t)}{\sqrt{n}}\right)_{t \geq 0} \overset{\mathcal{D}}{\Rightarrow} \sqrt{\frac{\pi \lambda \delta^2}{D(f)}} B$$

where B is a Brownian motion, $\sqrt{D(F)} = \sqrt{\int_{\mathbb{R}_+^2} xy \; dF(x,y)}$, the geometric mean of the bid queue and ask queue sizes, is a measure of order book depth after a price change.

When observed at time scale $\tau_2 \gg \tau_0$ representing $n \log(n)$ orders, the price behaves as a diffusion with variance

$$\sigma^2 = \delta^2 \frac{\tau_2}{\tau_0} \frac{\pi \lambda}{D(f)}$$

Linking order flow and volatility

$$\sigma^2 = \delta^2 \frac{\pi\lambda}{D(f)}$$

- expresses the variance of the price increments in terms of **order flow statistics**: quantities whose estimation does NOT require to observe the price!
- a means 'microstructure' affects the volatility only through
  - the arrival rate of orders λ
  - $D(f)$ average market depth / queue size after a price change

Example : General Electric (GE), June 26 2008.
(Realized) volatility of 10-minute price changes (in annualized vol units):
$\sigma = 21.78\%$ with 95% confidence interval: [19.3 ; 23.2] \$
Our 'microstructure' volatility estimator: $\hat{\sigma} = \delta^2 \pi\lambda n / D(f) = 22.51\%$ \$
Not bad for such a simple model!
This is a first step towards incorporating information on order flow into estimators of intraday volatility.

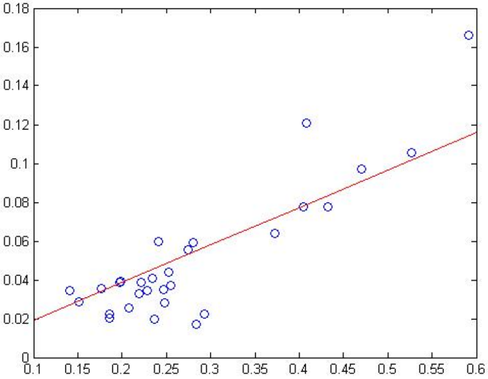


Figure: $\sqrt{\lambda / D(f)}$, estimated from tick-by-tick order flow (vertical axis) vs realized volatility over 10-minute intervals for stocks in the Dow Jones Index, June 26, 2008. Each point represents one stock.

Scaling of volatility with order frequency

$$\sigma^2 = \delta^2 \frac{\pi\lambda}{D(f)}$$

If we increase the intensity of order by a factor x,

- The intensity of limit orders becomes λx
- The intensity of market orders and cancelations becomes $(\mu + \theta)x$
- The *limit order book depth* becomes $x^2 D(f)$.

our model predicts that volatility is decreased by a factor $\sqrt{\frac{1}{x}}$.
Rosu (2010) shows the same dependence in $1/\sqrt{x}$ of price volatility using an equilibrium approach.

Diffusion limit of the price

Theorem

When $\lambda < \theta + \mu$ (market orders/ cancelations dominate limit orders),

$$\left(\frac{s(nt)}{\sqrt{n}}\right)_{t\geq 0} \overset{\mathcal{D}}{\Rightarrow} \delta\sqrt{\frac{1}{m(f,\theta+\mu,\lambda)}}B$$

where B is a Brownian motion and $m(f,\theta+\mu,\lambda) = \mathbb{E}[\tau_f]$ is the average time between two consecutive prices moves.

Diffusion limit of the price

Theorem

When $\lambda < \theta + \mu$ (market orders/ cancelations dominate limit orders),

$$\left(\frac{s(nt)}{\sqrt{n}}\right)_{t\geq 0} \overset{\mathcal{D}}{\Rightarrow} \delta\sqrt{\frac{1}{m(f,\theta+\mu,\lambda)}}B$$

where B is a Brownian motion and $m(f,\theta+\mu,\lambda) = \mathbb{E}[\tau_f]$ is the average time between two consecutive prices moves.

Remark

If τ_0 is the (UHF) time scale of incoming orders and $\tau_2 >> \tau_0$ the variance of the price increments at time scale τ_2 is

$$\sigma^2 = \frac{\tau_2}{\tau_0}\frac{\delta^2}{m(f,\lambda+\mu,\theta)}$$

Durations are not exponentially distributed..

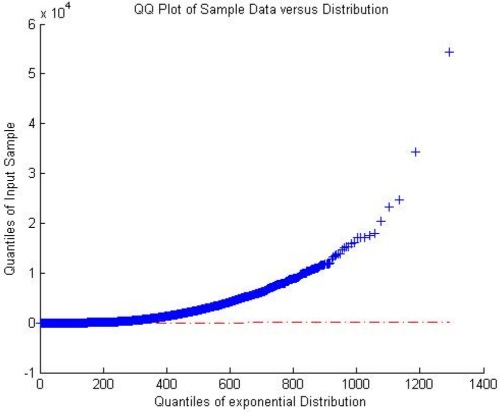


Figure: Quantile-Plot for inter-event durations, referenced against an exponential distribution. Citigroup June 2008.

Order sizes are heterogeneous

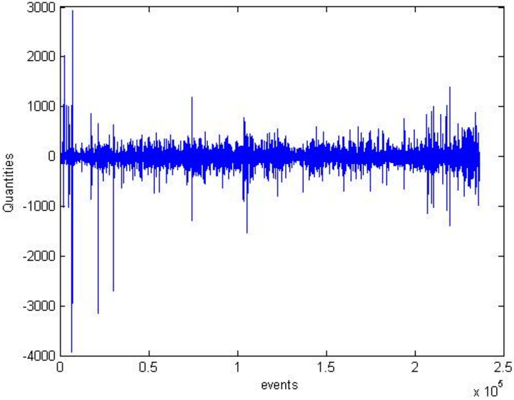


Figure: Number of shares per event for events affecting the ask. Citigroup stock, June 26, 2008.

Construction of queue sizes from net order flow

Key idea: (q_t^a, q_t^b) may be constructed from the **net order flow process**

$$X_t = (x_t^b, x_t^a) = \left(\sum_{i=1}^{N_t^b} V_i^b, \sum_{i=1}^{N_t^a} V_i^a \right)$$

where N_t^b (resp. N_t^a) is the number of events (i.e. orders or cancelations) occurring at the bid (resp. the ask) during $[0, t]$.

$$(q_t^a, q_t^b, t \in [0, T]) = \Psi(X_t, t \in [0, T], (R_n)_{n \geq 1}, (\tilde{R})_{n \geq 1})$$

where $\Psi(\omega, R, \tilde{R})$ is obtained from ω by "discontinuous reflection at the boundary of the positive quadrant": in between two exit times, the increments of $\Psi(\omega, R, \tilde{R})$ follow those of ω and each time the process attempts to exit the positive orthant by crossing the x-axis (resp. the y-axis), it jumps to a a new position inside the orthant, taken from the sequence $(R_n)_{n \geq 1}$ (resp. from the sequence $(\tilde{R}_n)_{n \geq 1}$).

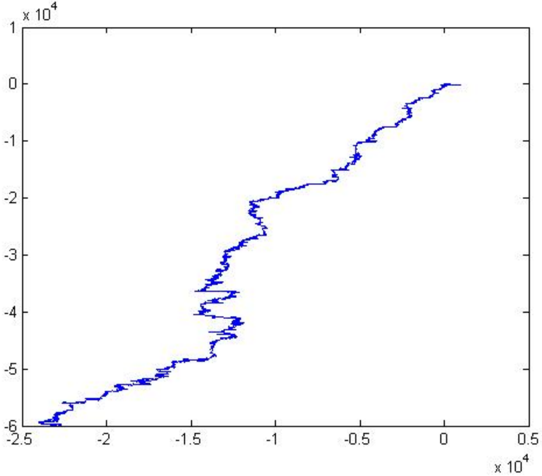
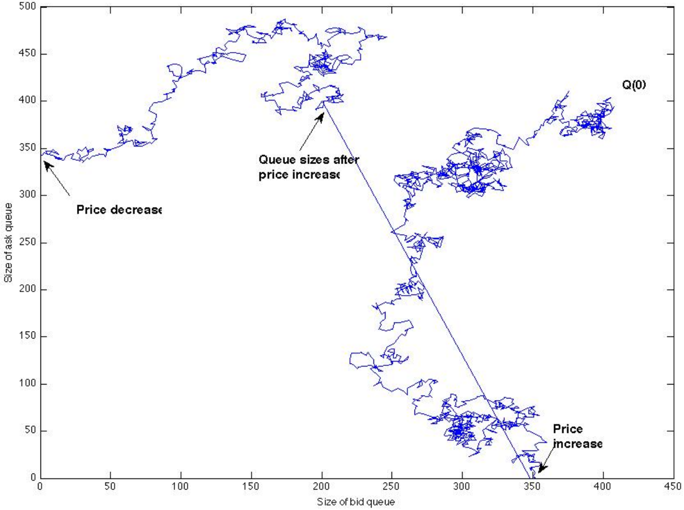


Figure: Intraday dynamics of net order flow (X^b, X^a): Citigroup, June 26, 2008.


Q(0)
Queue sizes after
price increase
Price decrease
Price
increase
Size of ask queue
Size of bid queue
0
50
100
150
200
250
300
350
400
450
500

Construction of queue sizes from net order flow

We endow $D([0,\infty),\mathbb{R}^2)$ with Skorokhod's J_1 topology and $(\mathbb{R}_+^2)^{\mathbb{N}}$ with the topology induced by

$$R^n \overset{n\to\infty}{\to} R \in (\mathbb{R}_+^2)^{\mathbb{N}} \iff \left(\forall k \geq 1, \quad \sup\{|R_1^n - R_1|, ..., |R_k^n - R_k|) \overset{n\to\infty}{\to} 0\right).$$

Theorem

Let $R = (R_n)_{n\geq 1}, \tilde{R} = (\tilde{R}_n)_{n\geq 1}$ be sequences in $]0,\infty[\times]0,\infty[$ which do not have any accumulation point on the axes. If $\omega \in C^0([0,\infty),\mathbb{R}^2)$ is such that

$$(0,0) \notin \Psi(\omega, R, \tilde{R})([0,\infty)\ ). \tag{6}$$

Then the map

$$\Psi : D([0,\infty),\mathbb{R}^2) \times (\mathbb{R}_+^2)^{\mathbb{N}} \times (\mathbb{R}_+^2)^{\mathbb{N}} \quad \to \quad D([0,\infty),\mathbb{R}_+^2) \tag{7}$$

is continuous at $(\omega, R, \tilde{R})$.

The relevance of asymptotics

	Average no. of orders in 10s	Price changes in 1 day
Citigroup	4469	12499
General Electric	2356	7862
General Motors	1275	9016

Table: Average number of orders in 10 seconds and number of price changes (June 26th, 2008).

These observations point to the relevance of *asymptotics* when analyzing the dynamics of prices in a limit order market where orders arrivals occur frequently.

From micro- to meso-structure: heavy traffic approximation

- Let τ_0 be the time scale of order arrivals (the millisecond)
- At the time scale $\tau_1 >> \tau_0$, the impact of one order is 'very small' compared to the total number of orders q^a and q^b.
- It is reasonable to approximate $q = (q^a, q^b)$ by a process whose state space is continuous $(\mathbb{R}_+^2)$
- More precisely we will show that the *rescaled* order book

$$Q_n(t) = \left(\frac{q^a(tn)}{\sqrt{n}}, \frac{q^b(tn)}{\sqrt{n}}\right)_{t\geq 0}$$

converges in distribution to a limit $Q = (Q^a, Q^b)$ of Q_n (*Heavy traffic approximation*)

Assumptions on the order arrivals

- $(T_i^a, i \geq 1)$ and $(T_i^b, i \geq 1)$ are stationary sequences with

$$\frac{T_1^a + T_2^a + \ldots + T_n^a}{n} \overset{n\to\infty}{\to} \frac{1}{\lambda^a} \qquad \frac{T_1^b + T_2^b + \ldots + T_n^b}{n} \overset{n\to\infty}{\to} \frac{1}{\lambda^b}$$

- Examples verifying these assumptions:
  - Independent Poisson processes for each order type (Cont Stoikov Talreja 2010)
  - Self exciting and mutually exciting Hawkes processes (Andersen, Cont & Vinkovskaya 2010)
  - Autoregressive Conditional Duration (ACD) model (Engle & Russell 1997)

Assumptions on order sizes

$(V_i^{n,a}, V_i^{n,b})_{i\geq 1}$ is a stationary, uniformly mixing array of random variables satisfying

$$\sqrt{n}\mathbb{E}[V_1^{a,n}] \stackrel{n\to\infty}{\to} \overline{V^a}, \qquad \sqrt{n}\mathbb{E}[V_1^{b,n}] \stackrel{n\to\infty}{\to} \overline{V^b}, \tag{8}$$

$$\lim_{n\to\infty} \mathbb{E}[(V_i^{n,a} - \overline{V^a})^2] + 2\sum_{i=2}^{\infty} \mathrm{cov}(V_1^{n,a}, V_i^{n,a}) = v_a^2 < \infty, \qquad \text{and}$$

$$\lim_{n\to\infty} \mathbb{E}[(V_i^{n,b} - \overline{V^b})^2] + 2\sum_{i=2}^{\infty} \mathrm{cov}(V_1^{n,b}, V_i^{n,b}) = v_b^2 < \infty.$$

Under this assumption one can define

$$\rho := \lim_{n\to\infty} \frac{2\max(\lambda^a,\lambda^b)\mathrm{cov}(V_1^{n,a}, V_1^{n,b}) + 2\sum_i \lambda^a \mathrm{cov}(V_1^{n,a}, V_i^{n,b}) + \lambda^b \mathrm{cov}(V_1^{n,b}, V_i^{n,a})}{v_a v_b}$$

$\rho \in (-1,1)$ may be interpreted as a measure of 'correlation' between event sizes at the bid and event sizes at the ask.

Theorem: Order book dynamics in a high-frequency order flow

$$Q_n = \left(\frac{q^a(tn)}{\sqrt{n}}, \frac{q^b(tn)}{\sqrt{n}}\right)_{t\geq 0} \overset{\mathcal{D}}{\Rightarrow} Q \quad on \quad (\mathcal{D}, J_1),$$

where Q is a right-continuous process which

- behaves like planar Brownian motion with covariance matrix

$$\begin{pmatrix} \lambda_a v_a^2 & \rho\sqrt{\lambda_a\lambda_b} v_a v_b \\ \rho\sqrt{\lambda_a\lambda_b} v_a v_b & \lambda_b v_b^2 \end{pmatrix} \tag{9}$$

  in the interior of the quarter plane $\{x > 0\} \cap \{y > 0\}$
- jumps to a value with the distribution F each time it reaches the x-axis i.e. when $Q^b_{t-} = 0$
- jumps to a value with distribution $\tilde{F}$ each time it reaches the y-axis i.e. when $Q^a_{t-} = 0$

Heavy traffic limit : technique of proof

Key idea: study the **net order flow process**

$$X_t^n = (x_t^b, x_t^a) = \left(\sum_{i=1}^{N_t^b} V_i^b, \sum_{i=1}^{N_t^a} V_i^a \right)$$

where N_t^b (resp. N_t^a) is the number of events (i.e. orders or cancelations) occurring at the bid (resp. the ask) during $[0, t]$.
Step 1: functional Central limit theorem for x: $\frac{X^n}{\sqrt{n}} \Rightarrow X$
Step 2: build$Q$ from X^n by a pathwise construction $Q = \Psi(X)$ where $\Psi : D([0,\infty), \mathbb{R}^2) \mapsto D([0,\infty), \mathbb{R}^2_+)$
Step 3: show continuity of Ψ for Skorokhod topology (D, J_1) at continuous paths which avoid (0,0).
Step 4: apply continuous mapping theorem $Q_n = \Psi(x) \Rightarrow Q = \Psi(X)$

Heavy traffic limit: description

Let τ_0 the time scale of incoming orders and $\tau_1 >> \tau_0$. Under the previous assumptions we can approximate the dynamics of the order book $a = (q^a, q^b)$ by the process Q whose dynamics between two price changes is described by a planar Brownian motion with covariance matrix

$$\Sigma = \begin{pmatrix} \lambda_a v_a^2 & \rho\sqrt{\lambda_a \lambda_b} v_a v_b \\ \rho\sqrt{\lambda_a \lambda_b} v_a v_b & \lambda_b v_b^2 \end{pmatrix} \tag{10}$$

- $\mathbb{E}[T_1^a] = 1/\lambda_a$ $\mathbb{E}[T_1^b] = 1/\lambda_b$: average duration between events
- $v_a^2 = \mathbb{E}[(V_1^a)^2] + 2\sum_{i=2}^{\infty} Cov(V_1^a, V_i^a)$: variance of order sizes at ask
- ρ "correlation" between the order sizes at the bid and at the ask.

If order sizes at bid and ask are symmetric and uncorrelated then $\rho = 0$.
Empirically we find that $\rho < 0$ for all data sets examined.

Intraday price dynamics

Proposition (Cont & De Larrard 2011)

Under the same assumptions

$$(s_{nt}, t \geq 0) \overset{n\to\infty}{\Rightarrow} S,$$

where

$$S_t = \sum_{0\leq s\leq t} \mathbf{1}_{Q^a(t-)=0} - \sum_{0\leq s\leq t} \mathbf{1}_{Q^b(t-)=0} \tag{11}$$

is a piecewise constant cadlag process which

- increases by one tick every time the process $(Q(t-), t \geq 0)$ hits the horizontal axis $\{y = 0\}$ and
- decreases by one tick every time $(Q(t-), t \geq 0)$ hits the vertical axis $\{x = 0\}$.

This characterization allows to compute in detail various probabilistic properties of price dynamics and relate them to order flow parameters.

Duration between price changes (R C & Larrard, 2010

If $\overline{V^a} = \overline{V^b} = 0$, $\mathbb{P}[\tau > t | Q_0^a = x, Q_0^b = y] =$

$$\sqrt{\frac{2U}{\pi t}} e^{-\frac{U}{4t}} \sum_{n=0}^{\infty} \frac{1}{(2n+1)} \sin \frac{(2n+1)\pi\theta_0}{\alpha} \left(I_{(\nu_n-1)/2}\left(\frac{U}{4t}\right) + I_{(\nu_n+1)/2}\left(\frac{U}{4t}\right)\right),$$

where $\nu_n = (2n+1)\pi/\alpha$, I_n is the nth Bessel function,

$$U = \frac{\left(\frac{x}{\lambda_a v_a^2}\right)^2 + \left(\frac{y}{\lambda_b v_b^2}\right)^2 - 2\rho \frac{xy}{\lambda_a \lambda_b v_a^2 v_b^2}}{(1-\rho)},$$

$$\alpha = \begin{cases} \pi + \tan^{-1}\left(-\frac{\sqrt{1-\rho^2}}{\rho}\right) & \rho > 0 \\ \tan^{-1}\left(-\frac{\sqrt{1-\rho^2}}{\rho}\right) & \rho < 0 \end{cases} \qquad \theta_0 = \begin{cases} \pi + \tan^{-1}\left(-\frac{y\sqrt{1-\rho^2}}{x-\rho y}\right) \\ \tan^{-1}\left(-\frac{y\sqrt{1-\rho^2}}{x-\rho y}\right) \end{cases}$$

Duration between price changes

- Spitzer (1958) computed the tail index of τ as a function of the correlation coefficient ρ
- The tail index of τ given x orders at the ask and y orders at the bid is $\dfrac{\pi}{\pi + 2\arcsin(\rho)}$
  - If $\rho = 0$, the tail index is 1. The tail index was the same for the Markovian order book when $\lambda = \theta + \mu$
  - If $\rho > 0$, the tail index is strictly less than one
  - If $\rho < 0$, the tail index is higher than one: The duration between consecutive price moves has a finite first moment

Forecasting price moves from the Level I order book

Proposition

(R C & Larrard, 2010): The probability $p_{\mathrm{up}}(x,y)$ that the next price move is an increase, given a queue of x shares on the bid side and y shares on the ask side is

$$p_{\mathrm{up}}(x,y) = \frac{1}{2} - \frac{\arctan\left(\sqrt{\frac{1+\rho}{1-\rho}}\,\frac{\frac{y}{\sqrt{\lambda_a v_a}} - \frac{x}{\sqrt{\lambda_b v_b}}}{\frac{y}{\sqrt{\lambda_a v_a}} + \frac{x}{\sqrt{\lambda_b v_b}}}\right)}{2\arctan\left(\sqrt{\frac{1+\rho}{1-\rho}}\right)}, \tag{13}$$

Avellaneda, Stoikov & Reed (2010) computed this for the case $\rho = -1$.
When $\rho = 0$ (independent flows at bid and ask)
$p_{\mathrm{up}}(x,y) = 2\arctan(y/x)/\pi$.

Probability of upward price move conditional on queue sizes

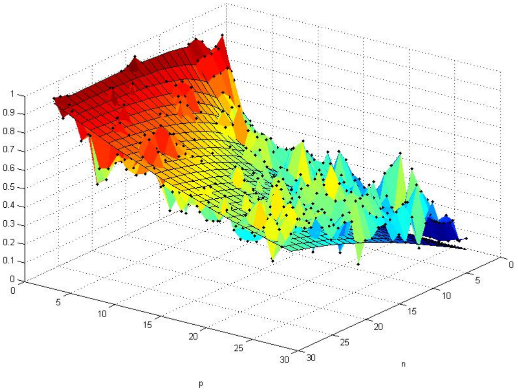


Figure: Conditional probability of a price increase, as a function of the bid and ask queue size (solid curve) compared with transition frequencies for CitiGroup

Many econometric models of intraday price dynamics assume the existence of a latent 'true' or 'efficient' price process -assumed to be a martingale- and such that the bid/ask prices are rounded/discretized version of this process.
In our model we can in fact *exhibit* this process: given the bid/ask queue dynamics, it not latent but a function of (Q_t^b, Q_t^a):

Proposition (Martingale price)

If $p^+ = p^- = 1/2$, then

$$P_t = S_t^b + \delta(2p^{up}(Q_t^b, Q_t^a) - 1)$$

is a continuous martingale.

If $\rho = -1$ this becomes an average of bid/ask prices weighted by queue size, an indicators used by many traders (Burghardt et al):

$$P_t = \frac{Q_t^a}{Q_t^a + Q_t^b} S_t^b + \frac{Q_t^b}{Q_t^a + Q_t^b} S_t^a.$$

Diffusion limit of the price

At a *tick* time scale the price is a piecewise constant, discrete process.
But over larger time scales, prices are observed to have "diffusive" dynamics and modeled as such.
Consider a time scale t_n over which n orders (limit, market, cancel) arrive.
Does the rescaled price process

$$s_t^n = \frac{s_{t^n}}{\sqrt{n}}$$

behave like a diffusion?
What is this diffusion limit?
How is the "low frequency" volatility of the price related to order flow statistics?
Approach: study low frequency description of price dynamics by deriving functional limit theorem for the price process $(s_t^n, t \geq 0)$ as $n \to \infty$

Link between intraday price trend and order flow

Probability of two successive price increases $p_+ = \int_{\mathbb{R}_+^2} p_{up}(x,y)F(dx\ dy)$
Probability of successive price decreases
$p_- = \int_{\mathbb{R}_+^2} (1 - p_{up}(x,y))\,\tilde{F}(dx\ dy)$
Empirically $p_+ < 1/2, p_- < 1/2$, due to **asymmetry** of $F, \tilde{F}$ which induces **mean reversion** in the price.

Theorem (Fluid limit)

$\frac{S(nt)}{n} \to \mu\ t$, where μ is an intraday trend/drift given by

$$\mu = \frac{\frac{p_+}{1-p_+} - \frac{p_-}{1-p_-}}{\frac{p_+}{1-p_+}\tau_F + \frac{p_-}{1-p_-}\tau_{\tilde{F}}}$$

where $\tau_F = E[\int_{\mathbb{R}_+^2} \tau(x,y)F(dx\ dy)]$ is the average duration between price changes after a price increase, $\tau_{\tilde{F}} = E[\int_{\mathbb{R}_+^2} \tau(x,y)\tilde{F}(dx\ dy)]$ is the average duration between price changes after a price decrease.

Diffusion limit of the price

Theorem (R.C, & de Larrard, 2010)

- *When* $\rho = 0$,

$$\left(\frac{s(n \log n \quad t)}{\sqrt{n}}\right)_{t \geq 0} \overset{\mathcal{D}}{\Rightarrow} \sigma B$$

where

$$\sigma^2 = \frac{\pi \delta^2 v^2 \lambda}{D(F)} \qquad D(F) = \int_{\mathbb{R}_+^2} xyF(dx, dy).$$

- *When* $\rho < 0$,

$$\left(\frac{s(n \quad t)}{\sqrt{n}}\right)_{t \geq 0} \overset{\mathcal{D}}{\Rightarrow} \sigma B$$

$$\text{where}\sigma^2 = \frac{\delta^2}{m(f, \sigma_Q, \rho)}, \qquad \text{and} \quad m(f, \sigma_Q, \rho) = \mathbb{E}[\tau_f]$$

is the expected hitting time of the axes by B.

Link between volatility and order flow: symmetric case

The variance of price increments at time scale $\tau_2 \gg \tau_1$ is thus given by

$$\sigma^2 = \frac{p}{1-p}\frac{\tau_2}{\tau_1}\frac{\pi\delta^2 v^2\lambda}{D(f)} \qquad D(f) = \int_{\mathbb{R}_+^2} xy \quad F(dx,dy).$$

So: intraday volatility emerges as a tradeoff between

- average rate of fluctuation of the order book : λv^2
- a measure of order book depth : (multiplicative) average of bid and ask queue size $D(f) = \int_{\mathbb{R}_+^2} xyF(dx,dy)$
- a measure of order book asymmetry : $p=$ probability of two consecutive price changes $\rightarrow$ mean reversion

Link between volatility and order flow: empirical test

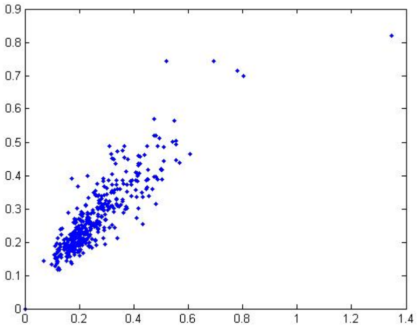


Figure: Empirical std deviation of 10 min returns vs theoretical prediction of volatility based on diffusion limit of queueing model for SP500 stocks.

Unbalanced order flow: "Flash Crash"

When sell orders exceed buy orders by an order of magnitude, the price acquires a negative trend and drops linearly and this the deterministic trend of the price dominates price volatility. If

$$\left(\mathbb{E}[V_1^{n,b}], n^{\beta}\mathbb{E}[V_1^{n,a}]\right) \stackrel{n\to\infty}{\to} (\Pi^b, \overline{V^a}) \quad \text{with} \quad \Pi^b < 0 \quad \text{and} \quad \overline{V^a} \geq 0,$$

$$\frac{T_1^{n,b} + \ldots + T_n^{n,b}}{n} \to \frac{1}{\lambda^b}, \quad \frac{T_1^{n,a} + \ldots + T_n^{n,a}}{n} \to \frac{1}{\lambda^a},$$

$$n^2 \tilde{f}_n(n., n.) \Rightarrow \tilde{F}.$$

then low-frequency price dynamics becomes 'balistic':

$$\left(\frac{S^b_{[nt]}}{n}, t \geq 0\right) \Rightarrow \left(\frac{\lambda^b \Pi^b}{\int_{\mathbb{R}^2} y\tilde{F}(dx, dy)} t, t \geq 0\right).$$

Conclusion

- Limit order book may be modeled as queueing systems
- Asymptotic methods(heavy traffic limit, Functional central limit theorems) give analytical insights into link between higher and lower frequency behavior, between order flow properties and price dynamics.
- General assumptions: finite second moment of order sizes, finite first moment of quote durations and weak dependence, allows for dependence in order arrival times and sizes
- Allows for **dependent** order durations, dependence between order size and durations, autocorrelation, ...
- Explicit expression of probability transitions of the price
- Distribution of the duration between consecutive price moves
- Different regimes for price behavior depending on the correlation between buy and sell order sizes
- Relates price volatility to orders flow statistics

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