Polynomial Preserving Diffusions on the Unit Ball

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joint work with Sergio Pulido

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ETH zürich

Polynomial models in finance

Factor models

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- **Task:** Find X and f to get tractable yet flexible class of models

Polynomial preserving diffusions

- Markov process X with state space $E \subseteq \mathbb{R}^d$
- (Extended) generator $\mathscr{G}$ given by

$$\mathscr{G} f(x) = b(x)^\top \nabla f(x) + \frac{1}{2} \operatorname{Tr}\left(a(x) \nabla^2 f(x)\right)$$

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Definition. X is called **Polynomial Preserving (PP)** if

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Lemma. X is (PP) if and only if

$$b_i \in \mathrm{Pol}_1(E) \qquad \text{and} \qquad a_{ij} \in \mathrm{Pol}_2(E)$$

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- **Key consequence:**

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- This only involves a matrix exponential as opposed to solving a PDE which leads to tractable pricing models

Polynomial preserving diffusions

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- Pearson diffusions (Forman and Sørensen, 2008), $E \subset \mathbb{R}$:

$$\mathrm{d}X_t = (\beta + bX_t)\mathrm{d}t + \sqrt{\alpha + aX_t + AX_t^2}\,\mathrm{d}W_t$$

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More generally:

- Existence theory for polynomial preserving diffusions is available when E is a **basic closed semialgebraic set**:

$$E = \{x \in \mathbb{R}^d \ : \ p_1(x) \geq 0, \ldots, p_m(x) \geq 0\}$$

where $p_1, \ldots, p_m \in \mathrm{Pol}(\mathbb{R}^d)$. See Filipović & L. (2014).

Literature

- Wong (1964)
- Mazet (1997)
- Zhou (2003)
- Forman and Sørensen (2008)
- Cuchiero, Keller-Ressel, Teichmann (2012)
- Filipović, Gourier, Mancini (2013)
- Filipović, L. (2014)
- Bakry, Orevkov, Zani (2014)
- Filipović, L., Trolle (2014)
- ...

(PP) diffusions on the unit ball

The role of the state space

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Example. Affine diffusions on $E = \mathbb{R}^d_+$:

$$dX_t = (b + BX_t)dt \; + \; \begin{pmatrix} \sigma_1\sqrt{X_{1t}} & 0 & \cdots & 0 \\ 0 & \sigma_2\sqrt{X_{2t}} & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \sigma_d\sqrt{X_{dt}} \end{pmatrix} dW_t$$

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- Compact state spaces useful for polynomial approximation

Theorem. If E is compact and X is an E-valued affine diffusion, then X is **deterministic**.

PP diffusions on the unit ball

Example.

$$dX_t = -X_t\, dt + \sqrt{1 - \|X_t\|^2}\, dW_t$$

where $W = (W^1, \ldots, W^d)$ is d-dimensional BM.

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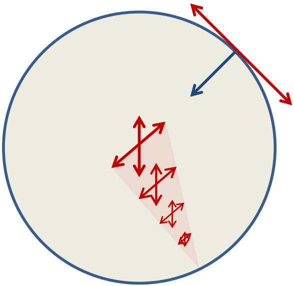
But richer diffusion dynamics is possible:



PP diffusions on the unit ball

Theorem. X is a PP diffusion on the unit ball if and only if

$$b(x) = b + Bx \qquad \text{and} \qquad a(x) = (1 - \|x\|^2)\alpha + c(x)$$

for some $b \in \mathbb{R}^d$, $B \in \mathbb{R}^{d\times d}$, $\alpha \in \mathbb{S}^d_+$, and $c \in \mathscr{C}_+$ such that

$$b^\top x + x^\top Bx + \frac{1}{2}\operatorname{Tr}(c(x)) \leq 0 \quad \text{for all} \quad x \in \mathscr{S}^{d-1}.$$

Here $\mathscr{S}^{d-1}$ is the unit sphere in $\mathbb{R}^d$, and

$$\mathscr{C}_+ = \left\{ c : \mathbb{R}^d \to \mathbb{S}^d \ : \ \begin{array}{l} c_{ij} \in \mathrm{Hom}_2 \text{ for all } i,j \\ c(x)x \equiv 0 \\ c(x) \in \mathbb{S}^d_+ \text{ for all } x \end{array} \right\}$$

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- Take $A_1 \in \mathrm{Skew}(d)$ and set

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- ... but is this exhaustive?

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$\mathscr{C}_+ \cong$ {all nonnegative biquadratic forms with vanishing diagonal}

$\stackrel{?}{=}$ {all **SOS** biquadratic forms with vanishing diagonal}

Nonnegativity vs. sum of squares

- **Hilbert (1888):** Every nonnegative homogeneous polynomial of degree k in d variables is **SOS** if and only if

$$d = 2 \qquad \text{or} \qquad k = 2 \qquad \text{or} \qquad (d, k) = (3, 4).$$

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- **Quarez (2010)** on biquadratic forms:
  - Every nonnegative biquadratic form in $3 + 3$ variables with at least 11 zeros is **SOS**
  - There exist nonnegative biquadratic forms in $4 + 4$ variables with infinitely many zeros that are not **SOS**

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- **László (2010):** There exist nonnegative, non-**SOS** biquadratic forms with vanishing diagonal in $6 + 6$ variables.

Nonnegativity vs. sum of squares

Theorem.

(i) If $d \leq 4$, then any nonnegative biquadratic form in $d + d$ variables vanishing on the diagonal is **SOS**. Equivalently, any $c \in \mathscr{C}_+$ is of the form

$$c(x) = \sum_{p=1}^{m} A_p x x^\top A_p^\top \qquad \text{for some} \qquad A_1, \ldots, A_m \in \mathrm{Skew}(d).$$

(ii) If $d \geq 6$, then there exist nonnegative biquadratic forms in $d + d$ variables vanishing on the diagonal that is not **SOS**. Equivalently, there exist $c \in \mathscr{C}_+$ that is not of the above form.

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Open question: What happens for $d = 5$?

Nonnegativity vs. sum of squares

Example. Let $d = 6$. The map $c : \mathbb{R}^d \to \mathbb{S}^d$ with components

$$c_{11} = (x_2 + x_3 + x_4 + x_5 + x_6)^2$$
$$c_{12} = (x_2 + x_3 + x_4 + x_5 + x_6)(-x_1 + x_3 + x_4 + x_5 + x_6)$$
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$$c_{66} = (x_1 + x_2 + x_3 + x_4 + x_5)^2$$

lies in $\mathscr{C}_+$ but $y^\top c(x) y$ is not **SOS**.

Consequences of **SOS**: PP diffusions on the unit sphere

- Let X be PP diffusion on $\mathscr{S}^{d-1} = \{x \in \mathbb{R}^d : \|x\| = 1\}$

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- $\mathscr{S}^{d-1}$ is an interesting state space for applications:
  - It is **compact**: polynomial approximation works well
  - It has **no boundary**: simulation works well
  - Let $\boldsymbol{Z}_t = [X_t^{(1)} \ \cdots \ X_t^{(n)}]$ be valued in $(\mathscr{S}^{d-1})^n$, $n \geq d$. Then

$$\boldsymbol{C}_t = \boldsymbol{Z}_t^\top \boldsymbol{Z}_t$$

    is an $n \times n$ **correlation matrix** of rank at most d.

Consequences of **SOS**: PP diffusions on the unit sphere

Theorem. Let X be a PP diffusion on $\mathscr{S}^{d-1}$. Equivalent are:

- $y^\top c(x) y$ is **SOS**.
- X can be realized as the unique strong solution to the SDE

$$dX_t = (\circ\, dY_t)\, X_t,$$

  where Y is correlated Brownian motion with drift on $\mathrm{Skew}(d)$:

$$Y_t = A_0\, t + A_1 W_t^1 + \cdots + A_m W_t^m$$

  with $A_0, \ldots, A_m \in \mathrm{Skew}(d)$ and m-dim BM $(W^1, \ldots, W^m)$.
- $\mathscr{G}$ can be expressed in Hörmander form as

$$\mathscr{G} = V_0 + \frac{1}{2} \sum_{p=1}^{m} V_p^2,$$

  where V_p is the linear vector field $V_p(x) = A_p x$, $A_p \in \mathrm{Skew}(d)$.

Consequences of **SOS**: PP diffusions on the unit sphere

Corollary (existence of density). Let X be a PP diffusion on $\mathscr{S}^{d-1}$ such that $y^\top c(x) y$ is **SOS**. The following are equivalent:

(i) X_t ($t > 0$) has a smooth density w.r.t. area measure on $\mathscr{S}^{d-1}$

(ii) $\mathrm{Lie}\{A_1, \ldots, A_m\} = \mathrm{Skew}(d)$

Conclusion

- (PP) processes can be used to build flexible and tractable models
- Geometry of the state space crucially affects factor dynamics
- The unit ball is an interesting example of a compact state space allowing for rich factor dynamics
- (PP) diffusions with the SOS property ...
  - ... can be completely parameterized
  - ... can be represented as strong solutions to SDE
  - ... admit simple conditions for existence of smooth density

Thank you!