Optimal Investment and Consumption with Small Transaction Costs

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Introduction

Portfolio Choice with Frictions

- Optimal portfolio selection is a key problem in finance.
  - Individual decision making.
  - Starting point for equilibrium models.
- Frictionless theory after Merton (1969, 1973) prescribes incessant rebalancing.
  - Not feasible with *frictions*.
- Optimal behavior should reflect tradeoff between:
  - Displacement from optimal allocation.
  - Costs of trading.
- Important?
- When and how?
- Simple and robust adjustments?

Introduction

Passive Investment

- This talk: *proportional* costs. Bid-ask spreads.
- Key insight (Magill/Constantinides, 1976):
  - *No-trade region* around the frictionless target.
  - Remain inactive while inside.
  - Start trading when boundaries are breached.
- Mathematically precise formulation?
  - Davis/Norman (1990). Shreve/Soner (1994).
  - Singular control. Reflected diffusions à la Harrison.
- Numerical results (Constantinides, 1986):
  - Even small costs have large effect on asset demand.
  - But welfare loss is small if trading is reduced optimally.
- Assumes constant investment opportunities.
  - Strategies almost passive. Rebalancing is only motive to trade.
  - What about more active trading strategies?

Introduction

Active Investment

- Complex models typically intractable with frictions.
- Numerical results in concrete settings:
  - Lynch/Tan (2011). Backward induction.
  - Dai, Li, Liu, Wang (2014). Coupled PDEs.
  - Frictions have much bigger effect.
  - But difficult to understand structure and comparative statics.
- Alternative: *asymptotics* for small costs.
  - Treat problem as perturbation of its frictionless counterpart.
  - Compute leading-order corrections of optimal policy and performance.
  - Constant investment opportunities: Shreve/Soner (1994). Janeček/Shreve (2004).
- Recent progress for active investment strategies..

Introduction

Active Investment ct'd

- Hedge a call option in Black-Scholes framework.
  - Whalley/Wilmott (1997). Almgren/Li (2011). Bichuch (2014).
- Maximize sum of one-period mean variance profits.
  - Mean-reversion strategies: Martin/Schöneborn (2011).
  - Trend following: Martin (2012). Bouchaud et al. (2012). Garleanu/Pedersen (2013). Collin-Dufresne et al. (2014).
- Infinite horizon investment/consumption problems.
  - Soner/Touzi (2013). Possamaï, Soner, Touzi (2014).
- Different concrete models and objectives.
- This talk: pass to *general* setting.
  - Uncovers underlying general structure.
  - Resulting formulas are easy to interpret and implement. Robust with respect to particular model specifications.

Results

Model

General asset prices:

- One safe asset. Normalized to one.
- One risky asset. Traded with proportional costs $\varepsilon_t = \varepsilon \mathcal{E}_t > 0$.
  Mid price:

$$dS_t = b_t^S dt + \sqrt{c_t^S} dW_t$$

- General diffusive dynamics.
- Can include heteroskedasticity and predictable returns leading to market timing.
- No Markovian structure required.
- Transaction costs can be random and time-varying.
  Itô process $\mathcal{E}_t$ rescaled by small parameter ε.

Results

Model

General investment/consumption problem:

- Investor solves:

$$E\left[\int_0^T u_1(t,\kappa_t^\varepsilon)dt + u_2(X_T^\varepsilon(\varphi^\varepsilon,\kappa^\varepsilon))\right] \to \max!$$

  over policies $(\varphi_t^\varepsilon, \kappa_t^\varepsilon)$ with wealth processes

$$X_t^\varepsilon(\varphi^\varepsilon,\kappa^\varepsilon) = X_0 + \int_0^t \varphi_s^\varepsilon dS_s - \int_0^t \kappa_s^\varepsilon ds + \Psi_t - \int_0^t \varepsilon_s d||\varphi^\varepsilon||_s$$

- Utility from intermediate consumption and terminal wealth. Random endowment stream Ψ_t.
- Covers hedging, lifecycle investing, market timing, etc.

Results

Approximately Optimal Policy

Adjustment of the frictionless optimal policy (Kallsen/M-K, 2014):

- Use frictionless consumption.
  - Robust. Only adjust for reduced wealth.
- Time- and state-dependent no-trade region:

$$[\overline{\mathrm{NT}}_t - \Delta\mathrm{NT}_t, \overline{\mathrm{NT}}_t - \Delta\mathrm{NT}_t]$$

- Midpoint $\overline{\mathrm{NT}}_t$ is frictionless target.
  - Also only adjusted for reduced wealth.
- Half-width $\Delta\mathrm{NT}_t$ is the crucial quantity:

$$\Delta\mathrm{NT}_t = \left(\frac{3R_t}{2}\frac{d\langle\varphi\rangle_t}{d\langle S\rangle_t}\varepsilon_t\right)^{1/3}$$

Results

Approximately Optimal Policy ct'd

- Half-width of optimal no-trade region:

$$\left(\frac{3R_t}{2}\frac{d\langle\varphi\rangle_t}{d\langle S\rangle_t}\varepsilon_t\right)^{1/3}$$

- Key driver: *portfolio gamma* $d\langle\varphi\rangle_t/d\langle S\rangle_t$.
  - Ratio of squared diffusion coefficients.
  - Active strategies require wide buffer.
  - Turbulent markets call for close tracking.
  - For delta-hedge: option gamma.
  - Sample from realized variance of frictionless benchmark.
- Only current spread ε_t matters for correction.
  - Future dynamics not hedged at the leading order.
- Preferences subsumed by *indirect risk-tolerance* R_t.

Results

Indirect Risk-Tolerance Process

Measure for risk tolerance?

- Risk tolerance $R_t = -\frac{U'(t,X_t)}{U''(t,X_t)}$ of the *indirect* utility:

$$U(t,x) = \sup_{(\varphi,\kappa)} E_t\left[\int_t^T u_1(s,\kappa_s)ds + u_2\left(x + \int_t^T \varphi_s dS_s - \int_t^T \kappa_s ds\right)\right]$$

- Current against future consumption. Average over scenarios.
- Quantifies wealth-dependence of preferences.
- Bound to appear in *any* perturbation of frictionless problems.
  - Utility-based prices and hedges for small claims (Kramkov/Sîrbu, 2006).
  - Sensitivity of optimal consumption streams w.r.t. perturbations of the endowment (Herdegen/M-K, 2015).
- Here: novel *dynamic* characterization by quadratic BSDE.

Results

Performance

- Performance loss due to trading costs?
- Maximal utilities: $U(x)$ without and $U^{\varepsilon}(x)$ with costs.
- Certainty equivalent loss (Kallsen/M-K, 2014):

$$U^{\varepsilon}(x) \sim U\left(x - E^{Q}\left[\int_0^T \frac{(\Delta\mathrm{NT}_t)^2}{2R_t} d\langle S\rangle_t\right]\right)$$

- Portfolio gamma $d\langle\varphi\rangle_t / d\langle S\rangle_t$ quantifies liquidity risk. Appealingly robust proxy: also central for..
  - ..discrete trading (Bertsimas, Kogan, Lo, 2000; Hayashi/Mykland, 2005)
  - ..optimal discretization (Fukasawa, 2011, 2013; Rosenbaum/Tankov, 2014)
  - ..other trading costs (Altarovici, M-K, Soner, 2013; Moreau, M-K, Soner, 2014)

Results

Performance ct'd

Performance loss:

- Portfolio gamma $d\langle \varphi \rangle_t / d\langle S \rangle_t$ determines magnitude.
  - Transaction costs matter for *active* trading!
- *Universal* scaling for welfare effect of small costs:
  - Two thirds caused by trading costs.
  - One third by displacement.
- For small transaction *tax* in the spirit of Tobin:
  - Two thirds of welfare loss paid to government.
    Can be redistributed.
  - One third dissipates. True "friction".
- Result surprisingly robust. Independent of asset price and cost dynamics, preferences, endowments.
- Only assumptions: diffusive prices, proportional cost structure.

Results

General Equilibrium

So far: partial equilibrium models. General equilibrium?

- Needed to analyze policies like a financial transaction tax.
  - 2/3-1/3 split of welfare losses robust?
- Finance literature: numerical solution of discrete models.
  - Buss/Dumas (2013). Buss, Uppal, Vilkov (2013).
- Only exception: Lo, Mamaysky, Wang (2004).
  - Asymptotic analysis of a particular model with fixed costs.
  - Bank account exogenous. No full equilibrium.
- Current work in progress with Martin Herdegen:
  - Endogenous asset returns and interest rates.
  - Two agents. Receive endowments. Invest and consume optimally in frictionless equilibrium.
  - Linear transaction tax paid to state. Consumes optimally.

Results

General Equilibrium ct'd

Effect of a *small* friction?

- Assume all agents use leading-order optimal strategies.
- No-trade regions have to match for stock market clearing:
  - Midpoints offset for exponential utilities.
  - Also need $\left(\frac{3R_t^1}{2}\frac{d\langle\varphi^1\rangle_t}{d\langle S\rangle_t}\varepsilon_t^1\right)^{1/3} = \left(\frac{3R_t^2}{2}\frac{d\langle\varphi^2\rangle_t}{d\langle S\rangle_t}\varepsilon_t^2\right)^{1/3}$
  - Frictionless market clearing implies $d\langle\varphi^1\rangle_t = d\langle\varphi^2\rangle_t$.
  - Split of tax $\varepsilon = \varepsilon_t^1 + \varepsilon_t^2$ determined by risk tolerances R_t^1, R_t^2.
- Consumptions of agents and state need to clear bond market.
  - Can be ensured using sensitivity analysis of optimal consumption streams (Herdegen/M-K, 2015).
- Final result: frictionless equilibrium robust.
  - Does not need to change because of small friction.
  - Partial equilibrium analysis justified for exponential utilities.

Summary

- Approximately optimal policy with small proportional transaction costs.
  - "Myopic" correction for small frictions.
  - Drivers: current trading cost, indirect risk tolerance, portfolio gamma.
- Leading order welfare loss.
  - 2/3 due to trading costs, 1/3 due to displacement.
  - Portfolio gamma $d\langle\varphi\rangle_t/d\langle S\rangle_t$ quantifies liquidity risk.
- Results are very robust.
  - General preferences, price and cost dynamics.
  - Random endowments.
  - No Markovian structure required.
  - Results extend to other optimization criteria and frictions.
  - Extension to general equilibrium for exponential utilities.

Derivations

Small-Cost Expansion

- How to derive the results summarized above?
  - General, non-Markovian, singular control problem.
  - Where do the myopic small-cost corrections come from?
- Let us sketch the idea on an informal level.
- For simplicity:
  - Utility from terminal wealth only.
  - Constant absolute risk tolerance $R = -u_2'/u_2''$.
- Perform second-order Taylor expansion around the frictionless optimal wealth process $x + \int_0^T \varphi_t dS_t$.
- Two pertuarbations:
  - Small trading cost ε_t.
  - Small adjustment $\Delta\varphi_t$ of the trading strategy.

Derivations

Transaction Costs and Displacement

$$
\begin{aligned}
& E\left[u_2\left(x+\int_0^T(\varphi_t+\Delta\varphi_t)dS_t-\int_0^T \varepsilon_t d||\varphi+\Delta\varphi||_t\right)\right] \\
& \approx E\left[u_2\left(x+\int_0^T \varphi_t dS_t\right)\right] \\
& \quad +\beta E_Q\left[\int_0^T \Delta\varphi_t dS_t-\int_0^T \varepsilon_t d||\varphi+\Delta\varphi||_t\right] \\
& \quad -\frac{1}{2}\beta E_Q\left[R^{-1}\left(\int_0^T \Delta\varphi_t dS_t-\int_0^T \varepsilon_t d||\varphi+\Delta\varphi||_t\right)^2\right]
\end{aligned}
$$

- Here: Q is the frictionless dual *martingale* measure with density $dQ/dP = u_2'(x+\int_0^T \varphi_t dS_t)/\beta$.

Derivations

Transaction Costs and Displacement ct'd

- Whence:

$$
\begin{aligned}
&E\left[u_2\left(x+\int_0^T(\varphi_t+\Delta\varphi_t)dS_t-\int_0^T\varepsilon_t d||\varphi+\Delta\varphi||_t\right)\right]\\
&\approx E\left[u_2\left(x+\int_0^T\varphi_t dS_t\right)\right]-\beta E_Q\left[\int_0^T\varepsilon_t d||\varphi+\Delta\varphi||_t\right]\\
&\quad-\frac{1}{2}\beta E_Q\left[R^{-1}\int_0^T(\Delta\varphi_t)^2 d\langle S\rangle_t\right]
\end{aligned}
$$

- First correction term represents expected transaction cost loss.
- Second corresponds to displacement loss.
- Computation?

Derivations
Homogenization

- Ansatz: optimal strategy remains close to frictionless target by reflection off trading boundaries.
- Whence: deviation follows reflected diffusion.
- Change of time, space: approximate by reflected Brownian motion with infinitesimal variance $d\langle\varphi\rangle_t/dt$ at the first order.
- Transaction costs = local time at boundaries.
  - Expectation given by $(d\langle\varphi\rangle_t/dt)/2\Delta\mathrm{NT}_t$.
- Stationary law uniform.
  - Ergodic theorem allows to replace squared deviation $\Delta\varphi_t^2$ by expectation $\Delta\mathrm{NT}^2/3$.
- Separation of time scales. Fast variable is "homogenized" out.

Derivations

Pointwise Optimization

In summary:

- Transaction cost loss:

$$\beta E_Q \left[\int_0^T \varepsilon_t \frac{d\langle \varphi \rangle_t / dt}{2\Delta\mathrm{NT}_t} dt \right]$$

- Displacement loss:

$$\frac{\beta}{3R} E_Q \left[\int_0^T \Delta\mathrm{NT}_t^2 \frac{d\langle S \rangle_t}{dt} dt \right]$$

- Optimal boundaries determined by *pointwise* maximization:

$$\Delta\mathrm{NT}_t = \left(\frac{3R}{2} \frac{d\langle \varphi \rangle_t}{d\langle S \rangle_t} \varepsilon_t \right)^{1/3}$$

Derivations

Other frictions

Other trading costs? ($\rightsquigarrow$ tonight's talk)

- Basic idea similar. But renormalized deviations differ:
  - Reflected Brownian motion for proportional costs.
  - Fixed costs: killed Brownian motion restarted at the origin.
  - OU-type process with quadratic costs.
- Trading costs scale differently.
- Asymptotic stationary law depends on control used:
  - Uniform for proportional costs.
  - "Hat function" for fixed costs.
  - Gaussian for quadratic costs.

For papers and preprints:

- http://www.math.ethz.ch/~jmuhleka/