Multivariate lack of memory for default times: EV Copulas & a new Marshall Olkin characterisation

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Content I

The problem of all-survival I

We write

$$\underline{\tau} = [\tau_1, \tau_2, \ldots, \tau_N]$$

a vector of random times (typically default times).

We are interested in the statistics of joint survival first, namely in the probability of events ($\underline{T} = [T_1, T_2, \ldots, T_N]$ deterministic times)

$$\mathbb{P}(\underline{\tau} > \underline{T}) := \mathbb{P}(\tau_1 > T_1,\ \tau_2 > T_2, \ldots,\ \tau_N > T_N)$$

Define this joint survival function as $G(T_1, \ldots, T_N)$.
When $T_1 = T_2 = \ldots T_N = T$ we simply write $\mathbb{P}(\underline{\tau} > T)$ for $\mathbb{P}(\underline{\tau} > T\underline{1})$.

The problem of all-survival II

PROBLEM 1

Find conditions under which sampling two independent G-distributed random vecors $\underline{\tau}^1$ for survival of all the components up to T and $\underline{\tau}^2$ for survival of all on a further time T is equivalent to sampling directly survival of all components of $\underline{\tau}$ up to $2T$.

"Can we reduce a single one-shot simulation of 'survival-of-all' up to $2T$ in simulation of 'survival-of-all' steps of size T using the same distribution?"

(generalizing later to n steps). Practical interest?

The problem of all-survival III

- **Consistency with "Brownian-driven" asset classes simulation.** Risk measure or valuation adjustment simulation. Evolve risk factors according to common controlled time steps. Natural for asset models that are Brownian driven but harder when trying to include defaults. This is because default times, typically in intensity models, should be simulated just once, being static random variables as opposed to random processes.
- **Basel III requirement for risk horizons:** BIS suggests *"The Committee has agreed that the differentiation of market liquidity across the trading book will be based on the concept of liquidity horizons. It proposes that banks' trading book exposures be assigned to a small number of liquidity horizon categories. [10 days, 1 month, 3 months, 6 months, 1 year] [...]".* A bank will need to simulate the risk factors of the portfolio across a grid including the standardized holding periods above.

The univariate case I

In the univariate case $N = 1$ we have the following results.
Definition. We say that the distribution of τ has lack of memory (LOM) when for all $S, U > 0$

$$\mathbb{P}(\tau > S + U | \tau > S) = \mathbb{P}(\tau > U) \quad (\iff \quad G(S + U) = G(S)G(U)).$$

If we assume G strictly positive and less than one we can take logs, we get Cauchy's functional equation and by assuming continuity in at least one point we get the solution $\ln G(t) = -\lambda t$, and hence

$$G(t) = \exp(-\lambda t)$$

Of course, as we know well the above is a characterization of the exponential distribution.

The univariate case II

Proposition. In the case $N = 1$ lack of memory of G, resulting in an exponential distribution for τ, solves Problem 1.

The proof is immediate by noticing that

$$\mathbb{P}(\tau > S + U) = \mathbb{P}(\tau > S + U | \tau > S)\mathbb{P}(\tau > S) = \mathbb{P}(\tau^1 > U)\mathbb{P}(\tau^2 > S)$$

where the last equality follows from LOM.

The univariate case has not given us anything surprising, as all this is well known. The multivariate case is more interesting and to that we turn now.

The multivariate case I

Definition. We say that the distribution of $\underline{\tau}$ has multivariate homogeneous lack of memory (MHLOM) when, given $T > 0$, for any two integers i, j, $i > j$

$$\mathbb{P}(\underline{\tau} > jT | \underline{\tau} > iT) = \mathbb{P}(\underline{\tau} > (j - i)T) \;\; (\iff \;\; G((jT)\underline{1}) = G(iT\underline{1})G((j - i)T\underline{1})$$

The right hand side of the iff follows immediately by definition of conditional probability. The definition is formally the same as for the univariate case when $S = iT, U = jT$, which we refer to as "homogeneity" (of the time step T).

The multivariate case II

One might try to adopt a more general definition of lack of memory, namely for all $\underline{S} = [S_1, \ldots, S_N]$, $\underline{U} = [U_1, \ldots, U_N]$ deterministic times

$$\mathbb{P}(\underline{\tau} > \underline{S} + \underline{U} | \underline{\tau} > \underline{S}) = \mathbb{P}(\underline{\tau} > \underline{U}).$$

This however is too strong and results in the trivial case of independence of exponential univariates, see Marshall and Olkin [20].

The most general definition of multivariate lack of memory, without collapsing into indepedence, assumes uniformity in $\underline{S}$ but not in $\underline{U}$:
Definition. MLOM: Every subvector $\underline{\tau}_I$ of τ with $I \subset \{1, 2, \ldots, N\}$ satisfies

$$\mathbb{P}(\underline{\tau}_I > S\underline{1} + \underline{U} | \underline{\tau}_I > S\underline{1}) = \mathbb{P}(\underline{\tau}_I > \underline{U}) \ \ (\iff \ G_I(S\underline{1} + \underline{U}) = G_I(S\underline{1}) G_I(\underline{U})).$$

The multivariate case: Marshall Olkin

Theorem (Marhall Olkin [20]).

$$\underline{\tau} \text{ satisfies MLOM} \iff \underline{\tau} \sim \text{Marshall Olkin multivariate distribution.}$$

[Note: the MO COPULA with arbitrary exponential margins is not enough]

The homogeneous multivariate case: EV Copulas I

It is immediate to see that, for the same reasons as in the univariate case, MHLOM solves Problem 1 in the multivariate case. Since MHLOM is weaker than MLOM=Marshall-Olkin, we can hope for solutions different from the Marshall Olkin multivariate distribution. We need to understand what the condition

$$G((jT)\underline{1}) = G(iT\underline{1})G((j-i)T\underline{1})$$

entails. To investigate this, we assume that G is associated with a *survival copula* function C. Then

$$G(iT\underline{1}) = \mathbb{P}(\underline{\tau} > iT) = \mathbb{P}(G_m(\underline{\tau}) < G_m(iT)) = C(G_m(iT))$$

where $G_m(t)$ is the vector of the marginal survival functions of the components of $\underline{\tau}$, all computed in t.

The homogeneous multivariate case: EV Copulas II

Hence we can rewrite MHLOM as

$$G(jT\underline{1}) = G(iT\underline{1})\, G((j-i)T\underline{1}) \text{ iff } C(G_m(jT)) = C(G_m(iT))\, C(G_m((j-i)T))$$

We require that the marginal distributions satisfy lack of memory, so this means, due to the univariate characterization, that $G_m(kT) = G_m(T)^k$ (they are exponential functions), and hence the MHLOM condition reads

$$C(G_m(T)^j) = C(G_m(T)^i)\, C(G_m(T)^{j-i})$$

where the product of the G_m is component-wise.
Write the above equation for $i = 1, j = 2$ to get

$$C(G_m(T)^2) = C(G_m(T))^2 \; .$$

The homogeneous multivariate case: EV Copulas III

Then $i = 1, j = 3$, substituting the one just found, and iterating, gives

$$C(G_m(T)^k) = C(G_m(T))^k \ , \quad k \in \mathbb{N}.$$

Given the arbitrariness of the marginal intensities in G_m, we conclude

$$C(\underline{x}^t) = C(\underline{x})^t \iff \boxed{C(\underline{x}) = (C(\underline{x}^t))^{1/t}} \text{ for all } t > 0, \quad \underline{x} \in [0,1]^N$$

This is a characterization of **extreme value copulas**.

Theorem (B. Chourdakis [7]). In the multivariate setting, and under a common time step, Problem 1 is solved by G characterized by exponential margins and an extreme value survival copula (self-chaining copulas).

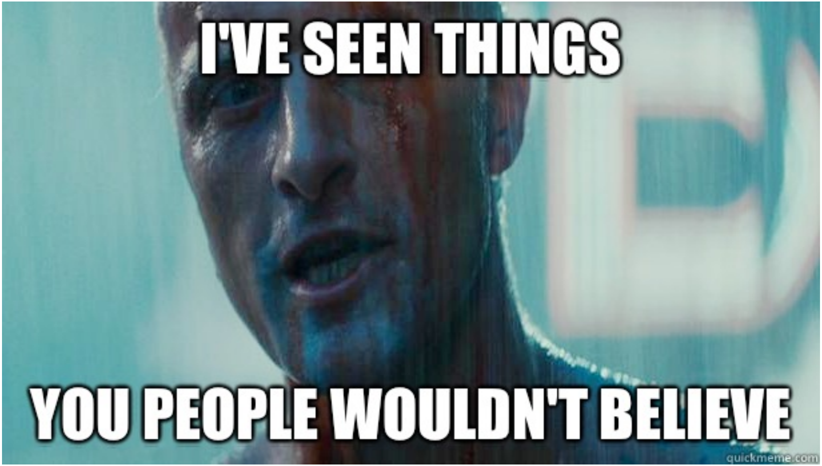
The homogeneous multivariate case: EV Copulas IV

Corollary: Given exponential margins, the only solution in the archimedean sub-family is the Gumbel Copula.

Corollary: Marshall Olkin copula with exponential margins solves this problem.

Corollary: In dimension 2 this is solved by Pickands functions and exponential margins.

Beware of iterating the Gaussian Copula I



Beware of iterating the Gaussian Copula II

Iterating a Gaussian copula kills dependence. Consider two exponentially distributed default times connected by a Gaussian copula with dependence parameter ρ and intensities λ_1 and λ_2,

$$\tau_1 = -\ln(1 - \Phi(X_1))/\lambda_1, \quad \tau_2 = -\ln(1 - \Phi(X_2))/\lambda_2,$$

$[X_1, X_2]$ bivariate Gaussian with standard marginals and correlation ρ. Assume $\lambda_1 = \lambda_2 = 0.02, \quad \rho = 0.5$, and either

a) Sample directly $\tau_1 > 30y \cap \tau_2 > 30y$. Get the probability of this event from a simulation with one million scenarios.

b) Iterate $\Delta_i\tau_1 > 1y \cap \Delta_i\tau_2 > 1y$ 30 times, where $[\Delta_i\tau_1, \Delta_i\tau_2]$, $i = 1, 2, \ldots, 30$ are independent copies of $[\tau_1, \tau_2]$ to be used to check survival in every year.

$$a): \ 0.386 \pm 0.0015 \qquad b): \ 0.328 \pm 0.0015 \quad (18\% diff)$$

A second and more-ambitious problem I

Problem 1: when can we split a terminal "survival of all" sampling in several equal time steps? Exponential margins and EV copula.

PROBLEM 2. Don't look just at the "survival of all" event but consider any possible mix of states (including removal of defaulted/liquidated components) and check when a terminal simulation of this can be split into different time steps for $\underline{\tau}$ and its sub-vectors.

We have already seen the theorem where M-O satisfies the most general multivariate lack of memory, including removal of components, so we may guess that M-O will play a key role here and will be a solution. However, we can say more. Define

$$Z_t = [1_{\tau_1 > t}, 1_{\tau_2 > t}, \ldots, 1_{\tau_N > t}]$$

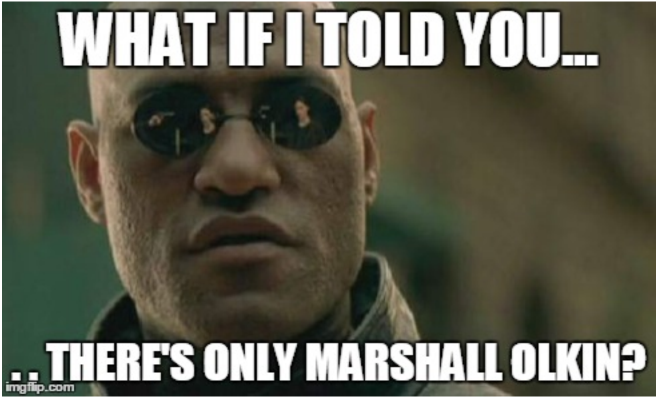
(notice that earlier we were considering $1_{\tau_1 > t \cap \tau_2 > t \ldots \cap \tau_N > t}$). Suppose Z is Markovian. This would be a great step forward for us.

A second and more-ambitious problem II

Previous literature studying credit risk via Z:

- Multi-variate phase distributions (MPH). Developed in late 70s by Neuts & co-workers, survey [6]. [2], see also [10]: default times are the first points in time at which components of an underlying multivariate Markov chain reach absorbing states.
- MPH Very difficult to work with in high-dimensional applications.
- A more practical, proper subclass of MPH is given by the family of default times whose Z is a continuous-time Markov chain (MCZ). Credit-risk modeling with MCZ [12, 5].
- MCZ accounts for looping defaults, default contagion in the sense of Jarrow and Yu [14, 23]; uses matrix exponentials.
- However, the family MC still imposes serious challenges in a real world implementation, since it is not naturally equipped with a "nested margining" property.

Lack of memory & new characterization of MO I



Our Contribution: We now introduce a theorem showing that MO is characterized by nested margining within MCZ, giving a new characterization of MO in terms of MCZ.

Lack of memory & new characterization of MO II

Theorem (B. Mai & Scherer [8]) : A New Characterization of the MO Distribution.

The survival indicator processes $\mathbf{Z}_I$ are time-homogeneous Markovian for all subsets $\emptyset \neq I \subset \{1, \ldots, N\}$

$$\Leftrightarrow$$

$(\tau_1, \ldots, \tau_N)$ has a Marshall–Olkin distribution

"$\Rightarrow$" is a new and rather surprising result.

Lack of memory & new characterization of MO III

Proof:
"$\Rightarrow$" By the time-homogeneous Markov property, there is a transition function $p_{\mathbf{x},\mathbf{y}}(t)$ for $\mathbf{x},\mathbf{y} \in \{0,1\}^N$ and $t \geq 0$ such that

$$\mathbb{P}(\mathbf{Z}(t_n) = \mathbf{x}_n, \ldots, \mathbf{Z}(t_1) = \mathbf{x}_1) = p_{(1,\ldots,1),\mathbf{x}_1}(t_1) \prod_{l=2}^{n} p_{\mathbf{x}_{l-1},\mathbf{x}_l}(t_l - t_{l-1})$$

for $t_n > \ldots > t_1 > 0$ and $\mathbf{x}_1, \ldots, \mathbf{x}_n \in \{0,1\}^N$. Let $t, s_1, \ldots, s_N \geq 0$ be arbitrary and denote by π a permutation s.t. $s_{\pi(1)} \leq s_{\pi(2)} \leq \ldots \leq s_{\pi(N)}$ is the ordered list of $s_1, \ldots, s_N$. Define the following subsets of $\{0,1\}^N$:

$$A_1 := \{(1,\ldots,1)\}, \; A_k := \left\{\mathbf{x} \in \{0,1\}^N \; : \; x_{\pi(l)} = 1 \text{ for all } l \geq k\right\}, \; k = 2..N.$$

In words, A_k denotes the subset of $\{0,1\}^N$ in which all components $\pi(k), \ldots, \pi(N)$ are still alive.

Lack of memory & new characterization of MO IV

There is a finite number M of distinct paths $(\mathbf{x}_2^{(i)},\dots,\mathbf{x}_N^{(i)}) \in A_2 \times \dots \times A_N$, $i = 1,\dots,M$, that avoid inconsistent patterns in time (such as default resurrections etc), i.e. such that

$$0 < \mathbb{P}\{\mathbf{Z}(t+s_{\pi(1)}) = (1,\dots,1), \mathbf{Z}(t+s_{\pi(2)}) = \mathbf{x}_2^{(i)},\dots,\mathbf{Z}(t+s_{\pi(N)}) = \mathbf{x}_N^{(i)}\}$$

This set of paths depends on $s_1,\dots,s_N$, but it does not depend on t by the time-homogeneity property of $\mathbf{Z}$. We have

$$\mathbb{P}(\tau_1 > t,\dots,\tau_N > t)\,\mathbb{P}(\tau_1 > s_1,\dots,\tau_N > s_N)$$
$$= \mathbb{P}(\mathbf{Z}(t) \in A_1)\,\mathbb{P}\big(\mathbf{Z}(s_{\pi(1)}) \in A_1,\, \mathbf{Z}(s_{\pi(2)}) \in A_2,\dots,\mathbf{Z}(s_{\pi(N)}) \in A_N\big)$$

$$= \mathbb{P}(\mathbf{Z}(t) \in A_1) \sum_{i=1}^{M} \mathbb{P}(\mathbf{Z}(s_{\pi(1)}) = (1..1), \mathbf{Z}(s_{\pi(2)}) = \mathbf{x}_2^{(i)},\dots,\mathbf{Z}(s_{\pi(N)}) = \mathbf{x}_N^{(i)})$$

Lack of memory & new characterization of MO V

$$= p_{(1,\dots,1),(1,\dots,1)}(t) \sum_{i=1}^{M} p_{(1,\dots,1),(1,\dots,1)}(s_{\pi(1)})\, p_{(1,\dots,1),\mathbf{x}_2^{(i)}}(s_{\pi(2)} - s_{\pi(1)}) \cdot$$

$$\cdot \prod_{k=3}^{N} p_{\mathbf{x}_{k-1}^{(i)},\mathbf{x}_k^{(i)}}(s_{\pi(k)} - s_{\pi(k-1)})$$

$$= \sum_{i=1}^{M} p_{(1,\dots,1),(1,\dots,1)}(t + s_{\pi(1)})\, p_{(1,\dots,1),\mathbf{x}_2^{(i)}}(t + s_{\pi(2)} - (t + s_{\pi(1)}))\cdot$$

$$\cdot \prod_{k=3}^{N} p_{\mathbf{x}_{k-1}^{(i)},\mathbf{x}_k^{(i)}}(t + s_{\pi(k)} - (t + s_{\pi(k-1)}))$$

$$= \mathbb{P}(\mathbf{Z}(t + s_{\pi(1)}) \in A_1, \mathbf{Z}(t + s_{\pi(2)}) \in A_2, \dots, \mathbf{Z}(t + s_{\pi(N)}) \in A_N)$$

$$= \mathbb{P}(\tau_1 > t + s_1, \dots, \tau_N > t + s_N)$$

Lack of memory & new characterization of MO VI

Repeating the above derivation for every subset $I \subset \{1, \ldots, N\}$ we obtain the equation

$$\mathbb{P}(\tau_{i_1} > t+s_{i_1}, \ldots, \tau_{i_k} > t+s_{i_k}) = \mathbb{P}(\tau_{i_1} > t, \ldots, \tau_{i_k} > t)\, \mathbb{P}(\tau_{i_1} > s_{i_1}..\tau_{i_k} > s_{i_k})$$

for arbitrary $1 \leq i_1, \ldots, i_k \leq N$ and $t, s_{i_1}, \ldots, s_{i_k} \geq 0$. This is precisely the functional equality describing the multi-variate lack-of-memory property, which is well-known to characterize the Marshall–Olkin exponential distribution, see [20, 21].

The following result is not new but we prove it anyway in our setting:

"$\Leftarrow$" Assume $(\tau_1, \ldots, \tau_N)$ has a Marshall–Olkin distribution with parameters $\{\lambda_I\}$, $\emptyset \neq I \subset \{1, \ldots, N\}$ satisfying $\sum_{I:k \in I} \lambda_I > 0$ for all $k = 1, \ldots, N$. We prove Markovianity of $\mathbf{Z}_I$ for an arbitrary non-empty subset I of components. Without loss of generality, we may assume

Lack of memory & new characterization of MO VII

that $(\tau_1, \ldots, \tau_N)$ is defined on the following probability space, as first considered in [1]: we consider an iid sequence $\{E_n\}_{n\in\mathbb{N}}$ of exponential random variables with rate $\lambda := \sum_{\emptyset \neq K \subset \{1,\ldots,N\}} \lambda_K$ and an independent iid sequence $\{Y_n\}_{n\in\mathbb{N}}$ of set-valued random variables with distribution given by

$$\mathbb{P}(Y_1 = K) = p_K := \frac{\lambda_K}{\lambda}, \quad \emptyset \neq K \subset \{1, \ldots, N\}.$$

The random vector $(\tau_1, \ldots, \tau_N)$ is then defined as $\tau_k := E_1 + \ldots + E_{\min\{n\,:\,k\in Y_n\}}$, $k = 1, \ldots, N$. Let us introduce the notation

$$N_t := \sum_{k=1}^{\infty} 1_{\{E_1+\ldots+E_k \leq t\}}, \quad t \geq 0,$$

Lack of memory & new characterization of MO VIII

which is a Poisson process with intensity λ. Fix a non-empty set $I \subset \{1, \ldots, N\}$, say $I = \{i_1, \ldots, i_k\}$ with $1 \leq i_1 < \ldots < i_k \leq N$. Denoting the power set of $\{1, \ldots, N\}$ by $\mathcal{P}_N$, we define the function $f_I : \{0,1\}^k \times \mathcal{P}_N \to \{0,1\}^k$ as follows:

$$j\text{-th component of } f_I(\vec{x}, J) := 1_{\{x_j = 1 \text{ and } i_j \notin J\}}, \quad j = 1, \ldots, k,$$

for $\vec{x} = (x_1, \ldots, x_k) \in \{0,1\}^k$ and $J \in \mathcal{P}_N$. It is now readily observed – in fact just a rewriting of Arnold's model – that

$$\mathbf{Z}_I(t) = f_I\Big(\mathbf{Z}_I(s), \bigcup_{k=N_s+1}^{N_t} Y_k\Big), \quad t \geq s \geq 0. \tag{1}$$

This stochastic representation implies the claim, since the second argument of f_I is independent of $\mathcal{F}_I(s) := \sigma(\mathbf{Z}_I(u) \,:\, u \leq s)$ by the

Lack of memory & new characterization of MO IX

Poisson property of $\{N_t\}$. To see this, it suffices to observe that $Z_I(s)$ is a function of N_s and $Y_1, \ldots, Y_{N_s}$ (which can be seen by setting $t = s$ and $s = 0$ in (1)), whereas the second argument is a function of $Y_{N_s+1}, \ldots, Y_{N_t}$. Consequently, the independent random variables N_s and $N_t - N_s$ only serve as a random pick of two independent (because disjoint) partial sequences of the iid sequence $Y_1, Y_2, \ldots$.

Thank you for your attention!

Questions?

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