Mean-Field Games

Lectures at the Imperial College London

2nd Lecture: Formulation of the Mean-Field Games

François Delarue (Nice – J.-A. Dieudonné)

May 7 2015

Joint works with R. Carmona; J.F. Chassagneux and D. Crisan; P. Cardaliaguet

Part I. Equilibria within a finite system

Part I. Equilibria within a finite system

a. Several notions

General formulation

- Controlled system of N interacting particles with symmetric mean-field interaction through the global state of the population
  - dynamics of particle number $i \in \{1, \dots, N\}$

$$\underbrace{dX_t^i}_{\in \mathbb{R}^d} = b(X_t^i, \text{global state of the collectivity}, \alpha_t^i)dt$$

$$+ \sigma(X_t^i, \text{global state}) \underbrace{dW_t^i}_{\text{idiosyncratic noises}}$$

$$+ \sigma^0(X_t^i, \text{global state}) \underbrace{dW_t^0}_{\text{common noise}}$$

- Rough description of the probabilistic set-up
  - $(W_t^0, W^1, \dots, W^N)_{0 \le t \le T}$ independent B.M. with values in $\mathbb{R}^d$
  - $(\alpha_t^i)_{0 \le t \le T}$ progressively-measurable processes with values in A
  - simplicity $\rightsquigarrow$ same deterministic initial conditions

Empirical measure

- Encode the global state of the population at time t through

$$\bar{\mu}_t^N = \frac{1}{N}\sum_{i=1}^{N} \delta_{X_t^i} \quad \rightsquigarrow \quad \text{probability measure on } \mathbb{R}^d$$

  - $\mathcal{P}(\mathbb{R}^d) \rightsquigarrow$ set of probability measures on $\mathbb{R}^d$
  - $\mathcal{P}_2(\mathbb{R}^d) \rightsquigarrow$ set of probability measures on $\mathbb{R}^d$ with second order moments

- Express the coefficients as

$$b : \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \times A \to \mathbb{R}^d, \quad \sigma, \sigma^0 : \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}^{d\times d}$$

  - example 1: $b(x, \mu, \alpha) = b\Big(x, \int_{\mathbb{R}^d} \varphi d\mu, \alpha\Big)$, $\varphi = \mathrm{Id} \rightsquigarrow$ mean
  - example 2: $b(x, \mu, \alpha) = \int_{\mathbb{R}^d} b(x, v, \alpha) d\mu(v)$

Cost functionals

- Rewrite the dynamics of the particles

$$dX_t^i = b(X_t^i, \bar{\mu}_t^N, \alpha_t^i)dt + \sigma(X_t^i, \bar{\mu}_t^N)dW_t^i + \sigma^0(X_t^i, \bar{\mu}_t^N)dW_t^0$$

- Cost functional to player $i \in \{1, \ldots, N\}$

$$J^i(\alpha^1, \alpha^2, \ldots, \alpha^N) = \mathbb{E}\Big[g(X_T^i, \bar{\mu}_T^N) + \int_0^T f(X_t^i, \bar{\mu}_t^N, \alpha_t^i)dt\Big]$$

  - take the same f and g for all players to symmetrize
  - pay attention that J^i depends on the other controls through empirical measure
  - same kind of example for f and g as above

Nash equilibrium

- Each player is willing to minimize its own cost functional
  - no chance that everybody can minimize at the same time
  - need for a consensus $\rightsquigarrow$ Nash equilibrium
- Say that a N-tuple of strategies $(\alpha^{1,\star},\dots,\alpha^{N,\star})$ is a consensus if
  - no interest for any player to leave the consensus
  - change $\alpha^{i,\star} \rightsquigarrow \alpha^i \Rightarrow J^i \nearrow$

$$J^i(\alpha^{1,\star},\dots,\alpha^{i,\star},\dots,\alpha^{N,\star}) \leq J^i(\alpha^{1,\star},\dots,\alpha^{i},\dots\alpha^{N,\star})$$

- Existence $\rightsquigarrow$ fixed point argument (see later on)
- Meaning of freezing $\alpha^{1,\star},\dots,\alpha^{i-1,\star},\alpha^{i+1,\star},\alpha^{N,\star}$
  - freezing the processes $\rightsquigarrow$ Nash equilibrium in open loop
  - means that the players observe the noises $\rightsquigarrow$ what about if the players only observe the states?

Markov loop

• PDE $\rightsquigarrow$ require that each α_t^i is a function of the private states $X_t^1, \dots, X_t^N$ at time t

∘ $\alpha_t^i = \alpha^i(t, X_t^1, \dots, X_t^N)$

∘ each function α^i is called a Markov feedback $\rightsquigarrow$ notion of Markov loop

• New notion of Nash equilibrium

∘ freeze the Markov feedback function $\alpha^{\star,1}, \dots, \alpha^{\star,N}$

∘ if change $\alpha^{\star,i}$ into $\alpha^i \Rightarrow$ all the players may move

∘ with this notion of Nash, the Markov feedback are frozen but not the control processes

∘ leads to different equilibria!

• In the framework of MFG, expect that there is no difference in the asymptotic setting

∘ when N tends to $+\infty$ and $\alpha^{\star,i}$ changed into $\alpha^i \Rightarrow$ other players hardly feel the modification

Social optimization and Pareto

• May also optimize the global wealth of the society

$$\sum_{i=1}^{N} J^i(\alpha^1, \dots, \alpha^N)$$

◦ a social optimizer is a Pareto equilibrium $\rightsquigarrow$ no way to decrease one's cost without increasing somebody else's cost

• Example: one center of decision for one big company with small agencies all over an area

◦ center decides of the general policy, for instance

$$\alpha_t^i = \alpha^i(t, X_t^1, \dots, X_t^N) \quad \text{or} \quad \alpha_t^i = \alpha^i(t, X_t^i, \bar{\mu}_t^N)$$

◦ choose $\alpha^i = \alpha$ symmetric $\Rightarrow ((X_t^i, \alpha_t^i, W_t^i)_{0 \le t \le T})_{1 \le i \le N}$ are exchangeable (invariance by permutation)

◦ may optimize the global wealth of the company over strategies $(\alpha^1, \dots, \alpha^N)$ such that $((\alpha_t^i, W_t^i)_{0 \le t \le T})_{1 \le i \le N}$ are exchangeable

Part I. Equilibria within a finite system

b. Examples

Exhaustible resources

- N producers of oil $\rightsquigarrow X_t^i$ (estimated reserve) at time t

$$dX_t^i = -\alpha_t^i dt + \sigma X_t^i dW_t^i$$

  - $\alpha_t^i \rightsquigarrow$ instantaneous production rate
  - σ common volatility for the perception of the reserve
  - should be a constraint $X_t^i \geq 0$

- Optimize the profit of a producer

$$J^i(\alpha^1, \ldots, \alpha^N) = \mathbb{E} \int_0^T (\alpha_t^i P_t - c(\alpha_t^i))dt$$

  - P_t is selling price
  - mean-field constraint $\rightsquigarrow$ selling price is a function of the

mean-state of the reserves

$$P_t = P(\frac{1}{N} \sum_{i=1}^{N} X_t^i)$$

Growth model

- Consider the labor productivity $(Z^1, \ldots, Z^N)$ and the wealth $(A^1, \ldots,)$ of N workers

$$dZ_t^i = b(Z_t^i)dt + \sigma(Z_t^i)dW_t^i$$
$$dA_t^i = (w_t Z_t^i + r_t A_t^i - c_t^i)dt$$

  - $w_t \rightsquigarrow$ wage rate
  - $r_t \rightsquigarrow$ interest rate
  - $c_t \rightsquigarrow$ consumption

- Optimize utility of consumption and final wealth

$$J^i(\boldsymbol{c}^1, \ldots, \boldsymbol{c}^N) = \mathbb{E}\Big[\int_0^T u(c_t^i)dt + U(A_T)\Big]$$

  - may impose state constraint on $(A_t)_{0 \le t \le T}$
  - utility functions u and U
  - mean-field constraint

$$w_t = F_W\Big(\frac{1}{N}\sum_{i=1}^N A_t^i\Big), \quad r_t = F_R\Big(\frac{1}{N}\sum_{i=1}^N A_t^i\Big)$$

Carbon markets

- N producers of energy
  - Producer i: X_T^i global emissions of carbon on $[0, T]$
  - Λ: number of permits received by producer i
- Cap rule
  - if $N^{-1} \sum_{j=1}^{N} X_T^j > \Lambda$
  - penalty for i: $\lambda (X_T^i - \Lambda)^+ \mathbf{1}_{(\Lambda,\infty)}\big(N^{-1} \sum_{j=1}^{N} X_T^j\big)$
- Dynamics of 'perceived' emissions

$$dX_t^i = (b_t - \alpha_t^i)dt + \sigma dW_t^i$$

  - $\alpha^i \rightsquigarrow$ abatement by investment in green technology
- Minimize

$$\mathbb{E}\Big[\int_0^T c(\alpha_t^i)dt + \lambda(X_T^i - \Lambda)^+ \mathbf{1}_{(\Lambda,\infty)}\Big(N^{-1} \sum_{j=1}^{N} X_T^j\Big)\Big]$$

Part I. Equilibria within a finite system

c. Seeking equilibria

Reminder from the first chapter

- Hamiltonian

$$H(x,\mu,\alpha,z) = b(x,\mu,\alpha)\cdot z + f(x,\mu,\alpha)$$

  ◦ $\alpha^\star(x,\mu,z) = \text{argmin}_{\alpha\in A} H(x,\mu,\alpha,z)$

- Two ways to handle stochastic optimal control

- Interpretation of the value function $\leftrightsquigarrow$ interpretation of the HJB equation

  ◦ sounds like a PDE method $\rightsquigarrow$ reformulate it in the framework of Nash equilibria with Markov closed loop

- Use of the stochastic Pontryagin principle

  ◦ very much demanding in terms of assumption but very robust (no need of a Markov structure behind)

  ◦ implement it in the framework of Nash equilibria with open loop

Hamiltonian associated with Markov loop

- As in last part of Chapter 1 $\rightsquigarrow$ assume that $\sigma^0 \equiv 0$
- Assume that $\alpha^{1,\star}, \ldots, \alpha^{N,\star}$ Nash equilibrium in Markov feedback form

$$dX_t^j = b(X_t^j, \bar{\mu}_t^N, \alpha^{j,\star}(t, X_t^1, \ldots, X_t^N))dt + \sigma(X_t^j, \bar{\mu}_t^N)dW_t^j$$

  - ◦ change feedback function $\alpha^{i,\star}$ into α^i
  - ◦ just facing a standard optimization problem but with a diffusion process with values in $(\mathbb{R}^d)^N$ $\rightsquigarrow$ the control is just through the player number i

- Write the Hamitonian

$$b\Big(x_i, \frac{1}{N}\sum_{j=1}^N \delta_{x_j}, \alpha\Big) \cdot z_i + f\Big(x_i, \frac{1}{N}\sum_{j=1}^N \delta_{x_j}, \alpha\Big)$$

$$+ \sum_{\ell \neq i} b\Big(x_\ell, \frac{1}{N}\sum_{j=1}^N \delta_{x_j}, \alpha^{\ell,\star}(t, x_1, \ldots, x_N)\Big) \cdot z_\ell$$

  - ◦ can forget the second line!

FBSDE associated with Markov loop

• Forget the cut-off function discussed in Chapter 1 and write the FBSDE

$$dX_t^i = b\Big(X_t^i, \bar{\mu}_t^N, \alpha^\star(X_t^i, \bar{\mu}_t^N, Z_t^{i,i}\sigma^{-1}(X_t^i, \bar{\mu}_t^N))\Big)dt + \sigma(X_t^i, \bar{\mu}_t^N)dW_t^i$$

$$dY_t^i = -f\Big(X_t^i, \bar{\mu}_t^N, \alpha^\star(X_t^i, \bar{\mu}_t^N, Z_t^{i,i}\sigma^{-1}(X_t^i, \bar{\mu}_t^N))\Big)dt + \sum_{j=1}^N Z_t^{i,j}dW_t^j$$

with $Y_T^i = g(X_T^i, \mu_T^N)$ as terminal condition

∘ may discuss sufficient conditions (won't do it in the lectures)

∘ part of the difficulty again consists in controlling the smoothness of the decoupling field

$$(Y_t^1, \ldots, Y_t^N) = u(t, X_t^1, \ldots, X_t^N)$$

∘ difficulty to handle the quadratic setting as Y is multi dimensional (series of works due to Bensoussan and Frehse)

∘ within MFG $\rightsquigarrow$ deterioration of the smoothness as $N \nearrow \infty$

Open loop

• Consider a very simple case when $b(x, \mu, \alpha) = b(x, \alpha)$, σ and σ^0 constant (typical framework for stochastic Pontryagin principle)

• When freezing $\boldsymbol{\alpha}^{1,\star}, \dots, \boldsymbol{\alpha}^{i-1,\star}, \boldsymbol{\alpha}^{i+1,\star}, \dots, \boldsymbol{\alpha}^{N,\star}$

∘ $(X_t^{1,\star}, \dots, X_t^{i-1,\star}, X_t^{i+1,\star}, X_t^{N,\star})$ remain the same (would be false with Markov loop)

∘ again, we are facing a standard optimization problem $\rightsquigarrow$ optimization of $\boldsymbol{\alpha}^i$

∘ may use the same Hamiltonian H

$$dX_t^i = b\Big(X_t^i, \bar{\mu}_t^N, \alpha^\star(X_t^i, \bar{\mu}_t^N, Y_t^i)\Big)dt + \sigma dW_t^i + \sigma^0 dW_t^0$$

$$dY_t^i = -\partial_x H\Big(X_t^i, \bar{\mu}_t^N, \alpha^\star(X_t^i, \bar{\mu}_t^N, Y_t^i)\Big)dt + \sum_{j=0}^N Z_t^{i,j} dW_t^j$$

with $Y_T^i = \partial_x g(X_T^i, \bar{\mu}_T^N)$

∘ if Lipschitz coefficients (and growth conditions) and $\sigma \neq 0 \rightsquigarrow$ unique solution

Part II. From propagation of chaos to MFG

Part II. From propagation of chaos to MFG

a. Handling an example

Systemic risk model

- Recall the dynamics of the (log)-reserve

$$dX_t^i = a\,(\bar{X}_t^N - X_t^i)\,dt + \alpha_t^i\,dt + \sigma\,dW_t^i + \sigma^0 dW_t^0$$

- Recall the cost functional

$$J^i(\alpha^1, \ldots, \alpha^N) = \mathbb{E}\Big[g(X_T^i, \bar{X}_T^N) + \int_0^T f(X_t^i, \bar{X}_t^N, \alpha_t^i)dt\Big]$$

  ◦ $f(x, m, \alpha) = \alpha^2 + \epsilon^2(m - x)^2 - 2q\epsilon\alpha(m - x), \quad q \le \epsilon^2$

  ◦ $g(x, m) = c^2(x - m)^2$

- Linear quadratic $\Rightarrow$ explicitly solvable

  ◦ ansatz $\rightsquigarrow$ seek optimal Markov feedback (both in the open loop and Markov closed loop case) of the linear form (derivative of quadratic functions)

$$\alpha_t^{\star,i} = \eta_t X_t^i + \chi_t$$

  ◦ by symmetry, expect same coefficients η and χ

Solving the systemic risk model

- Inject the ansatz into the FBSDE and proceed
- Nash equilibria over Markov loop
  - $(\eta_t)_{0\leq t\leq T}$ solves Riccati equation

$$\dot{\eta}_t = 2(a+q)\eta_t + (1-N^{-2})\eta_t^2 + q^2 - \epsilon, \quad \eta_T = c$$

  - equilibrium has the shape

$$\alpha_t^{\star,i} = (q + (1-\frac{1}{N})\eta_t)\left(\frac{1}{N}\sum_{j=1}^{N} X_t^j - X_t^i\right)$$

- Nash equilibria over open loop
  - $(\eta_t)_{0\leq t\leq T}$ solves Riccati equation

$$\dot{\eta}_t = (2(a+q) - \frac{1}{N}q)\eta_t + (1-N^{-1})\eta_t^2 + q^2 - \epsilon, \quad \eta_T = c$$

  - equilibrium has the same shape but with the solution of the new Riccati equation

About the Riccati equation

- Convexity of the coefficients $\Rightarrow$ Riccati equation is unique solvable

  ◦ solution depends upon N and differs according to the sense given to the Nash equilibrium

  ◦ explicitly solvable (combination of exponentials)

- Riccati equations have the same asymptotic behavior

  ◦ label η with superscript $N \Rightarrow (\eta_t^N)_{0 \le t \le T}$ (whatever the sense of the Nash equilibrium is)

  ◦ $\eta_t^N \to \eta_t^\infty$, $t \in [0, T]$

$$\dot{\eta}_t^\infty = 2(a+q)\eta_t^\infty + (\eta_t^\infty)^2 + q^2 - \epsilon, \quad \eta_T = c$$

  ◦ explicitly solvable as well

Particle system for the Nash equilibrium

- Inject the shape of the optimal feedback into the particle system

$$dX_t^i = (a + q + (1 - \frac{1}{N})\eta_t^N)\,(\bar{X}_t^N - X_t^i)\,dt + \sigma\,dW_t^i + \sigma^0 dW_t^0$$

  - whatever the meaning of the Nash equilibrium is

- Take the empirical mean $\bar{X}_t^N = \frac{1}{N}\sum_{i=1}^N X_t^i$

$$\bar{X}_t^N = \bar{X}_t^0 + \frac{\sigma}{N}\sum_{i=1}^N W_t^i + \sigma^0 W_t^0$$

  - choose $X_0^i = x_0 \Rightarrow$

$$\bar{X}_t^N \to x + \sigma^0 W_t^0 =: m_t$$

- Expect in the limit

$$dX_t^i = (a + q + \eta_t^\infty)\,(m_t - X_t^i)\,dt + \sigma\,dW_t^i + \sigma^0 dW_t^0$$

  - particles are exchangeable and independent given $(W_t^0)_{0\le t\le T}$
  - m_t is conditional mean of any X_t^i given common noise

Part II. From propagation of chaos to MFG

b. McKean-Vlasov SDEs

General uncontrolled particle system

- Remove the control in the original particle system!

$$dX_t^i = b(X_t^i, \bar{\mu}_t^N)dt + \sigma(X_t^i, \bar{\mu}_t^N)dW_t^i + \sigma^0(X_t^i, \bar{\mu}_t^N)dW_t^0$$

  - $X_0^1, \ldots, X_N^i$ i.i.d. (and independent of the noises)
  - $\bar{\mu}_t^N = \frac{1}{N}\sum_{i=1}^{N} \delta_{X_t^i}$ (empirical measure)

- Assume the coefficients are Lipschitz in all the variables
  - need to say what it means in terms of the measure (connection with Lipschitz property with respect to the measure argument)
  - unique solution!

- Find the asymptotic behavior of the particle system as N tends to ∞

Wasserstein distance

- Several distances on the space of probability measures
- Here distance on $\mathcal{P}_2(\mathbb{R}^d)$ probability measures μ with a second order moment)

$$\int_{\mathbb{R}^d} |x|^2 d\mu(x) < \infty$$

  - use the Wasserstein distance

$$\mu, \nu \in \mathcal{P}_2(\mathbb{R}^d), \quad W_2(\mu, \nu) = \left(\inf_{\pi} \int_{\mathbb{R}^d \times \mathbb{R}^d} |x - y|^2 d\pi(x, y) \right)^{1/2},$$

where π has μ and ν as marginals on $\mathbb{R}^d \times \mathbb{R}^d$

  - X and X' two r.v.'s $\Rightarrow W_2(\mathcal{L}(X), \mathcal{L}(X')) \leq \mathbb{E}[|X - X'|^2]^{1/2}$
  - CV in Wasserstein $\Leftrightarrow$ weak CV + square unif. integrability

- Example $W_2\Big(\frac{1}{N}\sum_{i=1}^N \delta_{x_i}, \frac{1}{N}\sum_{i=1}^N \delta_{x'_i}\Big) \leq \Big(\frac{1}{N}\sum_{i=1}^N |x_i - x'_i|^2\Big)^{1/2}$
  - yields the required Lipschitz property

McKean-Vlasov SDE

- Start with the case without common noise

  ◦ on the model of (II a) expect some decorrelation in the particle system as $N \nearrow \infty$

  ◦ replace the empirical measure by the theoretical measure of the solution

$$dX_t = b(X_t, \mathcal{L}(X_t))dt + \sigma(X_t, \mathcal{L}(X_t))dW_t$$

- Cauchy-Lipschitz theory

  ◦ assume b and σ Lipschitz continuous on $\mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \Rightarrow$ unique solution for any given initial condition in L^2

  ◦ proof works as in the standard case taking advantage of

$$\mathbb{E}\Big[\big|(b,\sigma)(X_t, \mathcal{L}(X_t)) - (b,\sigma)(X_t', \mathcal{L}(X_t'))\big|^2\Big] \leq C\mathbb{E}[|X_t - X_t'|^2]$$

  ◦ permits to exhibit a contraction

Propagation of chaos

- Prove that the solution of the particle system converges to the solution of the MKV SDE when $\sigma^0 \equiv 0$

- Main statement

  - each $(X_t^i)_{0 \le t \le T}$ converges in law to the solution of MKV SDE
  - particles get independent in the limit $\rightsquigarrow$ for k fixed:

$$(X_t^1, \ldots, X_t^k)_{0 \le t \le T} \underset{\mathcal{L}}{\longrightarrow} \mathcal{L}(\text{MKV})^{\otimes k} = \mathcal{L}((X_t)_{0 \le t \le T})^{\otimes k} \quad \text{as } N \nearrow \infty$$

  - $\lim_{N \nearrow \infty} \sup_{0 \le t \le T} \mathbb{E}[(W_2(\bar{\mu}_t^N, \mathcal{L}(X_t))^2] = 0$

- Proof relies on a coupling argument

  - N copies $(\tilde{X}_t^1, \ldots, \tilde{X}_t^N)_{0 \le t \le T}$ of MKV SDE with $(W_t)_{0 \le t \le T}$ replaced by $((W_t^i)_{0 \le t \le T})_{1 \le i \le N}$

$$\mathbb{E}[\sup_{0 \le t \le T} |X_t^i - \tilde{X}_t^i|^2] \to 0 \Rightarrow \sup_{0 \le t \le T} \mathbb{E}\Big[\Big(W_2\Big(\bar{\mu}_t^N, \frac{1}{N}\sum_{i=1}^N \delta_{\tilde{X}_t^i}\Big)\Big)^2\Big] \to 0$$

  - LLN may replace $\frac{1}{N}\sum_{i=1}^N \delta_{\tilde{X}_t^i}$ by $\mathcal{L}(X_t)$

Case with a common noise

- MKV SDE $\rightsquigarrow$ conditional MKV SDE

$$
\begin{aligned}
dX_t &= b(X_t, \mathcal{L}(X_t|\boldsymbol{W}^0))dt \\
&\qquad + \sigma(X_t, \mathcal{L}(X_t|\boldsymbol{W}^0))dW_t + \sigma^0(X_t, \mathcal{L}(X_t|\boldsymbol{W}^0))dW_t^0
\end{aligned}
$$

  - $\mathcal{L}(X_t|\boldsymbol{W}^0)$ conditional law of X_t given the realization of $(W_t^0)_{0 \le t \le T}$

- Set the equation on $(\Omega^0 \times \Omega^1, \mathbb{F}^0 \otimes \mathbb{F}^1, \mathbb{P}^0 \otimes \mathbb{P}^1)$
  - Ω^0 carries W^0 and Ω^1 carries W and X_0
  - $\mathcal{L}(X_t|\boldsymbol{W}^0) = \mathcal{L}_{(\Omega^1, \mathbb{F}^1, \mathbb{P}^1)}(X_t(\omega^0, \cdot))$
  - $\mathcal{L}(X_t|\boldsymbol{W}^0)$ is also $\mathcal{L}(X_t|(W_s^0)_{0 \le s \le t})$

- Propagation of chaos revisited
  - asymptotically $\rightsquigarrow$ conditional independence given $\boldsymbol{W}^0$ instead of independence
  - convergence of the empirical measure to the conditional law

Part II. From propagation of chaos to MFG

c. Formulation of the asymptotic problems

Ansatz

- Start with the case when $\sigma^0 \equiv 0$
- Ansatz $\rightsquigarrow$ at equilibrium

$$\alpha_t^{i,\star} = \alpha^N(t, X_t^i, \bar{\mu}_t^N) \approx \alpha(t, X_t^i, \bar{\mu}_t^N)$$

  - particle system at equilibrium

$$dX_t^i \approx b\Big(X_t^i, \bar{\mu}_t^N, \alpha(t, X_t^i, \bar{\mu}_t^N)\Big)dt + \sigma\Big(X_t^i, \alpha(t, X_t^i, \bar{\mu}_t^N\Big)dW_t^i$$

  - particles should decorrelate as $N \nearrow \infty$
  - $\bar{\mu}_t^N$ should stabilize around some deterministic limit μ_t
- What about an intrinsic interpretation of μ_t?
  - should describe the global state of the population in equilibrium
  - in the limit setting, any particle that leaves the equilibrium

should not modify $\mu_t \rightsquigarrow$ leaving the equilibrium means that the cost increases $\rightsquigarrow$ any particle in the limit should solve an optimal control problem in the environment $(\mu_t)_{0 \le t \le T}$

Matching problem of MFG

- Assume again that $\sigma^0 \equiv 0$
- Define the asymptotic equilibrium state of the population as the solution of a fixed point problem

(1) fix a flow of probability measures $(\mu_t)_{0 \le t \le T}$ (with values in $\mathcal{P}_2(\mathbb{R}^d)$)

(2) solve the stochastic optimal control problem in the environment $(\mu_t)_{0 \le t \le T}$

$$dX_t = b(X_t, \mu_t, \alpha_t)dt + \sigma(X_t, \mu_t)dW_t$$

◦ with $X_0 = \xi$ being fixed on some set-up $(\Omega, \mathbb{F}, \mathbb{P})$ with a d-dimensional B.M.

◦ with cost $J(\alpha) = \mathbb{E}\Big[g(X_T, \mu_T) + \int_0^T f(X_t, \mu_t, \alpha_t)dt\Big]$

(3) let $(X_t^{\star,\mu})_{0 \le t \le T}$ be the unique optimizer (under nice assumptions)

$\rightsquigarrow$ find $(\mu_t)_{0 \le t \le T}$ such that

$$\mu_t = \mathcal{L}(X_t^{\star,\mu}), \quad t \in [0, T]$$

- Not a proof of the convergence!

MFG with a common noise

- Same probabilistic set-up as for conditional MKV

$$\Omega = \Omega^0 \times \Omega^1, \ \mathbb{F} = \mathbb{F}^0 \otimes \mathbb{F}^1, \ \mathbb{P} = \mathbb{P}^0 \otimes \mathbb{P}^1$$

(1) fix an adapted continuous process on $(\Omega^0, \mathbb{F}^0, \mathbb{P}^0)$

$$\boldsymbol{\mu} : [0, T] \ni t \mapsto \mu_t \in \mathcal{P}_2(\mathbb{R}^d)$$

(2) solve the stochastic optimal control problem in the random environment $(\mu_t)_{0 \le t \le T}$

$$dX_t = b(X_t, \mu_t, \alpha_t)dt + \sigma(X_t, \mu_t)dW_t + \sigma^0(X_t, \mu_t)dW_t^0$$

∘ with $X_0 = \xi \in L^2(\Omega^1, \mathcal{F}_0^1, \mathbb{P}^1; \mathbb{R}^d)$

∘ with $(\alpha_t)_{0 \le t \le T}$ -progressively measurable with values in A (square integrable) on Ω

∘ same cost functional (under the double expectation)

(3) let $(X_t^{\star,\boldsymbol{\mu}})_{0 \le t \le T}$ be the unique optimizer (under nice assumptions)
$\rightsquigarrow$ find $(\mu_t)_{0 \le t \le T}$ such that, $\mathbb{P}^0$ almost surely,

$$\mu_t(\omega^0) = \mathcal{L}_{\Omega^1}(X_t^{\star,\boldsymbol{\mu}}(\omega^0, \cdot)), \quad t \in [0, T]$$

Social optimization

• Assume again that $\sigma^0 \equiv 0$

• Recall that one center of decision imposes some Markov feedback function to all the agents

∘ the ansatz must be the same!

∘ the difference is in the interpretation of the measures $(\mu_t)_{0 \le t \le T}$

• In the social optimization, when one moves $\rightsquigarrow$ everybody moves!
No way to fix the flow of measures!

∘ the flow of measures describe the collective state of population under the decision of the center

$$dX_t = b(X_t, \mathcal{L}(X_t), \alpha_t)dt + \sigma(X_t, \mathcal{L}(X_t))dW_t$$

∘ optimize the cost $J(\alpha) = \mathbb{E}[g(X_T, \mathcal{L}(X_T)) + \int_0^T f(X_t, \mathcal{L}(X_t), \alpha_t)dt]$

∘ optimization of McKean-Vlasov diffusion processes!

Part III. McKean-Vlasov FBSDEs

Part III. McKean-Vlasov FBSDEs

a. Within the framework of MFG

New program without common noise

• Make use of the results from the first chapter in order to characterize the optimal paths in the fixed point

∘ in the FBSDE formulation of the optimization problem $\rightsquigarrow$ replace the environment by the law of the solution

∘ derive an FBSDE of the McKean-Vlasov type of the general form

$$
\begin{aligned}
X_t &= \xi + \int_0^t b\left(X_s, \mathcal{L}(X_s), Y_s, Z_s\right) ds \\
&\quad + \int_0^t \sigma(X_s, \mathcal{L}(X_s), Y_s) dW_s \\
Y_t &= g(X_T, \mathcal{L}(X_T)) + \int_t^T f\left(X_s, \mathcal{L}(X_s), Y_s, Z_s\right) ds \\
&\quad - \int_t^T Z_s dW_s
\end{aligned}
$$

• Choose the coefficients accordingly

New program with common noise

• Make use of the results from the first chapter in order to characterize the optimal paths in the fixed point

∘ in the FBSDE formulation of the optimization problem $\rightsquigarrow$ replace the environment by the conditional law of the solution

∘ derive an FBSDE of the McKean-Vlasov type of the general form

$$X_t = \xi + \int_0^t b\left(X_s, \mathcal{L}(X_s|W^0), Y_s, Z_s\right) ds$$
$$+ \int_0^t \sigma(X_s, \mathcal{L}(X_s|W^0), Y_s) dW_s + \sigma^0(X_s, \mathcal{L}(X_s|W^0), Y_s) dW_s^0$$
$$Y_t = g(X_T, \mathcal{L}(X_T|W^0)) + \int_t^T f\left(X_s, \mathcal{L}(X_s|W^0), Y_s, Z_s\right) ds$$
$$- \int_t^T Z_s dW_s - \int_t^T Z_s^0 dW_s^0$$

• Choose the coefficients accordingly

MKV FBSDE for the value function

- Consider, on $(\Omega, \mathbb{F}, \mathbb{P})$, the MKV FBSDE

$$
\begin{aligned}
X_t &= \xi \\
&\quad + \int_0^t b\left(X_s, \mathcal{L}(X_s), \alpha^\star(X_s, \mathcal{L}(X_s), Z_s\sigma^{-1}(X_s, \mathcal{L}(X_s)))\right) ds \\
&\quad + \int_0^t \sigma(X_s, \mathcal{L}(X_s)) dW_s \\
Y_t &= g(X_T, \mathcal{L}(X_T)) \\
&\quad + \int_t^T f\left(X_s, \mathcal{L}(X_s), \alpha^\star(X_s, \mathcal{L}(X_s), Z_s\sigma^{-1}(X_s, \mathcal{L}(X_s)))\right) ds \\
&\quad - \int_t^T Z_s dW_s
\end{aligned}
$$

  - $\alpha^\star(x, \mu, z)$ is the unique minimizer of $\alpha \mapsto H(x, \mu, \alpha, z)$

- Under assumptions of Chapter 1 $\rightsquigarrow$ solution to MKV FBSDE is MFG equilibrium

MKV FBSDE for the value function

- Consider, on $(\Omega, \mathbb{F}, \mathbb{P})$, the MKV FBSDE

$$
\begin{aligned}
X_t &= \xi \\
&\quad + \int_0^t b\Big(X_s, \mathcal{L}(X_s|\boldsymbol{W}^0), \alpha^\star(X_s, \mathcal{L}(X_s|\boldsymbol{W}^0), Z_s\sigma^{-1}(X_s, \mathcal{L}(X_s|\boldsymbol{W}^0)))\Big)\, ds \\
&\quad + \int_0^t \sigma(X_s, \mathcal{L}(X_s|\boldsymbol{W}^0))dW_s + \sigma^0(X_s, \mathcal{L}(X_s|\boldsymbol{W}^0))dW_s^0 \\
Y_t &= g(X_T, \mathcal{L}(X_T|\boldsymbol{W}^0)) \\
&\quad + \int_t^T f\,(X_s, \mathcal{L}(X_s|\boldsymbol{W}^0), \alpha^\star(X_s, \mathcal{L}(X_s|\boldsymbol{W}^0), Z_s\sigma^{-1}(X_s, \mathcal{L}(X_s|\boldsymbol{W}^0))))\, ds \\
&\quad - \int_t^T Z_s dW_s - \int_t^T Z_s^{0,} dW_s^0
\end{aligned}
$$

  - $\alpha^\star(x, \mu, z)$ is the unique minimizer of $\alpha \mapsto H(x, \mu, \alpha, z)$

- Under assumptions of Chapter 1 $\rightsquigarrow$ solution to MKV FBSDE is MFG equilibrium

MKV FBSDE for the Pontryagin principle

• Consider, on $(\Omega, \mathbb{F}, \mathbb{P})$, the MKV FBSDE

$$
\begin{aligned}
X_t &= \xi + \int_0^t b\left(X_s, \mathcal{L}(X_s), \alpha^\star(X_s, \mathcal{L}(X_s), Y_s)\right) ds \\
&\quad + \int_0^t \sigma(\mathcal{L}(X_s)) dW_s \\
Y_t &= \partial_x g(X_T, \mathcal{L}(X_T)) \\
&\quad + \int_t^T \partial_x H\left(X_s, \mathcal{L}(X_s), \alpha^\star(X_s, \mathcal{L}(X_s), Y_s), Y_s\right) ds \\
&\quad - \int_t^T Z_s dW_s
\end{aligned}
$$

• Under assumptions of Chapter 1 $\rightsquigarrow$ solution to MKV FBSDE is MFG equilibrium

MKV FBSDE for the Pontryagin principle

- Consider, on $(\Omega, \mathbb{F}, \mathbb{P})$, the MKV FBSDE

$$
\begin{aligned}
X_t &= \xi + \int_0^t b\left(X_s, \mathcal{L}(X_s|W^0), \alpha^\star(X_s, \mathcal{L}(X_s|W^0), Y_s)\right) ds \\
&\quad + \int_0^t \sigma(\mathcal{L}(X_s|W^0))dW_s + \sigma^0(\mathcal{L}(X_s|W^0))dW_s^0 \\
Y_t &= \partial_x g(X_T, \mathcal{L}(X_T|W^0)) \\
&\quad + \int_t^T \partial_x H\left(X_s, \mathcal{L}(X_s|W^0), \alpha^\star(X_s, \mathcal{L}(X_s|W^0), Y_s), Y_s\right) ds \\
&\quad - \int_t^T Z_s dW_s - \int_t^T Z_s^0 dW_s^0
\end{aligned}
$$

- Under assumptions of Chapter 1 $\rightsquigarrow$ solution to MKV FBSDE is MFG equilibrium

Seeking a solution

- New two-point-boundary-problem $\rightsquigarrow$
  - Cauchy-Lipschitz theory in small time only
  - if Lipschitz coefficients (including the direction of the measure) $\rightsquigarrow$ existence and uniqueness in short time

  $\rightsquigarrow$ existence and uniqueness of MFG equilibria in small time

- Third lecture $\rightsquigarrow$ what about arbitrary time?
  - existence $\rightsquigarrow$ fixed point over the measure argument by means of compactness arguments

  Schauder's theorem

  - uniqueness $\rightsquigarrow$ require additional assumption

- Other question $\rightsquigarrow$ what about social optimization?
  - don't write the HJB equation (infinite dimension)
  - use Pontryagin principle instead

Part III. McKean-Vlasov FBSDEs

b. Lions derivative overs $\mathcal{P}_2(\mathbb{R}^d)$

Differentiation on $\mathcal{P}_2(\mathbb{R}^d)$

- Consider $\mathcal{U} : \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$
- Lifted-version of $\mathcal{U}$

$$\hat{\mathcal{U}} : L^2(\Omega, \mathbb{P}) \ni X \mapsto \mathcal{U}(\text{Law}(X))$$

  - ∘ $\mathcal{U}$ differentiable if $\hat{\mathcal{U}}$ Fréchet differentiable (Lions)
  - ∘ independent of the choice of $(\Omega, \mathbb{P})$ (rich enough)
- Differential of $\mathcal{U}$
  - ∘ Fréchet derivative of $\hat{\mathcal{U}}$ with $\mu = \text{Law}(X)$

$$D\hat{\mathcal{U}}(X) = \partial_\mu \mathcal{U}(\mu)(X), \quad \partial_\mu \mathcal{U}(\mu) : \mathbb{R}^d \ni x \mapsto \partial_\mu \mathcal{U}(\mu)(x) \in \mathbb{R}^d.$$

  - ∘ Derivative of $\mathcal{U}$ at $\mu \rightsquigarrow \partial_\mu \mathcal{U}(\mu) \in L^2(\mathbb{R}^d, \mu; \mathbb{R}^d)$

Differentiation on $\mathcal{P}_2(\mathbb{R}^d)$

- Consider $\mathcal{U} : \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$
- Lifted-version of $\mathcal{U}$

$$\hat{\mathcal{U}} : L^2(\Omega, \mathbb{P}) \ni X \mapsto \mathcal{U}(\mathrm{Law}(X))$$

  - $\mathcal{U}$ differentiable if $\hat{\mathcal{U}}$ Fréchet differentiable (Lions)
  - independent of the choice of $(\Omega, \mathbb{P})$ (rich enough)
- Differential of $\mathcal{U}$
  - Fréchet derivative of $\hat{\mathcal{U}}$ with $\mu = \mathrm{Law}(X)$

$$D\hat{\mathcal{U}}(X) = \partial_\mu \mathcal{U}(\mu)(X), \quad \partial_\mu \mathcal{U}(\mu) : \mathbb{R}^d \ni x \mapsto \partial_\mu \mathcal{U}(\mu)(x) \in \mathbb{R}^d.$$

  - Derivative of $\mathcal{U}$ at $\mu \rightsquigarrow \partial_\mu \mathcal{U}(\mu) \in L^2(\mathbb{R}^d, \mu; \mathbb{R}^d)$
- Finite dimensional projection

$$\partial_{x_i}\Big[\mathcal{U}\Big(\frac{1}{N}\sum_{j=1}^N \delta_{x_j}\Big)\Big] = \frac{1}{N}\partial_\mu \mathcal{U}\Big(\frac{1}{N}\sum_{j=1}^N \delta_{x_j}\Big)(x_i).$$

Examples

- 1st example: $\mathcal{U}(\mu) = \int_{\mathbb{R}^d} h(x) d\mu(x)$
  - two r.v.'s X and Y with values in $\mathbb{R}^d$

$$\begin{aligned}\mathcal{U}(\mathcal{L}(X + \varepsilon Y)) &= \mathbb{E}[h(X + \varepsilon Y)] \\ &= \mathbb{E}[h(X)] + \varepsilon \mathbb{E}\big[\partial h(X) Y\big] + o(\varepsilon)\end{aligned}$$

  - $\partial_\mu \mathcal{U}(\mu)(v) = \partial h(v)$
- 2nd example: $\mathcal{U}(\mu) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} h(x - y) d\mu(x) d\mu(y)$
  - two r.v.'s X and Y with independent copies X' and Y'

$$\begin{aligned}&\mathcal{U}(\mathcal{L}(X + \varepsilon Y)) \\ &= \mathbb{E}[h(X - X' + \varepsilon(Y - Y'))] \\ &= \mathbb{E}[h(X - X')] + \varepsilon \mathbb{E}\big[\partial h(X - X')(Y - Y')\big] + o(\varepsilon) \\ &= \mathbb{E}[h(X - X')] + \varepsilon \mathbb{E}\big[\partial h(X - X') Y\big] - \varepsilon \mathbb{E}\big[\partial h(X' - X) Y\big] + o(\varepsilon)\end{aligned}$$

  - $\partial_\mu \mathcal{U}(\mu)(v) = \int_{\mathbb{R}^d} \partial h(v - y) d\mu(y) - \int_{\mathbb{R}^d} \partial h(y - v) d\mu(y)$

Part III. McKean-Vlasov FBSDEs

c. Control of McKean-Vlasov and potential games

Rough version of the Pontryagin principle

- Controlled MKV processes (no common noise)

$$dX_t = b(X_t, \mathcal{L}(X_t), \alpha_t)dt + \sigma(X_t, \mathcal{L}(X_t))dW_t$$

  - optimize the cost $J(\alpha) = \mathbb{E}[g(X_T, \mathcal{L}(X_T)) + \int_0^T f(X_t, \mathcal{L}(X_t), \alpha_t)dt]$

- Optimize w.r.t. the measure as well
  - Use the same H and the same $\hat{\alpha}(t, x, \mu, y)$
  - Adjoint equations:

$$\begin{aligned}
dX_t &= b(X_t, \mu_t, \hat{\alpha}(t, X_t, \mathcal{L}X_t, Y_t))dt + \sigma dW_t \\
dY_t &= -\partial_x H(X_t, \mathcal{L}(X_t), \hat{\alpha}(X_t, \mathcal{L}(X_t), Y_t), Y_t)dt \\
&\qquad - \text{''}\partial_\mu H(X_t, \mathcal{L}(X_t), \hat{\alpha}(X_t, \mathcal{L}(X_t), Y_t), Y_t)\text{''}dt + Z_t dW_t \\
Y_T &= \partial_x g(X_T, \mathcal{L}(X_T)) + \text{''}\partial_\mu g(X_T, \mathcal{L}(X_T))\text{''}
\end{aligned}$$

  - What do "$\partial_\mu H$" and "$\partial_\mu g$" mean?

Right version of the Pontryagin principle

- Adjoint equations take the form

$$\begin{aligned} dX_t &= b(X_t, \mathcal{L}(X_t), \hat\alpha(t, X_t, \mathcal{L}(X_t), Y_t))dt + \sigma dW_t \\ dY_t &= -\partial_x H(X_t, \mathcal{L}(X_t), \hat\alpha(t, X_t, \mathcal{L}(X_t), Y_t), Y_t)dt \\ &\qquad - \mathbb{E}'[\partial_\mu H(X'_t, \mathcal{L}(X_t), \hat\alpha(X'_t, \mathcal{L}(X_t), Y'_t))(X_t)]dt + Z_t dW_t \\ Y_T &= \partial_x g(X_T, \mathcal{L}(X_T)) + \mathbb{E}'[\partial_\mu g(X'_T, \mathcal{L}(X_T))(X_T)] \end{aligned}$$

  - (X'_t, Y'_t) independent copy of (X_t, Y_t) on $(\Omega', \mathbb{F}', \mathbb{P}')$
- example $f(\mu, \alpha) = \frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(x-y)d\mu(x)d\mu(y) + \frac{1}{2}|\alpha|^2$, f symmetric
  - $g(\mu) = \frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} g(x-y)d\mu(x)d\mu(y)$
  - $b(\alpha) = \alpha$

$$\partial_\mu H(\cdot) = \partial_\mu f(\mathcal{L}(X_t))(X_t) = \mathbb{E}'[\partial f(X_t - X'_t)] = \partial_{|x=X_t}\mathbb{E}'[f(x - X'_t)]$$

  - same as an MFG with $\int_{\mathbb{R}^d} f(x-y)d\mu(y) + \frac{1}{2}|\alpha|^2 \rightsquigarrow$ potential game!