LOG-HESSIAN FORMULA AND THE TALAGRAND CONJECTURE

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ABSTRACT. In 1989, Talagrand proposed a conjecture regarding the regularization effect on integrable functions of a natural Markov semigroup on the Boolean hypercube. While this conjecture remains unresolved, the analogous conjecture for the Ornstein-Uhlenbeck semigroup was recently resolved by Eldan-Lee and Lehec, by combining an inequality for the log-Hessian of this semigroup with a new deviation inequality for log-semiconvex functions under Gaussian measure. Our first goal is to explore the validity of both these ingredients for some diffusion semigroups in $\mathbb{R}^n$ as well as for the $M/M/\infty$ queue on the non-negative integers. Our second goal is to prove an analogue of Talagrand's conjecture for these settings, even in those cases where these ingredients are not valid.

1. INTRODUCTION

The aim of this paper is threefold. First, we give explicit formulas for the log-Hessian of some diffusion semigroups in $\mathbb{R}^n$, and explicit lower bounds on some discrete analogue of the log-Hessian for the $M/M/\infty$ queuing process on the non-negative integers $\mathbb{N} := \{0, 1, \dots\}$. Second, we investigate deviation bounds for log-semiconvex functions, in the above two settings. Third, we prove in each context an analogue of the Talagrand Conjecture by different means. In the continuous setting of some class of diffusion semigroups in dimension 1, we generalize the approach developed in [17, 14, 26] based on the log-Hessian and deviation bounds just mentioned; while for the $M/M/\infty$ queuing process, we use a direct computation.

We will now present the conjecture by Talagrand first in its original version on the discrete hypercube and then in the continuous setting of the Ornstein-Uhlenbeck process, before moving to a historical presentation of its resolution in the continuous setting and the presentation of our results.

Consider the following infinitesimal generator on the n-dimensional hypercube $\Omega_n := \{-1,1\}^n$, acting on functions as $Lf(\sigma) = \frac{1}{2}\sum_{i=1}^n (f(\sigma^i) - f(\sigma))$. Here σ^i is the configuration with the i-th coordinate flipped (*i.e.* $\sigma^i_j = \sigma_j$ for all $j \neq i$ and $\sigma^i_i = -\sigma_i$). Denote by $(P_s)_{s\geq 0}$ the associated semigroup (sometimes called "convolution by a biased coin" in the literature), and by $\mu_n \equiv 2^{-n}$ the uniform measure on Ω_n which is reversible for L. In [37], Talagrand conjectured (see Conjecture 1 in [37]) that for any $s > 0$, it holds that

$$\lim_{t\to\infty} t \sup_n \sup_{f\in\mathcal{F}_n} \mu_n(\{\sigma : P_s f(\sigma) \geq t\}) = 0,$$

where $\mathcal{F}_n := \{f\colon \Omega_n \to [0,\infty) \text{ with } \|f\|_1 = 1\}$, and $\|f\|_p := \left(\sum_{\sigma\in\Omega_n} |f(\sigma)|^p \mu_n(\sigma)\right)^{\frac{1}{p}}$ stands for the $\mathbb{L}^p(\mu_n)$-norm of f, $p \geq 1$. Moreover Talagrand formulated the following stronger

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statement (Conjecture 2 in [37]) :

$$t \sup_{f\in\mathcal{F}_n} \mu_n(\{\sigma : P_s f(\sigma) \geq t\}) \leq c\frac{1}{\sqrt{\log t}}, \qquad t>1 \tag{1.1}$$

for some constant $c = c_s$ depending only on s (and not on n). Both conjectures are still open.

Let us underline that the difficulty of these questions completely relies on the uniformity in the dimension n. For a fixed integer n, proving (1.1) with a constant c depending on s and on the dimension n is easy. This can be seen using for instance the following line of reasoning, that we will call the "strategy of the supremum" in the reminder of the paper. Observe that for all $f : \Omega_n \to \mathbb{R}$, it holds

$$P_s f(\sigma) = \int f(\eta) K_s(\sigma,\eta) d\mu_n(\eta),$$

with $K_s(\sigma,\eta) = \prod_{i=1}^n (1+e^{-s}\sigma_i\eta_i)$ and so

$$\sup_{f\in\mathcal{F}_n} P_s f(\sigma) = \sup_{\eta\in\Omega_n} K_s(\sigma,\eta) = (1+e^{-s})^n, \qquad \forall\sigma\in\Omega_n$$

Therefore, for $t\geq 0$,

$$t \sup_{f\in\mathcal{F}_n} \mu_n(\{\sigma : P_s f(\sigma) \geq t\}) \leq t\mu_n(\{\sigma : \sup_{f\in\mathcal{F}_n} P_s f(\sigma) \geq t\}) = t\mathbf{1}_{\{t\leq(1+e^{-s})^n\}}.$$

In particular,

$$t \sup_{f\in\mathcal{F}_n} \mu_n(\{\sigma : P_s f(\sigma) \geq t\}) = 0$$

as soon as $t > (1+e^{-s})^n$ and so, for any fixed $s>0$ and $n\in\mathbb{N}$ it clearly exists a constant $c = c_{s,n}$ such that (1.1) is satisfied.

If one assumes that $\|f\|_p = 1$ for some $p>1$, then Markov's inequality would give a universal upper bound or order $1/t^{p-1}$ which is much better than $1/\sqrt{\log t}$. The hypercontractivity property of the semigroup [9, 7] also ensures that, if $f:\Omega_n\to\mathbb{R}$ and $p\geq 1$, then $\|P_s f\|_q \leq \|f\|_p$ with $q = 1+(p-1)e^{2s}$. But this inequality does not say anything when $p=1$. Talagrand's conjecture can therefore be seen as a weak $\mathbb{L}^1$ type regularization property of the semigroup.

While the above problems (Conjectures 1 and 2) are still open, a recent series of papers deals with a natural continuous counterpart to the conjectures, related to the Ornstein-Uhlenbeck semigroup. Denote by γ_n the standard Gaussian (probability) measure in dimension n, with density

$$\mathbb{R}^n \ni x \mapsto (2\pi)^{-n/2} \exp\left\{-\frac{|x|^2}{2}\right\},$$

where $|\cdot|$ denotes the standard Euclidean norm on $\mathbb{R}^n$. For $p\geq 1$, let $\mathbb{L}^p(\gamma_n)$ be the set of measurable functions $f\colon \mathbb{R}^n\to\mathbb{R}$ such that $|f|^p$ is integrable with respect to γ_n. Then, given $g\in\mathbb{L}^1(\gamma_n)$, the Ornstein-Uhlenbeck semigroup is defined by the so-called Mehler representation as

$$P_t^{\mathrm{ou}} g(x) := \int g\left(e^{-t}x + \sqrt{1-e^{-2t}}y\right) d\gamma_n(y), \qquad x\in\mathbb{R}^n,\ t\geq 0. \tag{1.2}$$

By a change of variable, we may also write

$$P_t^{\mathrm{ou}} g(x) = \frac{1}{Z_t}\int g(z) M_t(x,z) dz, \qquad x\in\mathbb{R}^n,\ t\geq 0, \tag{1.3}$$

where

$$M_t(x,z) := \exp\left\{-\frac{|z-e^{-t}x|^2}{2(1-e^{-2t})}\right\} = e^{-c_t^2|e^t z - x|^2/2} \qquad x,z\in\mathbb{R}^n,\ t\geq 0$$

and $Z_t = (2\pi(1-e^{-2t}))^{n/2}$ is the normalizing constant and

$$c_t := \frac{e^{-t}}{\sqrt{1-e^{-2t}}} \qquad t > 0. \tag{1.4}$$

The semigroup $(P_t^{\mathrm{ou}})_{t\geq 0}$ is associated to the infinitesimal diffusion operator $L^{\mathrm{ou}} := \Delta - x\cdot\nabla$ and enjoys the exact same hypercontractivity property as the convolution by biased coin operator on the discrete hypercube defined above. It is therefore natural to ask for an upper bound for

$$S_s(t) := t \sup_{f\geq 0, \|f\|_1=1} \gamma_n(\{\sigma : P_s^{\mathrm{ou}} f(\sigma) \geq t\}), \qquad s > 0.$$

In [14, 26] Eldan, Lee and Lehec fully solved the problem by proving that for any $s > 0$ there exists a constant $c_s \in (0,\infty)$ (depending only on s and not on the dimension n) such that $S_s(t) \leq \frac{c_s}{\sqrt{\log t}}$ for all $t > 1$ and this bound is optimal in the sense that the factor $\sqrt{\log t}$ cannot be improved. In an earlier paper [4], Ball, Barthe, Bednorz, Oleszkiewicz and Wolff already obtained a similar bound but with a constant $c_{s,n}$ depending on the dimension n plus some extra $\log\log t$ factor in the numerator. Later Eldan and Lee [14], using tools from stochastic calculus, proved that the above bound holds with a constant c_s independent on n but again with the extra $\log\log t$ factor in the numerator. Finally Lehec [26], following [14], removed the $\log\log t$ factor.

In both Eldan-Lee and Lehec's papers, the two key ingredients are the following:

(1) for any $s > 0$, the Ornstein-Uhlenbeck semigroup satisfies, for all non-negative function $g \in \mathbb{L}^1(\gamma_n)$,
$$\mathrm{Hess}\,(\log P_s^{\mathrm{ou}} g) \geq -c_s^2 \mathrm{Id},$$
where Hess denotes the Hessian matrix and Id the identity matrix of $\mathbb{R}^n$.

(2) for any positive function g with $\mathrm{Hess}\,(\log g) \geq -\beta \mathrm{Id}$, for some $\beta \geq 0$, and $\int g\, d\gamma_n = 1$, one has
$$\gamma_n(\{g \geq t\}) \leq \frac{C_\beta}{t\sqrt{\log t}} \qquad \forall t > 1,$$
with $C_\beta = \alpha \max(1,\beta)$.

More recently, following the above strategy, four of the authors [17] gave an alternative elementary proof of Eldan-Lee-Lehec's result in dimension 1, opening new lines of investigation. At this point we note that problem (1) is also at the heart of some of the fundamental problems in the Analysis of Loop Spaces. A program of Gross [21] is to prove Logarithmic Sobolev and Poincaré inequalities from Gaussian measures to Brownian motion and conditioned Brownian motion measures. The main problem involves constructing an Ornstein-Uhlenbeck process on the space of loops, obtaining integration by parts formula for these measures, and Poincaré inequalities. The latter is notoriously difficult, with counter examples by Eberle [13] and defective inequalities by Gong-Ma [16]. The Poincaré inequality is only proven to hold for very few classes of manifolds: see Aida [2] for asymptotically flat manifolds and Chen-Li-Wu [11] for hyperbolic spaces. The idea is to compare $\log p_t$, its gradient and Hessian with that of the Heat kernel on the Euclidean space and one wishes to obtain information on $t\mathrm{Hess}\log p(t,x,y) + \mathrm{Hess}\left(\frac{d^2(x,y)}{2}\right)$.

In Section 2, we investigate Item (1) above for general diffusion semigroups $(P_s)_{s\geq 0}$, in any dimension. In fact, using the Feynman-Kac formula, we are able to give an explicit representation for $\mathrm{Hess}(\log P_s g)$ that leads, under some assumptions, to a similar bound as in Item (1). We also investigate Item (2) for diffusions, in dimension 1 only, of the form $L = \frac{d^2}{dx^2} - h'\frac{d}{dx}$ (which corresponds to the Ornstein-Uhlenbeck operator for the choice $h(x) = \frac{1}{2}x^2$), when $0 < c \leq h'' \leq C$. Our results might therefore be seen as perturbations (though potentially unbounded) of the Ornstein-Uhlenbeck setting. Then we apply the approach developed in [17] to prove the Talagrand Conjecture for such diffusions in dimension 1.

In Section 3, we investigate Item (1) and Item (2) in the discrete setting of the $M/M/\infty$ queuing process on the integers. We will prove that, in that setting, a result similar to that of Item (1) still holds. On the other hand it appears that the picture is very different for Item (2) in the discrete setting. In fact, if g is "log-convex", in the sense that $\Delta \log g \geq 0$, where Δ is a discrete analogue of the Laplacian (see Section 3 for the definition), then a statement similar to Item (2) holds. In contrast, we will construct counterexamples of the result of Item (2) for g satisfying $\Delta \log g = -\beta$, with $\beta > 0$. The first property transfers to Talagrand's conjecture for the $M/M/\infty$ queuing process. More precisely, if g is "log-convex" (in the discrete sense), then the strategy developed in [17] leads to a positive conclusion regarding the Talagrand Conjecture but *restricted* to convex functions. However, as shown in Section 3.5, the strategy of the supremum presented above appears to be powerful in the case of the $M/M/\infty$ semigroup on the integers and will allow us to (fully) prove the conjecture in this setting.

It should be noticed here that the strategy of the supremum holds in the case of the Ornstein-Uhlenbeck semigroup in dimension 1 [4], but does not seem to apply to the perturbations of the Ornstein-Uhlenbeck considered in this paper. Therefore the situation is very different between the continuous and the discrete setting, and somehow in opposition (at least for the $M/M/\infty$ queuing process and the family of diffusion semigroups we consider): the strategy of the supremum works in the discrete, but not in the continuous; in contrast, the strategy consisting of proving (1) and (2) above works in the continuous, but fails in the discrete setting.

In fact, as will be shown in Section 4 by considering yet another class of semigroups (namely, the Laguerre semigroups on $(0, \infty)$), the picture can be different from the two previous ones. Namely we will show that neither Item (1) nor item (2) holds for the Laguerre semigroup, but the analogue of Talagrand's conjecture still holds.

We may summarize the different situations in the following diagram (in the present paper we investigate and prove results in the last three columns):

Semi-group:	Ornst.-Uhl.	$0 \leq c \leq h'' \leq C$	$M/M/\infty$	Laguerre
Item (1): Lower bound on $(\log P_t f)''$	Yes	Yes (under some assumptions on h)	Yes	No
Item (2): Deviation bounds for semi-log-convex functions $((\log f)'' \geq -\beta)$	Yes	Yes (under some assumptions on h)	No ($\beta > 0$) Yes ($\beta = 0$)	No ($\beta > 0$)
Talagrand's conjecture dim $n = 1$	(1) + (2) [14, 26, 17] or strat. sup. [4]	(1) + (2) $\frac{1}{t\sqrt{\log t}}$	strat. sup. $\frac{\sqrt{\log \log t}}{t\sqrt{\log t}}$	strat. sup. $\frac{1}{t\sqrt{\log t}}$
Talagrand's conjecture dim $n > 1$	(1) + (2) [14, 26]	unknown	unknown	unknown

The paper is organized as follows. In the next section, we deal with the continuous setting of diffusion semigroups (we give a formula for the log Hessian and apply our result to the Talagrand Conjecture in dimension 1). Section 3 is dedicated to the $M/M/\infty$ queuing process: again bounds on some discrete analogue of the log-Hessian are given, with a partial application to Talagrand's conjecture (restricted to log-convex functions).

In Section 3.5, we develop the strategy of the supremum to fully prove the conjecture. Finally, in the last section we quickly deal with Laguerre semigroups.

Contents

2. Diffusion semigroups

In this section, we derive an explicit formula for the Hessian (second-order space derivative) of $\log P_t f$ for general diffusion semigroups. As a warm up, we start with the easy case of the Ornstein-Uhlenbeck semigroup itself, in dimension 1, in Section 2.1. Section 2.2 contains the proof of the formula for the Hessian, which is the technical heart of our results on diffusion semigroups. In the subsequent subsections, focusing on dimension 1, we use this formula to first show semi-log-convexity of $P_t f$ (Section 2.3), then explore deviation inequalities for semi-log-convex functions (Section 2.4), and finally put these together to prove that the Talagrand Conjecture holds (Section 2.5) for a wide class of diffusion semigroups.

Notation:

- $\mathcal{C}_K^\infty$ denotes the set of $\mathcal{C}^\infty$ real valued functions with compact support.
- $\mathcal{D}^{(n)}$, $n = 0, 1, ...$, denotes the set of $\mathcal{C}^n$ real valued functions whose derivatives and the function itself have polynomial growth.

2.1. **Bounds on the Ornstein-Uhlenbeck semigroup in dimension 1.** In this section we deal with the dimension 1 for simplicity, and set $\gamma := \gamma_1$ with density denoted $\varphi(x) = (2\pi)^{-1/2} e^{-x^2/2}$, $x \in \mathbb{R}$. Denote by

$$H_n(x) := e^{x^2/2}(-1)^n \frac{d^n}{dx^n}\left(e^{-x^2/2}\right)$$

the Hermite polynomial of degree $n = 0, 1, \ldots$, with the convention that $H_0 \equiv 1$. It is well known that the family of Hermite polynomial is an orthonormal basis of $\mathbb{L}^2(\gamma)$. Simple computations lead to $H_1(x) = x$, $H_2(x) = x^2 - 1$, $H_3(x) = x^3 - 3x$ etc. Now, by a direct induction argument, the following identities hold:

$$(P_t^{\mathrm{ou}} g)^{(n)}(x) = c_t^n \int g\left(e^{-t}x + \sqrt{1-e^{-2t}}y\right) H_n(y) d\gamma(y), \qquad n \in \mathbb{N},$$

with c_t defined by (1.4). Fix a positive integrable function g and, for any $x \in \mathbb{R}$, denote by $\mathbb{E}_x$ the expectation with respect to the probability measure with density

$$y \mapsto g\left(e^{-t}x + \sqrt{1-e^{-2t}}y\right) / \int g\left(e^{-t}x + \sqrt{1-e^{-2t}}y\right) d\gamma(y)$$

with respect to the Gaussian measure γ. The above identities then read

$$d_n(x) := \frac{(P_t^{\mathrm{ou}} g)^{(n)}(x)}{P_t^{\mathrm{ou}} g(x)} = c_t^n \mathbb{E}_x(H_n(Y)), \qquad x \in \mathbb{R}, n \in \mathbb{N}.$$

Our next step is to explore the first derivatives of $x \mapsto u_t(x) := \log P_t^{\mathrm{ou}} g(x)$. Letting for simplicity $g_t(x) := P_t^{\mathrm{ou}} g(x)$, we get after simple algebra

$$u_t'(x) = \frac{g_t'}{g_t}(x) = d_1(x) = c_t \mathbb{E}_x[H_1(Y)] = c_t \mathbb{E}_x[Y]$$

$$u_t''(x) = \frac{g_t''}{g_t}(x) - \left(\frac{g_t'}{g_t}\right)^2 (x) = d_2(x) - d_1^2(x) = c_t^2 \left(\mathbb{E}_x[H_2(Y)] - \mathbb{E}_x[H_1(Y)]^2\right) = c_t^2 \left(-1 + \mu_2(x)\right),$$

where $\mu_2(x) = \mathbb{E}_x[Y^2] - \mathbb{E}_x[Y]^2 \geq 0$. In particular,

$$(\log P_t^{\mathrm{ou}} g)''(x) = u_t''(x) \geq -c_t^2$$

which corresponds to Item (1) in the Introduction.

2.2. **Representation for the Hessian of perturbed Ornstein-Uhlenbeck semi-groups.** In this section, we give an explicit formula for the Hessian of $\log P_t$ for a wide class of diffusion operators. We need to introduce some additional notation. For $a, \sigma > 0$, consider the general Ornstein-Uhlenbeck operator $L_{\sigma,a}^{\mathrm{ou}}$ on $\mathbb{R}^n$

$$L_{\sigma,a}^{\mathrm{ou}} = \frac{1}{2}\sigma^2 \Delta - ax \cdot \nabla,$$

where the dot stands for the scalar product. Observe that the Ornstein-Uhlenbeck operator given in the introduction corresponds to $\sigma = \sqrt{2}$ and $a = 1$. In what follows, we will write L^{ou} instead of $L_{\sigma,a}^{\mathrm{ou}}$ in order not to overload the notation. Let $(B_t)_{t \geq 0}$ be a standard Brownian motion on $\mathbb{R}^n$ on a (filtered) probability space $(\Omega, \mathbb{P})$ which we fix. For any $x \in \mathbb{R}^n$ let $(X_s^x)_{s \geq 0}$ be the (unique strong) solution to

$$X_t^x = x + \sigma B_t - a \int_0^t X_s^x ds.$$

This is the so-called Ornstein-Uhlenbeck process (with parameters a, σ) starting at x ; its infinitesimal generator is L^{ou}. For any $t > 0$, the law of X_t^x will be denoted by γ_t^x and is given by the (general) Mehler formula

$$d\gamma_t^x(y) = \frac{1}{Z_t} M_t(x,y)\, dy$$

with

$$M_t(x,y) = M_t^{\sigma,a}(x,y) = \exp\left(-\frac{a|y - e^{-at}x|^2}{\sigma^2(1-e^{-2at})}\right), \qquad y \in \mathbb{R}^n,$$

and Z_t a normalizing constant. We will denote by γ the equilibrium measure of the process given by

$$\gamma(dy) = \frac{1}{Z} \exp\left(-\frac{a|y|^2}{\sigma^2}\right) dy, \qquad Z = \left(\frac{\pi\sigma^2}{a}\right)^{n/2}.$$

Note that when $a = 1$ and $\sigma = \sqrt{2}$, then $\gamma = \gamma_n$ is the standard Gaussian distribution on $\mathbb{R}^n$.

We also consider the following perturbation of the Ornstein-Uhlenbeck operator

$$L^V = L^{\mathrm{ou}} - V$$

where $V\colon \mathbb{R}^n \to \mathbb{R}$ is a potential that acts multiplicatively, namely $L^V f = L^{\mathrm{ou}} f - V f$. The associated semigroup will be denoted by $(P_t^V)_{t\geq 0}$. We recall that P_t^V can be represented by the Feynman-Kac formula:

Proposition 2.1. *Suppose that $V : \mathbb{R}^n \to \mathbb{R}$ is continuous and bounded from below and define for $t \geq 0$ the operator P_t^V by*

$$P_t^V f(x) = \mathbb{E}\left[f(X_t^x) e^{-\int_0^t V(X_s^x)\,ds}\right], \qquad \forall x \in \mathbb{R}^n, \qquad \forall f \in \mathbb{L}^2(\gamma).$$

Then $(P_t^V)_{t\geq 0}$ is a semigroup on $\mathbb{L}^2(\gamma)$ with infinitesimal generator L^V.

In the sequel we will need the following definition.

Definition 2.2. *Let $t > 0$, $x \in \mathbb{R}$ and let $f \in \mathbb{L}^2(\gamma) \setminus \{0\}$ be a non-negative function. We define the probability measure $\mathbb{Q}_{f,x}$ on Ω (which depends also on t and V) by*

$$\mathbb{Q}_{f,x}(\Gamma) = \frac{1}{P_t^V f(x)} \int_\Gamma f(X_t^x) e^{-\int_0^t V(X_s^x) ds}\, d\mathbb{P}$$

and use $\mathbb{E}_{f,x}$ for the expectation with respect to $\mathbb{Q}_{f,x}$.

The following result gives an explicit representation for the Hessian of $\log P_t^V f$:

Theorem 2.3. *Suppose that $V\colon \mathbb{R}^n \to \mathbb{R}$ is bounded from below and in $\mathcal{D}^{(2)}$. For $t > 0$ and $x \in \mathbb{R}^n$, set*

$$A_t^x := -\int_0^t \nabla V(X_s^x) \frac{\sinh(a(t-s))}{\sinh(at)}\, ds + \frac{2ae^{-at}}{\sigma^2(1-e^{-2at})}(X_t^x - e^{-at}x).$$

Let $f \in \mathbb{L}^2(\gamma) \setminus \{0\}$ be non-negative ; with the notation of Definition 2.2, it holds

$$\nabla(P_t^V f)(x) = \mathbb{E}_{f,x}(A_t^x)$$

and

(2.1)

$$\begin{aligned}
&\mathrm{Hess}(\log P_t^V f)(x) + \frac{2ae^{-2at}}{\sigma^2(1-e^{-2at})}\mathrm{Id} \\
&= -\int_0^t \left(\frac{\sinh(a(t-s))}{\sinh(at)}\right)^2 \mathbb{E}_{f,x}(\mathrm{Hess}\, V(X_s^x))ds + \mathbb{E}_{f,x}(A_t^x \otimes A_t^x) - \mathbb{E}_{f,x}(A_t^x) \otimes \mathbb{E}_{f,x}(A_t^x).
\end{aligned}$$

The notation ∇ denotes the gradient with respect to the standard Euclidean metric (note that the Riemannian metric, intrinsic to the equation, is $\widetilde{\nabla} = \sigma^2 \nabla$).

The interested reader may find a series of articles on first/second order Feynman-Kac formulas for general elliptic diffusions on manifolds in [36, 15, 33, 3]. Moreover Hessian estimates can be found in [28, 30] under general conditions that are non-trivial to check (exchanging orders of operators, non-explosion, existence of global smooth flows). In the proof of Theorem 2.3, we are able to compute the derivatives thanks to an explicit formulation of Ornstein-Uhlenbeck bridge (which appears to be linear in its initial position) and the introduction of the probability $\mathbb{Q}_{f,x}$ (see [31, 29] for more on elliptic diffusion bridges).

Remark 2.4. *Observe that, when $V \equiv 0$, $a = 1$ and $\sigma^2 = 2$, P_t^V is the Ornstein-Uhlenbeck semigroup. In dimension 1, after a change of variable, (2.1) reads*

$$\begin{aligned}(\log P_t^V f)''(x) &= -c_t^2 + \mathrm{Var}_{f,x}(A_t^x) - \int_0^t \alpha_t(s)^2 \mathbb{E}_{f,x}(V''(X_s^x))ds \\ &= u_t'' = c_t^2\left(-1 + \mu_2(x)\right),\end{aligned}$$

using the notation of Section 2.1.

Proof of Theorem 2.3. Fix $x \in \mathbb{R}$ and $t \geq 0$. According to Proposition 2.1, it holds

$$P_t^V f(x) = \mathbb{E}\left(f(X_t^x)e^{-\int_0^t V(X_s^x)ds}\right) = Z^{-1}\int_{\mathbb{R}^n} f(y)\mathbb{E}\left(e^{-\int_0^t V(X_s^x)ds}|X_t^x = y\right) M_t(x,y)\,dy,$$

where $Z = Z_t$ is the normalisation constant for $M_t(x,y)$ that does not depend on x.

Conditioning on $X_t^x = y$, $(X_s^x)_{0\leq s\leq t}$ is distributed as the Ornstein-Uhlenbeck bridge $(Y_s^{x,y})_{0\leq s\leq t}$, which begins at x and ends at y at the final time t. To determine the dependence of the functions with respect to the variable x, we use an explicit representation of $Y_s \equiv Y_s^{x,y}$ as solution of the following equation

$$dY_s = \sigma dB_s - aY_s ds + \sigma^2 \nabla_x \log M_{t-s}(Y_s, y)ds,$$

with the initial value $Y_0 = x$ and where ∇_x stands for the derivative with respect to the x variable. It has a singular drift at the terminal time t and so it is initially defined for $s < t$, and then extended by continuity to $X_s = z$ for $s \geq t$. We have

$$\nabla_x \log M_t(x,y) = (y - e^{-at}x)\frac{2ae^{-at}}{\sigma^2(1-e^{-2at})} = d_t(y - e^{-at}x),$$

(which is a drift pulling toward y), where we set $d_t := \frac{2ae^{-at}}{\sigma^2(1-e^{-2at})}$. Thus we get

$$dY_s = \sigma dB_s + \frac{2aye^{-a(t-s)}}{1-e^{-2a(t-s)}}ds - a\frac{1+e^{-2a(t-s)}}{1-e^{-2a(t-s)}}Y_s ds. \tag{2.2}$$

The difference $Y_s^{x,y} - Y_s^{0,y}$ solves a time dependent linear equation and is given, for all $s \in [0,t]$, by

$$Y_s^{x,y} - Y_s^{0,y} = \alpha_t(s)x, \qquad \text{where} \qquad \alpha_t(s) := \frac{\sinh(a(t-s))}{\sinh(at)}. \tag{2.3}$$

Therefore we have

$$\begin{aligned}P_t^V f(x) &= \int_{\mathbb{R}^n} f(y)\mathbb{E}\left(e^{-\int_0^t V(Y_s^{x,y})ds}\right) M_t(x,y)dy \\ &= \int_{\mathbb{R}^n} f(y)\mathbb{E}\left(e^{-\int_0^t V(\alpha_t(s)x + Y_s^{0,y})ds}\right) M_t(x,y)dy,\end{aligned} \tag{2.4}$$

Take $f \in \mathcal{C}_K^\infty$; since $Y_s^{0,y}$ does not depend on x, it holds

$$\nabla_x\left(e^{-\int_0^t V(Y_s^{x,y})ds}\right) = -e^{-\int_0^t V(Y_s^{x,y})ds}\int_0^t \alpha_t(s)\nabla V(Y_s^{x,y})ds. \tag{2.5}$$

So,

$$\begin{aligned}\nabla(P_t^V f)(x) = &-\int_{\mathbb{R}^n} f(y)\mathbb{E}\left(e^{-\int_0^t V(Y_s^{x,y})ds}\int_0^t \nabla V(Y_s^{x,y})\alpha_t(s)\,ds\right) M_t(x,y)dy \\ &+ \int_{\mathbb{R}^n} f(y)\mathbb{E}\left(e^{-\int_0^t V(Y_s^{x,y})ds}\right)\nabla_x M_t(x,y)dy.\end{aligned} \tag{2.6}$$

Plugging in the expression for $\nabla_x \log M_t(x,y)$ and reversing the conditioning process, we see that

$$\nabla(P_t^V f)(x) = -\int_0^t \mathbb{E}\left(f(X_t^x)\,e^{-\int_0^t V(X_s^x)ds}\nabla V(X_s^x)\right)\alpha_t(s)\,ds + d_t\mathbb{E}\left(e^{-\int_0^t V(X_s^x)ds}f(X_t^x)(X_t^x - e^{-at}x)\right).$$

Therefore, for $0 \le s \le t$,

$$\nabla(\log P_t^V f)(x) = \mathbb{E}_{f,x}(A_t^x).$$

In the calculations above, we have taken liberty to differentiate under the integration sign, which holds for any smooth functions with compact support. Since $Y_s^{x,y}$ is Gaussian and has moments of all order and $|\nabla V|$ growth at most polynomially, if a sequence $f_n \in \mathcal{C}_K^\infty$ converges to $f \in \mathbb{L}^2(\gamma)$ then the right hand side of the latter converges uniformly. Hence $P_t^V f$ is differentiable and the identity holds for any $f \in \mathbb{L}^2(\gamma)$.

Using the same conditioning strategy, we can similarly compute the second order derivative of $P_t^V f$, treated as a symmetric matrix. For this we go back to (2.6) and differentiate under the integral signs: for any $w \in \mathbb{R}^n$,

$$\begin{aligned}
&\langle \mathrm{Hess}(P_t^V f)(x), w\otimes w\rangle \\
&= -\int f(y)\mathbb{E}\left(e^{-\int_0^t V(Y_s^{x,y})ds}\int_0^t \alpha_t(s)^2\langle \mathrm{Hess}\, V(Y_s^{x,y}), w\otimes w\rangle \, ds\right) M_t(x,y)dy \\
&\quad + \int f(y)\mathbb{E}\left(e^{-\int_0^t V(Y_s^{x,y})ds}\left(\int_0^t \alpha_t(s)\langle \nabla V(Y_s^{x,y}), w\rangle \, ds\right)^2\right) M_t(x,y)dy \\
&\quad - 2\int f(y)\mathbb{E}\left(e^{-\int_0^t V(Z_s^{x,y})ds}\int_0^t \langle \nabla V(Y_s^{x,y}), w\rangle \alpha_t(s)ds\right)\langle \nabla_x M_t(x,y), w\rangle dy \\
&\quad + \mathbb{E}\left(e^{-\int_0^t V(X_s^x)ds} f(X_t^x)\langle \mathrm{Hess}_x M_t(x,z), w\otimes w\rangle dy\right).
\end{aligned}$$

The differentiation procedure holds for f in $\mathcal{C}_K^\infty$, and the same approximation argument as before shows that it holds also for any $f \in \mathbb{L}^2(\gamma)$. Next we observe that the following identity holds

$$\mathrm{Hess}_x \log M_t(x,y) = -d_t e^{-at}\mathrm{Id}$$

where Id is the $n\times n$ identity matrix. Therefore,

$$\begin{aligned}
\frac{\mathrm{Hess}_x M_t(x,y)}{M_t(x,y)} &= \mathrm{Hess}_x \log M_t(x,y) + \nabla_x \log M_t(x,y)\otimes \nabla_x \log M_t(x,y) \\
&= -d_t e^{-at}\mathrm{Id} + d_t^2(y - e^{-at}x)\otimes(y - e^{-at}x).
\end{aligned}$$

Using (2.3) and (2.5) we get

$$\begin{aligned}
&\frac{\langle \mathrm{Hess}(P_t^V f)(x), w\otimes w\rangle}{P_t^V f(x)} \\
&= -\int_0^t \alpha_t(s)^2 \mathbb{E}_{f,x}(\langle \mathrm{Hess}\, V(X_s^x), w\otimes w\rangle)ds + \mathbb{E}_{f,x}\left(\left(\int_0^t \alpha_t(s)\langle \nabla V(X_s^x), w\rangle ds\right)^2\right) \\
&\quad - 2\mathbb{E}_{f,x}\left(\int_0^t \langle \nabla V(X_s^x), w\rangle \alpha_t(s)ds\, d_t\langle X_t^x - e^{-at}x, w\rangle\right) + d_t^2\mathbb{E}_{f,x}\left(\langle X_t^x - e^{-at}x, w\rangle^2\right) - d_t e^{-at}|w|^2 \\
&= -\int_0^t \alpha_t(s)^2 \mathbb{E}_{f,x}(\langle \mathrm{Hess}\, V(X_s^x), w\otimes w\rangle)ds \\
&\quad + \mathbb{E}_{f,x}\left(\left(-\int_0^t \alpha_t(s)\langle \nabla V(X_s^x), w\rangle ds + d_t\langle X_t - e^{-at}x, w\rangle\right)^2\right) - d_t e^{-at}|w|^2 \\
&= -\int_0^t \alpha_t(s)^2 \mathbb{E}_{f,x}(\langle \mathrm{Hess}\, V(X_s^x), w\otimes w\rangle)ds + \mathbb{E}_{f,x}(\langle A_t^x, w\rangle^2) - d_t e^{-at}|w|^2.
\end{aligned}$$

We then use the identity

$$\mathrm{Hess}(\log P_t^V f)(x) = \frac{\mathrm{Hess}(P_t^V f)(x)}{P_t^V f(x)} - \frac{\nabla(P_t^V f)(x)\otimes \nabla(P_t^V f)(x)}{P_t^V f(x)^2}$$

to obtain the following

$$\begin{aligned}\mathrm{Hess}(\log P_t^V f)(x) = &- d_t e^{-at}\mathrm{Id} - \int_0^t \alpha_t(s)^2 \mathbb{E}_{f,x}(\mathrm{Hess}\, V(X_s^x))ds \\ &+ \mathbb{E}_{f,x}(A_t^x \otimes A_t^x) - \mathbb{E}_{f,x}(A_t^x) \otimes \mathbb{E}_{f,x}(A_t^x).\end{aligned}$$

This completes the proof. □

Remark 2.5. *Using Equation (2.2) we see that the Ornstein-Uhlenbeck starting from x conditioned to reach z at time t has the following explicit reprersentation:*

$$Z_s^{x,z} = \alpha_t(s)x + z\int_0^s \frac{a\ \sinh(a(t-s))}{\sinh^2(a(t-r))}dr + \sigma \int_0^s \frac{\sinh(a(t-s))}{\sinh(a(t-r))}dB_r.$$

The one dimensional case can be found in [12], see also [6] and the reference therein.

Remark 2.6. *In some situations, it might be also useful to control the second order derivative of the semigroup by the derivatives of f themselves. For example, in [18, 19], the authors deal with log-semi-convex functions in order to get a characterization of transport inequalities. The result below shows how such a log-semi-convexity transfers to the semigroup. More precisely, assume that, in addition of the hypotheses of the theorem, $f\colon \mathbb{R}^n \to \mathbb{R}$ is in $\mathcal{D}^{(2)}$ then, with the notation in the proof of the theorem, it also holds*

$$\begin{aligned}\mathrm{Hess}(\log P_t^V f)(x) = &e^{-2at}\mathbb{E}_{f,x}(\mathrm{Hess}(\log f)(X_t^x)) - \int_0^t \mathbb{E}_{f,x}(\mathrm{Hess}(V)(X_s^x))e^{-2as}ds \\ &+ \mathbb{E}_{f,x}(\tilde{A}_t^x \otimes \tilde{A}_t^x) - \mathbb{E}_{f,x}(\tilde{A}_t^x) \otimes \mathbb{E}_{f,x}(\tilde{A}_t^x)\end{aligned} \tag{2.7}$$

where $\tilde{A}_t^x := \int_0^t \nabla V(X_s^x)e^{-as}ds$.

Observe that $X_s^x = e^{-as}x + \sigma \int_0^s e^{-a(s-r)}dB_r$ so that, using the Feynman-Kac formula,

$$P_t^V f(x) = \mathbb{E}\left(f\left(e^{-at}x + \sigma\int_0^t e^{-a(t-r)}dB_r\right)e^{-\int_0^t V(e^{-as}x+\sigma\int_0^s e^{-a(s-r)}dB_r)ds}\right)$$

it holds

$$\nabla P_t^V f(x) = e^{-at}\mathbb{E}\left(\nabla f(X_t^x)e^{-\int_0^t V(X_s^x)ds}\right) - \mathbb{E}\left(f(X_t^x)\int_0^t \nabla V(X_s^x)e^{-as}ds e^{-\int_0^t V(X_s^x)ds}\right).$$

Differentiating one more time, we get

$$\begin{aligned}\mathrm{Hess}(P_t^V f)(x) = &e^{-2at}\mathbb{E}\left(\mathrm{Hess}(f)(X_t^x)e^{-\int_0^t V(X_s^x)ds}\right) \\ &- 2\mathbb{E}\left(\nabla f(X_t^x) \otimes \int_0^t \nabla V(X_s^x)e^{-as}ds e^{-\int_0^t V(X_s^x)ds}\right) \\ &- \mathbb{E}\left(f(X_t^x)\int_0^t \mathrm{Hess}V(X_s^x)e^{-2as}ds e^{-\int_0^t V(X_s^x)ds}\right) \\ &+ \mathbb{E}\left(f(X_t^x)\int_0^t \nabla V(X_s^x)e^{-as}ds \otimes \int_0^t \nabla V(X_s^x)e^{-as}ds\ e^{-\int_0^t V(X_s^x)ds}\right)\end{aligned}$$

from which the expected result follows.

2.3. **Semi-log-convexity for diffusion semigroups.** Thanks to the result of the previous section and with the help of the h-transform, we can obtain explicit formula for the log-Hessian of general diffusion semigroups. In turn we may obtain explicit lower bounds that will be useful for applications.

Given $W\colon \mathbb{R}^n \to \mathbb{R}$, smooth enough, and the operator $L^V = \Delta - x\cdot\nabla - V$ on $\mathbb{L}^2(\gamma_n)$, we define the operator $\mathcal{L}^W$ on $\mathbb{L}^2(e^{W/2}\gamma_n)$ by the unitary transform (h-transform) below:

$$\begin{aligned}\mathcal{L}^W f &:= e^{-W/2}L^V(e^{W/2}f) \\ &= \Delta f - (x - \nabla W)\cdot\nabla f + \left(\frac{1}{2}\Delta W + \frac{1}{4}|\nabla W|^2 - \frac{1}{2}x\cdot\nabla W - V\right)f,\end{aligned}$$

whose associated semi-group $\mathcal{P}_t^W$ is intertwined with P_t^V by

$$\mathcal{P}_t^W f = e^{-W/2} P_t^V (e^{W/2} f).$$

Let $h : \mathbb{R}^n \to \mathbb{R}$ and define $d\mu_h(x) = e^{-h}dx$. If we seek a representation for the reversible operator

$$L_h := \Delta - \nabla h \cdot \nabla$$

on $\mathbb{L}^2(\mu_h)$ of the form $L_h f = \mathcal{L}^W f = e^{-W/2} L^V (e^{W/2} f)$, we choose W and then V so that

$$\nabla \left(\frac{|x|^2}{2} - W \right) = \nabla h \qquad \text{and} \qquad \frac{1}{2}\Delta W + \frac{1}{4}|\nabla W|^2 - \frac{1}{2} x \cdot \nabla W - V = 0.$$

A function f belongs to $\mathbb{L}^2(\mu_h)$ if and only if $f e^{W/2}$ belongs to $\mathbb{L}^2(\gamma_n)$. We denote by $(\mathcal{P}_t)_{t\geq 0}$ the semi-group associated to the diffusion operator $L_h := \Delta - \nabla h \cdot \nabla$. The operator L_h is essentially self-adjoint on $\mathcal{C}_K^\infty$, see [27]. Theorem 2.3 and Remark 2.6 (with $a = 1$ and $\sigma = \sqrt{2}$) then admits the following immediate corollary.

Corollary 2.7. *Let $h \colon \mathbb{R}^n \to \mathbb{R}$ belongs to $\mathcal{D}^{(4)}$. Set $\mu_h(dx) = e^{-h(x)}dx$ and*

$$W(x) := \frac{|x|^2}{2} - h(x), \qquad V(x) := \frac{1}{2}(n - \Delta h) - \frac{1}{4}(|x|^2 - |\nabla h|^2), \quad x \in \mathbb{R}^n.$$

Assume that V is bounded from below. Let $f \in \mathbb{L}^2(\mu_h) \setminus \{0\}$ be non negative and, for all $x \in \mathbb{R}^n$, denote by $\mathbb{E}^W_{f,x}$ the expectation with respect to the probability measure $\mathbb{Q}_{e^{\frac{W}{2}} f, x}$ introduced in Definition 2.2. Then

$$\begin{aligned} \text{Hess}(\log \mathcal{P}_t f)(x) &= -\frac{1}{2}(\text{Id} - \text{Hess}(h)(x)) + \mathbb{E}^W_{f,x}(A_t^x \otimes A_t^x) - \mathbb{E}^W_{f,x}(A_t^x) \otimes \mathbb{E}^W_{f,x}(A_t^x) \\ &\quad - c_t^2 \text{Id} - \int_0^t \left(\frac{\sinh(t-s)}{\sinh(t)} \right)^2 \mathbb{E}^W_{f,x}(\text{Hess}\, V(X_s^x)) ds \end{aligned} \tag{2.8}$$

where

$$A_t^x := -\int_0^t \nabla V(X_s^x) \frac{\sinh(t-s)}{\sinh(t)}\, ds + \frac{e^{-t}}{1 - e^{-2t}} (X_t^x - e^{-t} x)$$

and c_t is given by (1.4). Assume in addition $f \in \mathcal{D}^{(4)}$, then for $\tilde{A}_t^x := \int_0^t \nabla V(X_s^x) e^{-as} ds$, the following holds

(2.9)

$$\begin{aligned} \text{Hess}(\log \mathcal{P}_t f)(x) &= -\frac{1}{2}(\text{Id} - \text{Hess}(h)(x)) + e^{-2at} \mathbb{E}^W_{f,x}(\text{Hess}(\log f)(X_t^x)) \\ &\quad + \mathbb{E}^W_{f,x}(\tilde{A}_t^x \otimes \tilde{A}_t^x) - \mathbb{E}^W_{f,x}(\tilde{A}_t^x) \otimes \mathbb{E}^W_{f,x}(\tilde{A}_t^x) - \int_0^t \mathbb{E}^W_{f,x}(\text{Hess}(V)(X_s^x)) e^{-2as} ds. \end{aligned}$$

Observe that, as for Theorem 2.3, Corollary 2.7 contains the case of the Ornstein-Uhlenbeck semigroup which corresponds to the trivial case $W = V = 0$.

For applications, especially in dimension 1, it will be useful to get a lower bound on $(\log \mathcal{P}_t)''$. To that respect (2.8) is useful since we can remove the non negative (co)variance term. Then one is left with estimates on h'' and V''.

Proposition 2.8. *Let $h \colon \mathbb{R} \to \mathbb{R}$ be of class $\mathcal{D}^{(4)}$ with $\mu_h(dx) := e^{-h(x)}dx$ a finite probability measure. Set $V(x) := \frac{1}{2}(1 - h''(x)) - \frac{1}{4}(x^2 - h'^2(x))$, $x \in \mathbb{R}$ and assume that V is bounded from below. Then, given $f \in \mathbb{L}^2(\mu_h)$ non-negative, for all $x \in \mathbb{R}$ and $t \geq 0$, one has*

$$(\log \mathcal{P}_t f)''(x) \geq -c_t^2 - \frac{1}{2}(1 - h''(x)) - \frac{1}{2} \sup_{y \in \mathbb{R}} V''(y). \tag{2.10}$$

Proof. This is a direct consequence of Corollary 2.7 above and the fact that

$$\int_0^t \left(\frac{\sinh(t-s)}{\sinh(t)}\right)^2 ds = \frac{e^{2t}-e^{-2t}-4t}{2(e^{2t}+e^{-2t}-2)} \le \frac{1}{2}, \qquad t \ge 0.$$

□

As an example of application, one can consider $h(x) = \frac{x^2}{2} + (1+x^2)^{p/2}$, with $p \le 2$. Then it is easy to see that h satisfies the hypotheses of the latter and that $-\frac{1}{2}(1-h''(x)) - \frac{1}{2}\sup_{y\in\mathbb{R}} V''(y) \ge -c_p$ for some constant c_p depending only on p.

2.4. **Deviation bounds for semi-log-convex functions.** The aim of this section is to prove a deviation bound for semi-log-convex functions, following [17]. Namely, the following holds.

Theorem 2.9. *Let μ_h be a probability measure on $\mathbb{R}$ of the form $d\mu_h(x) = e^{-h(x)}\,dx$ with $h\colon \mathbb{R} \to \mathbb{R}$ a symmetric $\mathcal{C}^2$ function. Assume that there exist $c, C > 0$ such that $c \le h'' \le C$. Then, for any $\mathcal{C}^2$ function $f\colon \mathbb{R} \to (0,\infty)$ such that $(\log f)'' \ge -\beta$ for some $\beta \ge 0$, it holds*

$$\mu_h\Big(\Big\{f \ge t\int f\,d\mu_h\Big\}\Big) \le \left(\frac{C+\beta}{c}\right)\frac{1}{t\sqrt{\log t}}, \qquad \forall t \ge 2.$$

Remark 2.10. *The assumption h symmetric is here for simplicity. A similar statement would hold with h non symmetric. The special case $h(x) = x^2/2$ is given in [17] with a factor $(1+\beta)/\sqrt{2}$ which is slightly better than $1+\beta$ (since $c = C = 1$ when $h(x) = x^2/2$).*

The proof of Theorem 2.9 relies on the following technical lemma whose proof can be found at the end of this section.

Lemma 2.11. *Let μ_h be a probability measure on $\mathbb{R}$ of the form $d\mu_h(x) = e^{-h(x)}\,dx$ with $h\colon \mathbb{R} \to \mathbb{R}$ a symmetric $\mathcal{C}^2$ function. Assume that there exists $C > 0$ such that $0 \le h'' \le C$. Then, for any $\varphi\colon \mathbb{R} \to \mathbb{R}$ of class $\mathcal{C}^2$ such that $\varphi'' \ge -\beta$ for some $\beta \ge 0$, it holds*

$$\varphi(x) - \log\left(\int e^{\varphi}\,d\mu_h\right) \le \frac{1}{2}\log\left(\frac{C+\beta}{2\pi}\right) + h(x), \qquad \forall x \in \mathbb{R}.$$

Proof of Theorem 2.9. Set $\varphi = \log f$, which satisfies $\varphi'' \ge -\beta$. Without loss of generality one can assume that $\int e^{\varphi}\,d\mu_h = 1$. Define $a = \frac{1}{2}\log\left(\frac{C+\beta}{2\pi}\right)$. From Lemma 2.11 and by symmetry of h we have, for all $t > 2(a+h(0))$

$$\mu_h(\{\varphi \ge t\}) \le \mu_h(\{h(x) \ge t-a\}) \le 2\int_{h^{-1}(t-a)}^{\infty} e^{-h(x)}dx \le \frac{2e^a e^{-t}}{h'(h^{-1}(t-a)))}$$

where we used the following bound, valid for any $s > 0$ (recall that h' is increasing on $\mathbb{R}^+$)

$$\int_s^{\infty} e^{-h(x)}dx \le \int_s^{\infty}\frac{h'(x)}{h'(s)}e^{-h(x)}dx = \frac{e^{-h(s)}}{h'(s)}.$$

Now observe that, since h is smooth and symmetric, $h'(0) = 0$ so that $h(x) \le h(0) + \frac{1}{2}Cx^2$ and $h'(x) \ge cx$, $x \ge 0$. Therefore

$$h'(h^{-1}(x)) \ge h'\left(\sqrt{\frac{2(x-h(0))}{C}}\right) \ge c\sqrt{\frac{2(x-h(0))}{C}} \qquad \text{for any } x \ge h(0).$$

In turn, since we fixed $t \ge 2(a+h(0))$, $2\left((t-a)-h(0)\right) \ge t$ and thus, thanks to the latter

$$h'(h^{-1}(t-a))) \ge c\sqrt{\frac{2\left((t-a)-h(0)\right)}{C}} \ge c\sqrt{\frac{t}{C}}.$$

We conclude that, for any $t \geq 2(a+h(0))$,

$$\mu_h(\{\varphi \geq t\}) \leq \frac{2\sqrt{C}e^a}{c}\frac{e^{-t}}{\sqrt{t}} \leq 2\frac{C+\beta}{c\sqrt{2\pi}}\frac{e^{-t}}{\sqrt{t}}$$

Next we deal with $t \in (0, 2(a+h(0)))$. Using Markov's inequality, since $\int e^{\varphi} d\mu_h = 1$, we have

$$\mu_h(\{\varphi \geq t\}) \leq e^{-t} \leq \sqrt{2(a+h(0))}\frac{e^{-t}}{\sqrt{t}}.$$

Since $\int e^{-h}\,dx = 1$ and $h(0) + c\frac{x^2}{2} \leq h(x)$, we have $2h(0) \leq \log\frac{2\pi}{c}$ so that

$$\sqrt{2a + 2h(0)} \leq \sqrt{\log((C+\beta)/c)} \leq \frac{C+\beta}{c}$$

where the last inequality follows from a direct computation. $\square$

Proof of Lemma 2.11. We follow [17, Lemma 2.1]. The bound is trivial if $\int e^{\varphi}\,d\mu_h = +\infty$ so let us assume that $\int e^{\varphi} d\mu_h = 1$. Define $g(x) = \varphi(x) - h(x) + \alpha\frac{x^2}{2}$, $x \in \mathbb{R}$, with $\alpha = C + \beta$. The function g is convex on $\mathbb{R}$ and so, by Fenchel-Legendre duality, it holds $g(x) = \sup_{y\in\mathbb{R}}\{xy - g^*(y)\}$, $x \in \mathbb{R}$, where $g^*(y) := \sup_{x\in\mathbb{R}}\{yx - g(x)\}$, $y \in \mathbb{R}$, is the convex conjugate of g. Therefore, for all $y \in \mathbb{R}$,

$$1 = \int e^{\varphi(x)-h(x)}\,dx = \int e^{g(x)-\alpha\frac{x^2}{2}}\,dx \geq e^{-g^*(y)}\int e^{xy-\alpha\frac{x^2}{2}}\,dx = e^{-g^*(y)}\sqrt{\frac{2\pi}{\alpha}}e^{\frac{y^2}{2\alpha}}.$$

So $g^*(y) \geq \frac{1}{2}\log\left(\frac{2\pi}{\alpha}\right) + \frac{y^2}{2\alpha}$, for all $y \in \mathbb{R}$. Therefore,

$$g(x) \leq \frac{1}{2}\log\left(\frac{\alpha}{2\pi}\right) + \sup_{y\in\mathbb{R}}\left\{xy - \frac{y^2}{2\alpha}\right\} = \frac{1}{2}\log\left(\frac{\alpha}{2\pi}\right) + \alpha\frac{x^2}{2},$$

which proves the claim. $\square$

2.5. **The Talagrand Conjecture for a class of diffusion, in dimension 1.** In this section we prove that for some class of potentials h, the associated diffusion semigroup satisfies the Talagrand Conjecture, in dimension 1.

Theorem 2.12. *Let μ_h be a probability measure on $\mathbb{R}$ of the form $d\mu_h(x) = e^{-h(x)}\,dx$ with $h\colon \mathbb{R} \to \mathbb{R}$ a symmetric function of class $\mathcal{D}^4$ such that $c \leq h'' \leq C$ where c, C are positive numbers. Set $V(x) := \frac{1}{2}(1-h'') - \frac{1}{4}(x^2 - h'^2)$. Assume V is bounded below, with $\sup_{x\geq 0} V''(x) < \infty$. Finally denote by $(\mathcal{P}_t)_{t\geq 0}$ the semi-group associated to the diffusion operator $L_h := \frac{\partial^2}{\partial x^2} - h'(x)\frac{\partial}{\partial x}$ symmetric in $\mathbb{L}^2(\mu_h)$.*

Then, for all $s > 0$, there exists a constant D (that depends only on s, c, C and $\sup_{x\geq 0} V''(x)$) such that for all non-negative $g \in \mathbb{L}^1(\mu_h)$

$$\mu_h(\{\mathcal{P}_s g \geq t\int g d\mu_h\}) \leq D\frac{1}{t\sqrt{\log t}} \qquad \forall t \geq 2.$$

Example 2.13. *As an example of application, one can consider $h(x) = \frac{x^2}{2} + (1+x^2)^{p/2}$, with $p \leq 2$ which satisfies the assumption of the Theorem. Note that this example corresponds to an unbounded perturbation of the Gaussian potential.*

Many bounded perturbations of the Gaussian potential also enter the framework of the above theorem. However, due to the assumption V bounded below, even apparently very tiny perturbation of the Gaussian potential does not enter the framework of the theorem, as for example $h(x) = \frac{x^2}{2} + \cos(x)$! We believe that the reason is technical and that the Talagrand's conjecture should hold also in this case.

Proof. Fix $g \in \mathbb{L}^2(\mu_h)$ positive and $s > 0$. Thanks to Proposition 2.8,

$$(\log \mathcal{P}_s g)'' \geq -c_s^2 - \frac{1}{2}(1-c) - \frac{1}{2}\|V''\|_\infty \geq -\beta$$

with $\beta := \max(0, c_s^2 + \frac{1}{2}(1-c) + \frac{1}{2}\|V''\|_\infty) \geq 0$. Therefore, by Theorem 2.9 applied to $f = \mathcal{P}_s g$, one can conclude that, for all $t \geq 2$,

$$\mu_h(\{\mathcal{P}_s g \geq t \int g \, d\mu_h\}) \leq \frac{C+\beta}{c} \frac{1}{t\sqrt{\log t}}$$

which is the desired conclusion for $g \in \mathbb{L}^2(\mu_h)$. Applying the previous bound to $g \wedge n$, $n \geq 1$, for non-negative $g \in \mathbb{L}^1(\mu_h)$ and letting $n \to \infty$ completes the proof. □

3. The $M/M/\infty$ semigroup

In this section we deal with the $M/M/\infty$ queuing process, which is a discrete analogue of the Ornstein-Uhlenbeck process on the integers $\mathbb{N} := \{0, 1 \dots\}$. First we obtain lower bounds of $\Delta \log P_t f$, where Δ is the discrete Laplacian. Then, we investigate the deviation property of semi-log-convex functions and prove that such a property, contrary to the continuous setting, does not hold unless the function is log-convex. In the last subsection, we prove that the Talagrand Conjecture holds by means of the strategy of the supremum presented in the introduction. We start with the notation.

3.1. Notation and setting.

In all what follows, we will deal with the following classical probability distributions on $\mathbb{N}$:

- $\mathcal{B}(n,p)$ stands for the binomial probability measure of parameters $n \in \mathbb{N}$ and $p \in [0,1)$, with the convention that $\mathcal{B}(n,0) = \delta_0$ (the Dirac mass at 0) and $\mathcal{B}(n,1) = \delta_n$. When $n = 1$, we simply denote by $\mathcal{B}(p)$ the Bernoulli distribution of parameter p.
- $\mathcal{P}(\theta)$ stands for the Poisson probability measure of intensity θ whose probability distribution function will be denoted by π_θ and is given by $\pi_\theta(k) = e^{-\theta}\theta^k/k!$, $k \in \mathbb{N}$. At some points, we will make a slight abuse of notation and write $\pi_\theta(A) = \sum_{a \in A} \pi_\theta(a)$, for $A \subset \mathbb{N}$.

The $M/M/\infty$ queuing process is defined through its infinitesimal generator L, acting on functions on the integers as

$$Lf(n) := n\mu\,[f(n-1) - f(n)] + \lambda\,[f(n+1) - f(n)]\,, \quad n \in \mathbb{N} \tag{3.1}$$

where $\lambda, \mu > 0$ are fixed parameters. In the above expression, there is no need to define $f(-1)$ since it is multiplied by 0. We use the following notation for the discrete derivative:

$$Df(n) := f(n+1) - f(n), \qquad n \in \mathbb{N},$$

and for the discrete second order derivative (Laplacian):

$$\Delta f(n) := f(n+1) + f(n-1) - 2f(n) = D(Df)(n-1), \qquad n \in \mathbb{N} \setminus \{0\}. \tag{3.2}$$

Then $Lf(n) = \lambda \Delta f(n) + (n\mu - \lambda) Df(n-1)$.

Denote by $(X_t)_{t \geq 0}$ the Markov (jump) process associated to L, so that for all (say) bounded function f it holds $P_t f(n) = \mathbb{E}(f(X_t)|X_0 = n)$, $n \in \mathbb{N}$. A remarkable feature of the $M/M/\infty$ queuing process is that

$$\mathcal{L}(X_t|X_0 = n) = \mathcal{B}(n, p(t)) \star \mathcal{P}(\rho q(t))$$

where $\star$ stands for the convolution,

$$p(t) := e^{-\mu t}, \quad q(t) = 1 - p(t), \quad \rho = \frac{\lambda}{\mu}.$$

In other words

$$P_t f(n) = \mathbb{E}(f(Y_t + Z_t))$$

with $Y_t \sim \mathcal{B}(n, p(t))$ independent of $Z_t \sim \mathcal{P}(\rho q(t))$ which can be seen as an analogue of the Mehler Formula (1.2) for the Ornstein-Uhlenbeck semigroup.

Finally, we recall that the $M/M/\infty$ queuing process is reversible with respect to the Poisson measure $\mathcal{P}(\rho)$.

In the next section we deal with estimates on the Laplacian of $\log P_t f$.

3.2. **Semi-log-convexity of the queuing process.** In this section we investigate the behavior of $\Delta \log P_t f$. The main result of the section (Proposotion 3.1) is that for any starting function f on the integers, as for the Ornstein-Uhlenbeck semigroup, $\Delta \log P_t f$ is bounded below by some universal constant depending only on t and on the parameters of the process (namely λ and μ), but not on f.

Proposition 3.1. *Let $f\colon \mathbb{N} \to \mathbb{R}^+$ not identically vanishing. Then for all $t > 0$,*

$$\Delta \log P_t f(n) \geq \log\left(\frac{1}{12}\left(1 - \frac{p^2}{(p+\rho(1-p)^2)^2}\right)\right) \qquad n = 1, 2 \dots \tag{3.3}$$

with $p = p(t) = e^{-\mu t}$ and $\rho = \lambda/\mu$.

Remark 3.2. *Notice the right hand side of (3.3) tends to $-\infty$ when $t \to 0^+$, as it should be, since f can be any function. On the other hand, the right hand side of (3.3) tends to $-\log(12)$ as t tends to ∞. This comes from the technicality of the proof, we believe however that there should exist a lower bound on $\Delta \log P_t f(n)$ that tends to 0 as t tends to infinity.*

The proof of Proposition 3.1 relies on the following lemma which asserts that a positive combination of log-convex (or more generally semi-log-convex) functions is log-convex (semi-log-convex).

Lemma 3.3. *Let $f_i\colon \mathbb{N} \to (0, \infty)$, $i = 1, \dots, N$, be a family of positive functions, with N possibly infinite. Assume that for all i and all $n = 1, \dots,$ $\Delta \log f_i(n) \geq -\beta_i$ for some $\beta_i \in \mathbb{R}$. Then, for all $\alpha_1, \dots, \alpha_N > 0$,*

$$\Delta \log\left(\sum_{i=1}^{N} \alpha_i f_i\right) \geq -\max_{1 \leq i \leq N} \beta_i.$$

The continuous counterpart of this result is classical and could be used to prove this discrete statement. For the sake of completeness we give below a direct proof.

Proof of Lemma 3.3. By induction, and possibly taking the limit, it suffices to prove the result for $N = 2$. Moreover, by homogeneity we can assume without loss of generality that $\alpha_1 = \alpha_2 = 1$. So let $f, g\colon \mathbb{N} \to (0, \infty)$ be two positive functions with $\Delta \log f \geq -\beta_f$ and $\Delta \log g \geq -\beta_g$ with $\beta_f, \beta_g \in \mathbb{R}$. Then, by definition

$$\begin{aligned}
\exp\{\Delta \log(f+g)(n)\} &= \frac{(f(n+1)+g(n+1))(f(n-1)+g(n-1))}{(f(n)+g(n))^2} \\
&= \frac{f(n+1)f(n-1)}{f(n)^2}\frac{f(n)^2}{(f(n)+g(n))^2} + \frac{g(n+1)g(n-1)}{g(n)^2}\frac{g(n)^2}{(f(n)+g(n))^2} \\
&\quad + \frac{f(n+1)g(n-1)}{f(n)g(n)}\frac{f(n)g(n)}{(f(n)+g(n))^2} + \frac{f(n-1)g(n+1)}{f(n)g(n)}\frac{f(n)g(n)}{(f(n)+g(n))^2} \\
&\geq e^{-\beta_f}\frac{f(n)^2}{(f(n)+g(n))^2} + e^{-\beta_g}\frac{g(n)^2}{(f(n)+g(n))^2} \\
&\quad + \left(e^{-\beta_f-\beta_g}\frac{f(n)g(n)}{f(n-1)g(n+1)} + \frac{f(n-1)g(n+1)}{f(n)g(n)}\right)\frac{f(n)g(n)}{(f(n)+g(n))^2}
\end{aligned}$$

where we used the hypotheses $\Delta \log f \geq -\beta_f$ and $\Delta \log g \geq -\beta_g$ on the one hand to bound the first two terms, and on the other hand to write $\frac{f(n+1)}{f(n)} \geq e^{-\beta_f} \frac{f(n)}{f(n-1)}$ and $\frac{g(n-1)}{g(n)} \geq e^{-\beta_g} \frac{g(n)}{g(n+1)}$. Next we observe that

$$e^{-\beta_f-\beta_g} \frac{f(n)g(n)}{f(n-1)g(n+1)} + \frac{f(n-1)g(n+1)}{f(n)g(n)} \geq \min_{x>0} \left(xe^{-\beta_f-\beta_g} + \frac{1}{x} \right) = 2e^{-(\beta_f+\beta_g)/2}.$$

It follows that

$$\begin{aligned}\exp\{\Delta \log(f+g)(n)\} &\geq \frac{(e^{-\beta_f/2}f(n))^2}{(f(n)+g(n))^2} + \frac{(e^{-\beta_g/2}g(n))^2}{(f(n)+g(n))^2} + \frac{2f(n)e^{-\beta_f/2}g(n)e^{-\beta_g/2}}{(f(n)+g(n))^2} \\ &= \frac{(f(n)e^{-\beta_f/2}+g(n)e^{-\beta_g/2})^2}{(f(n)+g(n))^2} \geq \min\left(e^{-\beta_f}, e^{-\beta_g}\right).\end{aligned}$$

This leads to the expected result. □

Proof of Proposition 3.1. Fix $t > 0$; we have

$$P_t f(n) = \mathbb{E}(f(X_t)|X_0 = n) = \sum_{k=0}^{\infty} f(k)\mathbb{P}(X_t = k|X_0 = n), \qquad \forall n \in \mathbb{N}.$$

For all $k \in \mathbb{N}$, denote by $F_k(n) = \mathbb{P}(X_t = k|X_0 = n)$, $n \in \mathbb{N}$. According to Lemma 3.3, it is enough to show that for all $k \in \mathbb{N}$, it holds

$$\Delta \log F_k(n) \geq \log\left(\frac{1}{12}\left(1 - \frac{p^2}{(p+\rho(1-p)^2)^2}\right)\right), \qquad \forall n \geq 1. \tag{3.4}$$

Since P_t is reversible with respect to $\mathcal{P}(\rho)$, it holds

$$\mathbb{P}(X_t = k|X_0 = n) = \pi_\rho(k)\frac{\mathbb{P}(X_t = n|X_0 = k)}{\pi_\rho(n)}, \qquad n \in \mathbb{N}.$$

Therefore,

$$\log F_k(n) = \log \pi_\rho(k) - \log \pi_\rho(n) + \log G_k(n),$$

where $G_k(n) = \mathbb{P}(Y_t + Z_t = n)$, with as above, $Y_t \sim \mathcal{B}(k,p)$ and $Z_t \sim \mathcal{P}(\rho(1-p))$. A simple calculation shows that, for any parameter $\theta > 0$, it holds for all $n \geq 1$

$$\Delta \log \pi_\theta(n) = \log\left(\frac{\pi_\theta(n+1)\pi_\theta(n-1)}{\pi_\theta(n)^2}\right) = \log \frac{(n!)^2}{(n+1)!(n-1)!} = \log \frac{n}{n+1}.$$

From this follows that $\Delta \log F_0(n) = 0$, $n \geq 1$, and that for $k \geq 1$, $\Delta \log F_k(n) \geq \Delta \log G_k(n)$, $n \geq 1$. So it is enough to show that the bound (3.4) is satisfied by G_k.

Let us first treat the case $k = 1$ and show the following slightly better lower bound:

$$\Delta \log G_1 \geq \log\left(\frac{1}{2}\left(1 - \frac{p^2}{(p+\rho(1-p)^2)^2}\right)\right) := -\alpha$$

or equivalently

$$G_1(n)^2 \leq e^\alpha G_1(n+1)G_1(n-1), \qquad \forall n \geq 1. \tag{3.5}$$

For all $n \geq 0$, it holds

$$G_1(n) = \left((1-p) + p\frac{n}{\rho(1-p)}\right)\frac{(\rho(1-p))^n}{n!}e^{-\rho(1-p)}, \qquad n \geq 1.$$

So, for $n \geq 1$,

$$\begin{aligned}\frac{G_1(n+1)G_1(n-1)}{G_1(n)^2} &= \frac{n}{n+1}\frac{\left((1-p)+p\frac{n+1}{\rho(1-p)}\right)\left((1-p)+p\frac{n-1}{\rho(1-p)}\right)}{\left((1-p)+p\frac{n}{\rho(1-p)}\right)^2} \\ &= \frac{n}{n+1}\frac{\left((1-p)+p\frac{n}{\rho(1-p)}\right)^2-\left(\frac{p}{\rho(1-p)}\right)^2}{\left((1-p)+p\frac{n}{\rho(1-p)}\right)^2} \\ &\geq \frac{1}{2}\left(1-\frac{p^2}{\left(\rho(1-p)^2+p\right)^2}\right)\end{aligned}$$

and so taking the log gives the announced lower bound for $\Delta \log G_1$.

Remark 3.4. *Note that one could be more accurate by keeping the $\frac{n}{n+1}$ factor which eventually yields to the bound*

$$\Delta \log F_1(n) \geq \log\left(1-\frac{p^2}{\left(\rho(1-p)^2+p\right)^2}\right), \qquad n \geq 1.$$

Now let us treat the case $k \geq 2$. It will be convenient to write $Y_t = Y_t' + \varepsilon_t$ with $Y_t' \sim \mathcal{B}(k-1,p)$ and $\varepsilon_t \sim \mathcal{B}(p)$ two independent random variables also independent of Z_t. Conditioning with respect to $Z_t + \varepsilon_t$ and using (3.5), we get

(3.6)

$$\begin{aligned}G_k(n) &= \sum_{j=0}^{n}\mathbb{P}(Y_t'=j)G_1(n-j) \\ &\leq \mathbb{P}(Y_t'=n)G_1(0)+e^{\alpha/2}\sum_{j=0}^{n-1}\mathbb{P}(Y_t'=j)G_1(n+1-j)^{1/2}G_1(n-1-j)^{1/2} \\ &\leq \mathbb{P}(Y_t'=n)G_1(0)+e^{\alpha/2}\left(\sum_{j=0}^{n-1}\mathbb{P}(Y_t'=j)G_1(n+1-j)\right)^{1/2}\left(\sum_{j=0}^{n-1}\mathbb{P}(Y_t'=j)G_1(n-1-j)\right)^{1/2} \\ &= \mathbb{P}(Y_t'=n)G_1(0)+e^{\alpha/2}\left(\sum_{j=0}^{n-1}\mathbb{P}(Y_t'=j)G_1(n+1-j)\right)^{1/2} G_k(n-1)^{1/2}\end{aligned}$$

Now let us treat separately the cases :

$$\text{(a) } n \geq k \geq 2, \qquad \text{(b) } 1 \leq n \leq k-2, k \geq 3 \qquad \text{(c) } n = k-1, k \geq 2.$$

(a) Suppose $n \geq k \geq 2$, then $\mathbb{P}(Y_t' = n) = 0$ and so (3.6) yields to

$$G_k(n) \leq e^{\alpha/2}G_k(n+1)^{1/2}G_k(n-1)^{1/2}.$$

(b) Fix $k \geq 3$. Let us admit for a moment that there exists $\beta > 0$ (independent of k) such that for all $1 \leq n \leq k-2$,

$$\mathbb{P}(Y_t'=n) \leq e^{\beta/2}\mathbb{P}(Y_t'=n-1)^{1/2}\mathbb{P}(Y_t'=n+1)^{1/2}, \qquad \forall 1 \leq n \leq k-2. \tag{3.7}$$

As we will see below, the optimal β is $\log 3$. If $1 \leq n \leq k-2$, then inserting (3.7) into (3.6) gives

$$\begin{aligned}
G_k(n) &\leq e^{\beta/2} \left(\mathbb{P}(Y_t' = n+1) G_1(0)\right)^{1/2} \left(\mathbb{P}(Y_t' = n-1) G_1(0)\right)^{1/2} \\
&\quad + e^{\alpha/2} \left(\sum_{j=0}^{n-1} \mathbb{P}(Y_t' = j) G_1(n+1-j) \right)^{1/2} G_k(n-1)^{1/2} \\
&\leq e^{\max(\alpha;\beta)/2} \left[\left(\mathbb{P}(Y_t' = n+1) G_1(0)\right)^{1/2} + \left(\sum_{j=0}^{n-1} \mathbb{P}(Y_t' = j) G_1(n+1-j) \right)^{1/2} \right] G_k(n-1)^{1/2} \\
&\leq \sqrt{2} e^{\max(\alpha;\beta)/2} G_k(n+1)^{1/2} G_k(n-1)^{1/2},
\end{aligned}$$

where the second inequality comes from $\mathbb{P}(Y_t' = n-1)G_1(0) \leq G_k(n-1)$ and the third inequality follows from $\sqrt{a} + \sqrt{b} \leq \sqrt{2}\sqrt{a+b}$, $a, b \geq 0$. To determine β in (3.7) note that

$$\binom{k-1}{n} p^n (1-p)^{k-1-n} \leq e^{\beta/2} \left(\binom{k-1}{n-1} p^{n-1} (1-p)^{k-n} \right)^{1/2} \left(\binom{k-1}{n+1} p^{n+1} (1-p)^{k-n-2} \right)^{1/2}$$

is equivalent to

$$\frac{1}{(n!(k-n-1)!)^2} \leq e^{\beta} \frac{1}{(n-1)!(k-n)!} \frac{1}{(n+1)!(k-n-2)!}$$

which is equivalent to

$$\frac{n+1}{n} \leq e^{\beta} \frac{k-n-1}{k-n}, \qquad \forall 1 \leq n \leq k-2.$$

Observe that

$$\frac{(n+1)(k-n)}{n(k-n-1)} = 1 + \frac{k}{n(k-1) - n^2}.$$

The minimal value of the function $n \mapsto n(k-1) - n^2$ on $\{1, \ldots, k-2\}$ is $k-2$ (reached at 1 and $k-2$). So $\max_{1 \leq n \leq k-2} \frac{(n+1)(k-n)}{n(k-n-1)} = 1 + \frac{k}{k-2} = 2 + \frac{1}{k-2} \leq 3$. Therefore, one can take $\beta = \log 3$.

(c) Finally, let us assume that $k \geq 2$ and $n = k-1$. Let us admit for a moment that

$$\mathbb{P}(Y_t' = k-1) G_1(0) \leq \left(\mathbb{P}(Y_t' = k-1) G_1(1)\right)^{1/2} \left(\mathbb{P}(Y_t' = k-2) G_1(0)\right)^{1/2}. \tag{3.8}$$

Then, inserting (3.8) into (3.6), and reasoning exactly as in the case (b) gives

$$G_k(k-1) \leq \sqrt{2} e^{\alpha/2} G_k(k)^{1/2} G_k(k-2)^{1/2}.$$

To prove (3.8), first observe that $\mathbb{P}(Y_t' = k-1) = p^{k-1}$, $\mathbb{P}(Y_t' = k-2) = (k-1)p^{k-2}(1-p)$ and so $\mathbb{P}(Y_t' = k-1) \leq \frac{p}{1-p} \mathbb{P}(Y_t' = k-2)$. Since, $G_1(0) = (1-p)e^{-\rho(1-p)}$ and $G_1(1) = \left((1-p) + p\frac{1}{\rho(1-p)}\right)(\rho(1-p))e^{-\rho(1-p)}$, we see that $G_1(0) = \frac{1}{(1-p)\rho + \frac{p}{1-p}} G_1(1)$. Therefore,

$$\mathbb{P}(Y_t' = k-1) G_1(0) \leq \frac{p}{(1-p)^2 \rho + p} \mathbb{P}(Y_t' = k-2) G_1(1) \leq \mathbb{P}(Y_t' = k-2) G_1(1)$$

which gives (3.8).

Putting everything together, one gets for all $k \geq 0$ and $n \geq 1$,

$$\Delta \log G_k(n) \geq -\max(\alpha;\beta) - \log 2 \geq -\alpha - \beta - \log 2 = \log \left(\frac{1}{12} \left(1 - \frac{p^2}{(\rho(1-p)^2 + p)^2} \right) \right)$$

which completes the proof. $\square$

3.3. **Remarks on the action of the $M/M/\infty$ semigroup on structured functions.** In this section, we collect some more facts about the action of P_t on log-convex (resp. log-concave) functions. The first statement, which is a simple application of Cauchy-Schwarz inequality, asserts that if f is log-semi-convex, then so is $P_t f$. The second statement is due to Johnson [22] and shows that P_t also leaves stable the class of log-concave functions.

Proposition 3.5. *Let f be a positive function on $\mathbb{N}$ such that, for some $\beta \geq 0$ and all $n = 1, 2 \dots,$ $\Delta \log f(n) \geq -\beta$. Then*

$$\Delta \log P_t f(n) \geq -\beta \qquad n = 1, 2, \dots, \quad t \geq 0.$$

Proof. Recall that $P_t f(n) = \mathbb{E}(f(\varepsilon_1 + \cdots + \varepsilon_n + Y))$ with $\varepsilon_i \sim \mathcal{B}(p)$ i.i.d. and independent of $Z \sim \mathcal{P}(\rho q)$, $q = 1 - p$, and similarly for $P_t f(n-1)$ and $P_t f(n+1)$. Hence, computing the expectation with respect to the Bernoulli random variables ε_n and ε_{n+1} respectively, we have

$$\begin{aligned} P_t f(n+1) &= p^2 \mathbb{E}(f(\varepsilon_1 + \cdots + \varepsilon_{n-1} + Z + 2)) + 2p(1-p)\mathbb{E}(f(\varepsilon_1 + \cdots + \varepsilon_{n-1} + Z + 1)) \\ &\quad + (1-p)^2 \mathbb{E}(f(\varepsilon_1 + \cdots + \varepsilon_{n-1} + Z)) \end{aligned}$$

and

$$P_t f(n) = p\mathbb{E}(f(\varepsilon_1 + \cdots + \varepsilon_{n-1} + Z + 1)) + (1-p)\mathbb{E}(f(\varepsilon_1 + \cdots + \varepsilon_{n-1} + Z)).$$

Letting $X := \varepsilon_1 + \cdots + \varepsilon_{n-1} + Z$, we get

$$\begin{aligned} \Delta \log P_t f(n) &= \log \left(\frac{P_t f(n+1) P_t f(n-1)}{P_t f(n)^2} \right) \\ &= \log \left(\frac{p^2 \mathbb{E}(f(X+2))\mathbb{E}(f(X)) + 2pq\mathbb{E}(f(X+1))\mathbb{E}(f(X)) + q^2 \mathbb{E}(f(X))^2}{p^2 \mathbb{E}(f(X+1))^2 + 2pq\mathbb{E}(f(X+1))\mathbb{E}(f(X)) + q^2 \mathbb{E}(f(X))^2} \right). \end{aligned}$$

Now, since $\Delta \log f \geq -\beta$, we infer that $e^{-\beta/2} f(n) \leq \sqrt{f(n+1)f(n-1)}$. Therefore, using the Cauchy-Schwarz Inequality,

$$e^{-\beta} \mathbb{E}(f(X+1))^2 \leq \mathbb{E}(\sqrt{f(X+2)f(X)})^2 \leq \mathbb{E}(f(X+2))\mathbb{E}(f(X)).$$

Hence

$$\begin{aligned} &p^2 \mathbb{E}(f(X+2))\mathbb{E}(f(X)) + 2pq\mathbb{E}(f(X+1))\mathbb{E}(f(X)) + q^2 \mathbb{E}(f(X))^2 \\ &\qquad \geq p^2 e^{-\beta} \mathbb{E}(f(X+1))^2 + 2pq\mathbb{E}(f(X+1))\mathbb{E}(f(X)) + q^2 \mathbb{E}(f(X))^2 \\ &\qquad \geq e^{-\beta} \left(p^2 \mathbb{E}(f(X+1))^2 + 2pq\mathbb{E}(f(X+1))\mathbb{E}(f(X)) + q^2 \mathbb{E}(f(X))^2 \right) \end{aligned}$$

which leads to the desired result. □

Recall that a function $f : \mathbb{N} \to (0, +\infty)$ is said log-concave if $\Delta \log f(n) \leq 0$, for all $n \geq 1$, or in other words if

$$f(n)^2 \geq f(n-1)f(n+1), \qquad \forall n \geq 1.$$

It is said ultra-log concave if $n \mapsto n! f(n)$ is log-concave, or equivalently

$$f(n)^2 \geq \frac{n+1}{n} f(n-1)f(n+1), \qquad \forall n \geq 1.$$

It is easily checked that f is ultra-log-concave if f/π_θ is log-concave for some (and thus all) $\theta > 0$.

The following result is due to Johnson [22].

Theorem 3.6. *Let $(X_t)_{t \geq 0}$ be the $M/M/\infty$ process with generator (3.1), associated semigroup $(P_t)_{t \geq 0}$ and reversible distribution π_ρ. For all $t \geq 0$, denote by h_t the distribution function of the law of X_t. If h_0 is ultra-log-concave, then for all $t > 0$, h_t is also ultra-log-concave. Equivalently, if $f_0 : \mathbb{N} \to (0, +\infty)$ is log-concave and integrable with respect to π_ρ then, for all $t > 0$, $f_t = P_t f$ is also log-concave.*

We note that the preservation of ultra-log-concavity by the $M/M/\infty$ process was proved in [22] en route to proving the maximum entropy property of the Poisson distribution; related properties connected to Poisson and compound Poisson approximation may also be found in [23, 5]. For the sake of completeness, we briefly sketch Johnson's proof (see [22] for details).

Sketch of proof. Fix some $t > 0$. The proof relies on the following explicit representation of X_t:

$$X_t = \sum_{k=1}^{X_0} \varepsilon_k + Z,$$

where, as in Section 3.1, the random variables X_0, Z, ε_k, $k \geq 1$, are independent, Z has law $\mathcal{P}(\rho(1-p))$ and ε_k has law $\mathcal{B}(p)$, $k \geq 1$, with $p = p(t) = e^{-\mu t}$. According to [22, Proposition 3.7], the random variable $\sum_{k=1}^{X_0} \varepsilon_k$ (which corresponds to a thinning of X_0) has an ultra-log-concave distribution. On the other hand, it is easily checked that Z has also an ultra-log-concave distribution. Since the class of ultra-log-concave functions is closed under convolution [38, 32], we conclude that the distribution function of X_t is ultra-log-concave. Finally, observe that if $f_0 \in \mathbb{L}^1(\pi_\rho)$ is a log-concave function such that (without loss of generality) $\int f_0 \, d\pi_\rho = 1$ and X_0 has distribution function $h_0 = f_0 \pi_\rho$, then h_0 is obviously ultra-log-concave and so, according to what precedes, the distribution function h_t of X_t is also ultra-log-concave. Since P_t is reversible with respect to π_ρ, it holds $h_t = (P_t f_0)\pi_\rho$. And so $f_t = P_t f_0$ is log-concave. □

3.4. **Deviation bounds for semi-log-convex functions.** In this section, we investigate deviation bounds of the type $\pi_\theta(\{n : f(n) \geq t \int f \, d\pi_\theta\})$ for log-convex, and more generally log-semi-convex, functions f. In other words, we address the analogue of Item (2) from the introduction for the Poisson distributions. As our results will reveal, in this discrete setting, an analogue of Item (2) does not hold in general, but it does hold if f is assumed to be log-convex. One reason for this spurious effect is that the tail of the measure $\sum_{k \geq n} \pi_\theta(k)$, in discrete, is of the same order as $\pi_\theta(n)$, *i.e.*, with no extra factor, while in the continuous, $\int_s^\infty e^{-t^2} dt \sim_{s \to \infty} \frac{e^{-s^2}}{2s}$.

In all what follows, will make a frequent use of a non asymptotic version of Stirling formula. More precisely, the following inequalities for the factorial are known (see [35]) to hold

$$\sqrt{2\pi} n^{n+\frac{1}{2}} e^{-n+\frac{1}{12n+1}} < n! < \sqrt{2\pi} n^{n+\frac{1}{2}} e^{-n+\frac{1}{12n}}, \qquad n \geq 1.$$

Hence,

$$n^{n+\frac{1}{2}} e^{-n} \leq n! \leq 3 n^{n+\frac{1}{2}} e^{-n} \tag{3.9}$$

for $n \geq 1$ (since $\sqrt{2\pi} e^{1/12n} \leq 3$).

Let us begin with a precise tail bound for the Poisson distributions.

Lemma 3.7. *Let $\theta > 0$ and define $\Phi_\theta(x) := x \log x - x \log \theta - x + \theta$, $x \geq 1$. Set $F_\theta(u) := \pi_\theta([u, \infty))$ for the tail of the distribution function of π_θ. For $u \geq 2\theta$, we have*

$$F_\theta(u) \leq \frac{2}{\sqrt{u}} \exp\left\{-\Phi_\theta(u)\right\}.$$

Proof. If $u \geq 2\theta$,

$$\begin{aligned} F_\theta(u) &= \sum_{k \geq u} \frac{\theta^k e^{-\theta}}{k!} = \frac{\theta^u e^{-\theta}}{u!} \sum_{k \geq u} \frac{\theta^{k-u}}{k(k-1)\dots(u+1)} \\ &\leq \frac{\theta^u e^{-\theta}}{u!} \sum_{k \geq u} 2^{u-k} = 2\frac{\theta^u e^{-\theta}}{u!} \leq \frac{2}{\sqrt{u}} \exp\left\{-\Phi(u)\right\} \end{aligned}$$

where we used (3.9). □

Proposition 3.8. *For any $\theta > 0$, there exists a constant c that depends only on θ such that for all $t \geq 4$ and all positive functions f on the integers satisfying $\Delta \log f \geq 0$, we have*

$$\pi_\theta(\{f \geq t \int f d\pi_\theta\}) \leq c \frac{\sqrt{\log \log t}}{t\sqrt{\log t}}.$$

Proof. We assume without loss of generality that $\int f d\pi_\theta = 1$ and we follow [17]. Define $\widetilde{f}\colon [0,\infty) \to (0,\infty)$ as the piecewise linear interpolation of f. Since $\Delta \log f \geq 0$, $\log \widetilde{f}$ is convex so that $\log \widetilde{f}(x) = \sup_{y\geq 0}\{xy - \widetilde{g}(y)\}$, $x \geq 0$, where $\widetilde{g}(y) = (\log \widetilde{f})^*(y) = \sup_{x\geq 0}\{yx - \log \widetilde{f}(x)\}$, $y \geq 0$, is the Legendre transform of $\log \widetilde{f}$. Then, since, for any $n \in \mathbb{N}$ and any $y \geq 0$, $\log f(n) = \log \widetilde{f}(n) \geq ny - \widetilde{g}(y)$, we have

$$1 = \int f d\pi_\theta \geq e^{-\widetilde{g}(y)} \int e^{ny} d\pi_\theta(n) = \exp\{-\widetilde{g}(y) + \theta(e^y - 1)\}.$$

Therefore

$$\widetilde{g}(y) \geq \theta(e^y - 1), \qquad y \geq 0,$$

and in turn

$$\begin{aligned} \log f(n) = \log \widetilde{f}(n) \leq \sup_{y\geq 0}\{ny - \theta(e^y - 1)\} &= \begin{cases} n(\log n - \log\theta) - n + \theta & \text{if } n \geq \theta \\ 0 & \text{if } n < \theta. \end{cases} \\ &= \max[n(\log n - \log\theta) - n + \theta, 0]. \end{aligned}$$

Hence, for $t \geq e^{\theta-1}/\theta$,

$$\begin{aligned} \pi_\theta(\{f \geq t\}) &\leq \pi_\theta(\{\max(n(\log n - \log\theta) - n + \theta, 0) \geq \log t\}) \\ &= \pi_\theta(\{n(\log n - \log\theta) - n + \theta \geq \log t\}) = \pi_\theta(\{\Phi_\theta(n) \geq \log t\}) \\ &= \pi_\theta(\{n \in \mathbb{N} : n \geq \Phi_\theta^{-1}(\log t)\}) \end{aligned}$$

where we set $\Phi_\theta(x) := x\log x - x\log\theta - x + \theta$, $x \geq 1$ and denoted by Φ_θ^{-1} its inverse function which is increasing on $[\theta - 1 - \log\theta, \infty)$. Using Lemma 3.7, we get for $t \geq c_\theta$, for some constant depending only on θ,

$$\pi_\theta(\{n \in \mathbb{N} : f(n) \geq t\}) \leq 2\frac{e^{-\Phi_\theta(\Phi_\theta^{-1}(\log t))}}{\sqrt{\Phi_\theta^{-1}(\log t)}} = \frac{2}{t\sqrt{\Phi_\theta^{-1}(\log t)}}.$$

To end the proof it suffices to observe that $\Phi_\theta(x/\log x) = x - x[\log\log x + \log\theta + 1]/\log x + \theta \leq x$ for x large enough so that $\Phi_\theta^{-1}(x) \geq x/\log x$ (for x large). $\square$

Remark 3.9. *Let us note that the bound in Proposition 3.8 is of optimal order. Indeed, consider the function $f_\lambda(n) = e^{\lambda n}c(\lambda)$, where $c(\lambda) = \exp\{1 - e^\lambda\}$, $\lambda \geq 0$, is taken to be the normalizing constant that makes $\int f_\lambda d\pi_1 = 1$. Observe that $\Delta \log f_\lambda = 0$ since $\log f_\lambda$ is linear. Now*

$$\pi_1(\{f_\lambda \geq t\}) = \pi_1\left(\left[\frac{1}{\lambda}\log\left(\frac{t}{c(\lambda)}\right), \infty\right)\right).$$

We are interested in lower bounds on this Poisson tail. Let us take $\lambda = \log k$ and $t = ek^k e^{-k}$, for some integer k, so that $\frac{1}{\lambda}\log\left(\frac{t}{c(\lambda)}\right) = k$. Observe that, using (3.9),

$$\pi_1([k, +\infty)) \geq \frac{1}{ek!} \geq \frac{1}{3e}k^{-k-\frac{1}{2}}e^k.$$

Therefore, after some calculations, we get

$$\frac{t\sqrt{\log t}}{\sqrt{\log\log t}}\pi_1(\{f_\lambda \geq t\}) \geq \frac{1}{3}\left(\frac{1 + k\log k - k}{k\log(1 + k\log k - k)}\right)^{1/2}$$

and the right hand side goes to $1/3$ as $k \to \infty$, which proves optimality.

The next proposition goes in the opposite direction to Proposition 3.8. It states that the log-semi-convex property is not enough to ensure a deviation bound better than just Markov's inequality. In what follows, $\theta > 0$ is fixed and we define for all $\beta \geq 0$

$$\mathcal{F}_\beta := \{f \colon \mathbb{N} \to \mathbb{R} \text{ such that } \Delta \log f \geq -\beta \text{ and } \int f d\pi_\theta = 1\}.$$

Proposition 3.10. *For all $\beta > 0$, the following holds*

$$\limsup_{t\to\infty} t \sup_{f\in\mathcal{F}_\beta} \pi_\theta(\{n : f(n) \geq t\}) > 0.$$

Proof. For $a \geq 0$, define f_a as

$$f_a(n) = \exp\{-\frac{\beta}{2}(n-a)^2 + Z(a)\}, \quad n \in \mathbb{N}, \text{ with } Z(a) := -\log \int \exp\{-\frac{\beta}{2}(n-a)^2\} d\pi_\theta(n)$$

so that $\int f_a d\pi_\theta = 1$. Moreover

$$\Delta \log f(n) = -\frac{\beta}{2}\left((n+1-a)^2 + (n-1-a)^2 - 2(n-a)^2\right) = -\beta.$$

Hence, for all $a \geq 0$, $f_a \in \mathcal{F}_\beta$. The expected result will follow if we are able to prove that there exists $T \colon [0,\infty) \to \mathbb{R}^+$ with $T(a) \to \infty$ as $a \to \infty$ such that

$$\limsup_{a\to\infty} T(a)\pi_\theta(\{f_a \geq T(a)\}) > 0 \tag{3.10}$$

since clearly $\limsup_{a\to\infty} T(a)\pi_\theta(\{f_a \geq T(a)\}) \leq \limsup_{t\to\infty} t \sup_{f\in\mathcal{F}_\beta} \pi_\theta(\{f \geq t\})$.

Set $\Psi_a \colon \mathbb{R}^+ \to \mathbb{R}$, $u \mapsto -\frac{\beta}{2}(u-a)^2 - \log\Gamma(u+1) + u\log\theta - \theta$ where $\Gamma(z) := \int_0^\infty t^{z-1}e^{-t}dt$, $z > 0$, is the Gamma functional. It is well known that $\log\Gamma$ is convex on $(0,\infty)$ so that Ψ_a is strictly concave on $\mathbb{R}^+$. Since $\lim_{u\to\infty}\Psi_a(u) = -\infty$, this guarantees that Ψ_a has a unique maximum on $\mathbb{R}^+$ achieved at a (unique) point we denote by $u_a \in [0,\infty)$.

We claim that $\mathcal{A} := \{a \geq 1 \text{ such that } u_a \in \mathbb{N}\}$ is infinite and unbounded and $u_a \to +\infty$, as $a \in \mathcal{A}$ tends to $+\infty$. We postpone the proof of the claim and continue with the proof of (3.10).

Set, for $a \in \mathcal{A}$,

$$T(a) := \exp\left(-\frac{\beta}{2}(u_a - a)^2 + Z(a)\right).$$

Now we observe that

$$\pi_\theta(\{f_a \geq T(a)\}) = \pi_\theta\left(\left\{n : -\frac{\beta}{2}(n-a)^2 \geq -\frac{\beta}{2}(u_a-a)^2\right\}\right) \geq \pi_\theta(u_a) = \frac{\theta^{u_a}e^{-\theta}}{u_a!}.$$

Therefore, since $u_a! = \Gamma(u_a+1)$ for $a \in \mathcal{A}$,

$$T(a)\pi_\theta(\{f_a \geq T(a)\}) \geq \exp\{\log(T(a)) - \log(u_a!) + u_a\log\theta - \theta\} = \exp\{\Psi_a(u_a) + Z(a)\}.$$

Our aim is to bound from below the right hand side of the latter. We notice that, by definition of Ψ_a and since $n! = \Gamma(n+1)$,

$$\begin{aligned}\int \exp\{-\frac{\beta}{2}(n-a)^2\}d\pi_\theta(n) &= \sum_{n=0}^{\infty}\exp\{-\frac{\beta}{2}(n-a)^2 - \log(n!) + n\log\theta - \theta\} \\ &= \sum_{n=0}^{\infty}\exp\{\Psi_a(n)\}.\end{aligned}$$

Since $\Psi_a'' \leq -\beta$ and $\Psi_a'(u_a) = 0$ we have

$$\Psi_a(n) \leq \Psi_a(u_a) + \Psi_a'(u_a)(n-u_a) - \frac{\beta}{2}(n-u_a)^2 = \Psi_a(u_a) - \frac{\beta}{2}(n-u_a)^2.$$

Hence

$$\int \exp\{-\frac{\beta}{2}(n-a)^2\}d\pi_\theta(n) \leq e^{\Psi_a(u_a)}\sum_{n=0}^{\infty}\exp\{-\frac{\beta}{2}(n-u_a)^2\} \leq 2e^{\Psi_a(u_a)}\sum_{n=0}^{\infty}e^{-\beta n^2/2}.$$

Setting $c_\beta = -\log\left(2\sum_{n=0}^{\infty} e^{-\beta n^2/2}\right)$, one gets

$$Z(a) = -\log \int \exp\{-\frac{\beta}{2}(n-a)^2\}) d\pi_\theta(n) \geq c_\beta - \Psi_a(u_a). \tag{3.11}$$

The latter implies two useful conclusions. First, for all $a \in \mathcal{A}$,

$$T(a)\pi_\theta(\{f_a \geq T(a)\}) \geq \exp\{\Psi_a(u_a) + Z(a)\} \geq e^{c_\beta}.$$

Second, $T(a) = -\frac{\beta}{2}(u_a - a)^2 + Z(a) \geq c_\beta + \log(u_a!) - u_a \log\theta + \theta \to \infty$ as $a \in \mathcal{A}$ tends to infinity.

The desired conclusion follows as soon as we prove the claim above. The equation $\Psi_a'(u_a) = 0$ shows that the map $a \mapsto u_a$ is continuous. Hence the claim will follow if we can prove that $u_a \to \infty$ as a goes to infinity. We observe that $\Psi_a'(u) = -\beta(u-a) - \psi(u+1) + \log\theta$ where $\psi(u) := \Gamma'(u)/\Gamma(u)$ is the digamma function, which is increasing on $[1,\infty)$. The following asymptotic is known, $\psi(u) = \log u + o(1)$, as u tends to infinity. Therefore $\Psi_a'(\sqrt{a}) \geq \beta(a - \sqrt{a}) - \log(\sqrt{a}) + c > 0$ for a large enough. In particular, for a large enough, $u_a \geq \sqrt{a}$ which proves the claim. $\square$

3.5. **The Talagrand Conjecture.** In this section we will prove the Talagrand's conjecture for the $M/M/\infty$ queuing process. This is one of the main result of this paper. We will use the strategy of the supremum presented in the Introduction. Recall that $\rho = \lambda/\mu$ and that the $M/M/\infty$ semigroup $(P_t)_{t\geq 0}$ is reversible with respect to the Poisson measure π_ρ of parameter ρ. For simplicity we will assume from now on that $\rho = 1$. All the results below remain valid for any $\rho > 0$, but at the price of more technicalities in the proofs, non essential for the purpose of the whole paper. As a motivation, it should be noticed that the $M/M/\infty$ semigroup enjoys some sort of hypercontractivity property, see [8, Section 7] (cf. [10, 24, 34]). Hence the question raised by Talagrand about the regularization property of the semigroup for functions in $\mathbb{L}^1$ makes perfect sense. Here is a positive answer.

Theorem 3.11 (Talagrand's conjecture for the $M/M/\infty$ queuing process)**.** *Let $(P_t)_{t\geq 0}$ be the $M/M/\infty$ semigroup (with $\rho = 1$). Then, for every $s > 0$, there exists a constant c (that depends only on s) such that, for all $t \geq 4$,*

$$\sup_{f\geq 0: \int f d\pi_1 = 1} \pi_1\left(\{n : P_s f(n) \geq t\}\right) \leq \frac{c\sqrt{\log\log t}}{t\sqrt{\log t}}.$$

Remark 3.12. *For any fixed $s > 0$, this bound is optimal for large values of t. Indeed, using the notation of Remark 3.9, it easily seen that $P_s f_\lambda = f_{\lambda(s)}$, with $\lambda(s) = \log(1 + e^{-s}(e^\lambda - 1))$. According to Remark 3.9, the deviation bound of Proposition 3.8 is optimal for the family $(f_\lambda)_{\lambda>0}$. Therefore the deviation bound of Theorem 3.11 is also optimal.*

The proof of the theorem is based on an estimate on the following quantity

$$\Psi_s(n) := \frac{1}{n!}\sup_{k\geq 0} \frac{\mathbb{P}(Y_{n,s} + Z_s = k)}{\pi_1(k)}, \qquad s \geq 0$$

where $Y_{n,s}$ is a binomial variable of parameter n and $p_s = e^{-t}$ and Z_s is a Poisson variable of parameter $q_s = 1 - p_s$.

Lemma 3.13. *For all $s > 0$, there exists a constant c (that depends only on s and ρ) such that for any $n \geq 1$, $\Psi_s(n) \leq \frac{c}{\sqrt{n}}$.*

Remark 3.14. *We observe that $1/\sqrt{n}$ is the correct order. Indeed, assume that $ne^s \in \mathbb{N}$. Considering the special case $k = n/p_s \in \mathbb{N}$ and then the sole term $j = n$ in the sum we get*

$$\Psi_s(n) \geq \frac{1}{n!} \frac{\mathbb{P}(Y_{n,s} + Z_s = n/p_s)}{\pi_1(n/p_s)} = e^{1-q_s} \frac{(n/p_s)!}{n!} \sum_{j=0}^{n} \binom{n}{j} \frac{p_s^j q_s^{n+(n/p_t)-2j}}{((n/p_s) - j)!}$$
$$\geq e^{p_s} \frac{(n/p_s)!}{n!(nq_s/p_s)!} p_s^n q_s^{nq_s/p_s}.$$

Therefore, using (3.9), we have

$$\log \Psi_s(n) \geq p_s + \left(\frac{n}{p_s} + \frac{1}{2}\right) \log\left(\frac{n}{p_s}\right) - \frac{n}{p_s} - \log 3 - \left(n + \frac{1}{2}\right) \log n + n$$
$$- \log 3 - \left(\frac{nq_s}{p_s} + \frac{1}{2}\right) \log\left(\frac{nq_s}{p_s}\right) + \frac{nq_s}{p_s} + n \log p_s + \frac{nq_s}{p_s} \log q_s$$
$$= p_s - 2\log 3 - \frac{1}{2} \log q_s - \frac{1}{2} \log n \geq -2 \log 3 - \frac{1}{2} \log n$$

from which we get $\Psi_s(n) \geq \frac{1}{9\sqrt{n}}$.

Proof of Lemma 3.13. Denoting by $X = (X_t)_{t\geq 0}$ the $M/M/\infty$ process, we know that $\mathbb{P}(Y_{n,s} + Z_s = k) = \mathbb{P}(X_s = k|X_0 = n)$. Since π_1 is reversible for X, we have

$$\pi_1(n) \frac{\mathbb{P}(X_s = k|X_0 = n)}{\pi_1(k)} = \mathbb{P}(X_s = n|X_0 = k) \tag{3.12}$$

and so $\Psi_s(n) = e \sup_{k\geq 0} \mathbb{P}(X_s = n|X_0 = k) = e \sup_{k\geq 0} \mathbb{P}(Y_{k,s} + Z_s = n)$. Using (3.12), one first sees that if $0 \leq k \leq n - 1$, then

$$\mathbb{P}(X_s = n|X_0 = k) \leq \frac{k!}{n!} \leq \frac{1}{n}.$$

Now, if $k \geq n$, then using Lemma 3.15 below, we see that

$$\mathbb{P}(Y_{k,s} + Z_s = n) = \sum_{i=0}^{k} \mathbb{P}(Y_{k,s} = i)\mathbb{P}(Z_s = n - i) \leq \sup_{0\leq i\leq k} \mathbb{P}(Y_{k,s} = i) \leq \frac{c_p}{\sqrt{k}} \leq \frac{c_p}{\sqrt{n}},$$

which completes the proof. □

Lemma 3.15. *For any $p \in (0, 1)$, there exists $c_p > 0$ such that*

$$\sup_{0\leq i\leq k} \binom{k}{i} p^i (1 - p)^{k-i} \leq \frac{c_p}{\sqrt{k}}, \qquad \forall k \geq 1. \tag{3.13}$$

Proof. When $p \in (0, 1)$, it is well known that the mode of the binomial distribution $\mathcal{B}(k, p)$ is $i_k := \lfloor (k + 1)p \rfloor$. In other words,

$$\sup_{0\leq i\leq k} \binom{k}{i} p^i (1 - p)^{k-i} = \binom{k}{i_k} p^{i_k} (1 - p)^{k-i_k}$$

Using (3.9), one gets that, when $1 \leq i \leq k - 1$

$$\binom{k}{i} p^i (1 - p)^{k-i} \leq 3\sqrt{\frac{k}{i(k - i)}} \frac{p^i (1 - p)^{k-i}}{\left(\frac{i}{k}\right)^i \left(1 - \frac{i}{k}\right)^{k-i}} \leq 3\sqrt{\frac{k}{i(k - i)}},$$

where the last inequality follows from the fact that the function $f(s) = s^i (1 - s)^{k-i}$, $s \in [0, 1]$, reaches its maximum at $s = \frac{i}{k}$. Therefore, if $1 \leq i_k \leq k - 1$, it holds

$$\binom{k}{i_k} p^{i_k} (1 - p)^{k-i_k} \leq 3\sqrt{\frac{k}{i_k(k - i_k)}} \leq c'_p \frac{1}{\sqrt{k}},$$

for some c'_p depending only on p. Now $i_k = 0$ or $i_k = k$ can only occur if $k \leq \max(\frac{1-p}{p}; \frac{p}{1-p}) := k_0$. So letting $c''_p = \sup_{0\leq i\leq k, k\leq k_0} \sqrt{k}\binom{k}{i}p^i(1-p)^{k-i}$, we see that (3.13) holds with $c_p = \max(c'_p; c''_p)$. $\square$

With Lemma 3.13 in hand, we are in position to prove Theorem 3.11.

Proof of Theorem 3.11. We first observe that (strategy of the supremum)

$$\sup_{f\geq 0:\int f d\pi_1 = 1} \pi_1\left(\{n : P_s f(n) \geq t\}\right) \leq \pi\left(\left\{n : \sup_{f\geq 0:\int f d\pi_1=1} P_s f(n) \geq t\right\}\right).$$

We claim that

$$\sup_{f\geq 0:\int f d\pi_1=1} P_s f(n) = \sup_{k\geq 0} \frac{\mathbb{P}(Y_{n,s} + Z_s = k)}{\pi_1(k)}$$

where we recall that $Y_{n,s}$ is a binomial variable of parameter n and $p_s = e^{-s}$ and Z_s is Poisson variable with parameter $q_s = 1 - p_s$. Indeed if one considers $f_o = \mathbb{1}_{k_o}/\pi_1(k_o)$, for some integer k_o, one immediately sees that

$$\sup_{f\geq 0:\int f d\pi_1=1} P_s f(n) \geq P_s f_o(n) = \sum_{k=0}^{\infty} f_o(k)\mathbb{P}(X_{n,s} + Y_s = k) = \frac{\mathbb{P}(X_{n,s} + Y_s = k_o)}{\pi_1(k_o)}.$$

Therefore $\sup_{f\geq 0:\int f d\pi_1=1} P_s f(n) \geq \sup_{k\geq 0} \frac{\mathbb{P}(X_{n,s}+Y_s=k)}{\pi_1(k)}$. On the other hand, for any f non-negative with $\int f d\pi_1 = 1$,

$$P_s f(n) = \sum_{k=0}^{\infty} f(k)\mathbb{P}(X_{n,s} + Y_s = k) = \sum_{k=0}^{\infty} f(k)\pi_1(k)\frac{\mathbb{P}(X_{n,s} + Y_s = k)}{\pi_1(k)} \leq \sup_{k\geq 0} \frac{\mathbb{P}(X_{n,s} + Y_s = k)}{\pi_1(k)}$$

which proves the claim.

Recall the definition of Ψ_s right before Lemma 3.13. From the claim and Lemma 3.13, we have

$$\sup_{f\geq 0:\int f d\pi_1=1} \pi_1\left(\{n : P_s f(n) \geq t\}\right) \leq \pi_1\left(\{n : n!\Psi_s(n) \geq t\}\right) \leq \pi_1\left(\{n : n!/\sqrt{n} \geq t/c\}\right)$$

for some constant c depending only on s. Using (3.9), we have $n!/\sqrt{n} \leq 3\exp\{n\log n - n\}$. Hence, setting $H(x) := x\log x - x$, which is an increasing function (hence one to one whose inverse we denote by H^{-1}),

$$\begin{aligned}\sup_{f\geq 0:\int f d\pi_1=1} \pi_1\left(\{n : P_s f(n) \geq t\}\right) &\leq \pi_1\left(\{n : e^{H(n)} \geq t/(3c)\}\right) \\ &= \pi_1\left(\{n : n \geq H^{-1}(\log(t/(3c)))\}\right) \\ &= \pi_1\left(\{n : n \geq \lceil H^{-1}(\log(t/(3c)))\rceil\}\right).\end{aligned}$$

(Here, as usual, $\lceil\cdot\rceil$ denotes the ceiling function, that maps x to the least integer greater than or equal x). Next we observe that, according to Lemma 3.7, for any integer $u \geq 2$,

$$\pi_1(\{n : n \geq u\}) \leq 2\frac{e^{-\Phi_1(u)}}{\sqrt{u}} \leq \frac{e^{-H(u)}}{\sqrt{u}}.$$

Since $x \mapsto e^{-H(x)}/\sqrt{x}$ is decreasing, for t large enough, we end up with

$$\begin{aligned}\sup_{f\geq 0:\int f d\pi_1=1} \pi_1\left(\{n : P_s f(n) \geq t\}\right) &\leq \frac{e^{-H(\lceil H^{-1}(\log(t/(3c)))\rceil)}}{\sqrt{\lceil H^{-1}(\log(t/(3c)))\rceil}} \leq \frac{e^{-H(H^{-1}(\log(t/(3c))))}}{\sqrt{H^{-1}(\log(t/(3c)))}} \\ &= \frac{3c}{t\sqrt{H^{-1}(\log(t/(3c)))}}.\end{aligned}$$

Finally, we observe that $H^{-1}(x) \geq \frac{x}{\log x}$ for x large enough, from which the expected result follows. □

4. Laguerre's semigroups

In this section we deal with the Laguerre semigroups on $(0,\infty)$. We may prove that both properties of Item (1), and item (2) in the introduction, do not hold. On the other hand, the strategy of the supremum applies and will allow us to prove the Talagrand Conjecture for the Gamma probability measures.

In the next subsection, we introduce the Laguerre operator in its full generality. However, in the subsequent sub-section we may, for simplicity, reduce to the sole case $\alpha = 3/2$ (see below) which is simpler to handle. Many computations could probably be done for general α but at the price of heavy technicalities. We preferred a simpler presentation rather than a complete one in order to present the phenomenon occurring in the setting of Laguerre's operators.

4.1. **Introduction.** On $(0,\infty)$ denote by ν_α, with $\alpha > 0$, the Gamma distribution with density

$$\varphi_\alpha(x) := \frac{1}{\Gamma(\alpha)} x^{\alpha-1} e^{-x}, \quad x > 0$$

with respect to the Lebesgue measure on $(0,\infty)$. It is the reversible measure of the diffusion operator L^α (which is negative), called Laguerre operator, defined on smooth enough functions f as

$$L^\alpha f(x) = x f''(x) + (\alpha - x) f'(x), \quad x > 0.$$

The Laguerre Operator is well-known and related to Laguerre's polynomials

$$Q_k^\alpha(x) := \frac{1}{k!} x^{-\alpha+1} e^x \frac{d^k}{dx^k}\left(x^{k+\alpha-1} e^{-x}\right), \quad k \in \mathbb{N}, \quad x > 0.$$

First Laguerre's polynomials are (we omit the super scripts α for simplicity) $Q_0(x) = 1$, $Q_1(x) = \alpha - x$, $Q_2(x) = \frac{\alpha(\alpha+1)}{2} - (\alpha+1)x + \frac{1}{2}x^2$. Moreover, the family $(Q_k^\alpha)_{k\geq 0}$ is an orthogonal decomposition of L^α in $\mathcal{L}^2((0,\infty),\gamma_\alpha)$: namely it is an orthogonal basis of $\mathcal{L}^2((0,\infty),\gamma_\alpha)$ and each Q_k^α is an eigenfunction of L^α with associated eigenvalue $-k$, $k = 0, 1, \ldots$. The associated semigroup, we denote by $(P_t^\alpha)_{t\geq 0}$, takes the form (see *e.g.* [1])

$$P_t^\alpha f(x) = \int G_t^\alpha(x,y) f(y) d\nu_\alpha(y)$$

for any $f \in \mathcal{L}^p((0,\infty),\gamma_\alpha)$ for some $p \geq 1$, with kernel

$$G_t^\alpha(x,y) := \frac{\Gamma(\alpha)e^t}{e^t - 1}\left(\frac{e^t}{xy}\right)^{\frac{\alpha-1}{2}} \exp\left\{-\frac{1}{e^t-1}(x+y)\right\} I_{\alpha-1}\left(\frac{2\sqrt{xye^t}}{e^t-1}\right).$$

Here I_β denotes the modified Bessel function of the first kind of order $\beta > -1$, defined as

$$I_\beta(x) := \sum_{n=0}^\infty \frac{1}{n!\Gamma(n+\beta+1)}\left(\frac{x}{2}\right)^{2n+\beta}, \quad x > 0.$$

4.2. **Semi-log-convexity for the Laguerre semigroups.** We will prove in this section that there does not exist any uniform lower bound (in x and f) on $(\log P_t^\alpha)''(x)$. For simplicity, and since $I_{1/2}(x) = \sqrt{2/\pi x}\sinh(x)$, $x > 0$, is explicit, we may focus only on the case $\alpha = 3/2$ for which we have (we omit the superscript $\alpha = 3/2$ all along this subsection)

$$P_t f(x) = \int G_t(x,y) f(y) d\nu(y), \quad G_t(x,y) := \frac{\Gamma(\frac{3}{2})e^t}{e^t-1}\left(\frac{e^t}{xy}\right)^{\frac{1}{4}} \exp\left\{-\frac{x+y}{e^t-1}\right\} I_{\frac{1}{2}}\left(\frac{2\sqrt{xye^t}}{e^t-1}\right)$$

with $d\nu(x) = \varphi(x)dx = \frac{\sqrt{x}}{\Gamma(3/2)}e^{-x}dx$. Now consider the special test function $f(y) = \delta_y/\varphi(y)$ so that $P_tf(x) = G_t(x,y)$ and therefore, setting $c_t := 2e^{t/2}/(e^t-1)$,

$$\log P_tf(x) = c_{y,t} - \frac{1}{4}\log x - \frac{x}{e^t-1} + \log I_{\frac{1}{2}}(c_t\sqrt{xy}) = c'_{y,t} - \frac{1}{2}\log x - \frac{x}{e^t-1} + \log\sinh(c_t\sqrt{xy})$$

where $c_{y,t}, c'_{y,t}$ are constants that depend only on y and t. It follows that

$$(\log P_tf)'(x) = -\frac{1}{2x} - \frac{1}{e^t-1} + \frac{c_t\sqrt{y}}{2\sqrt{x}}\coth(c_t\sqrt{xy})$$

and

$$\begin{aligned}
(\log P_tf)''(x) &= \frac{1}{2x^2} - \frac{c_t\sqrt{y}}{4x\sqrt{x}}\coth(c_t\sqrt{xy}) + \frac{c_t^2y}{4x}(1-\coth(c_t\sqrt{xy})^2) \\
&= \frac{1}{4x^2}\left(2 + c_t^2xy - c_t\sqrt{xy}\coth(c_t\sqrt{xy}) - [c_t\sqrt{xy}\coth(c_t\sqrt{xy})]^2\right) \\
&= \frac{1}{4x^2}\left(2 + z^2 - z\coth(z) - z^2\coth(z)^2\right) = \frac{1}{4x^2}\left(2 - \frac{z^2}{(\sinh z)^2} - z\coth(z)\right)
\end{aligned}$$

where in the third line we set $z = c_t\sqrt{xy}$. Since $z\coth z \to \infty$ as z tends to infinity, and since $z/\sinh z \le 1$ (for $z > 0$), the latter shows that $(\log P_tf)''(x)$ cannot be bounded below by a constant independent on f and x. Hence, Item (1) of the introduction does not hold.

4.3. **Deviation bounds for semi-log-convex functions.** In this section, we investigate item (2) in the introduction. We prove that, due to the weak tail of the measures ν_α, the log-semi-convexity property does not help to get a better bound than Markov's inequality. More precisely, setting $\mathcal{F}_{\beta,\alpha} := \{f \ge 0 : (\log f)'' \ge -\beta, \int f d\nu_\alpha = 1\}$, $\beta \in \mathbb{R}$, we have the following proposition.

Proposition 4.1. *Let $\alpha > 0$. Then, for all $\beta > 0$,*

$$\limsup_{t\to\infty} t \sup_{f\in\mathcal{F}_{\beta,\alpha}} \nu_\alpha(\{x : f(x) \ge t\}) > 0.$$

Proof. We proceed as in the proof of Proposition 3.10. Fix $\beta > 0$ and, for $a > 0$, define $f_a(x) = \exp\{-\frac{\beta}{2}(x-a)^2 + Z(a)\}$ where $Z(a) := -\log\int\exp\{-\frac{\beta}{2}(x-a)^2\}d\nu_\alpha(x)$ is devised so that $\int f_a d\nu_\alpha = 1$. It is easy to prove (we omit details) that $Z(a) \le c\varphi_\alpha(a)$ for some positive constant c that depends on β and α. Hence,

$$\nu_\alpha(f_a \ge t) \ge \nu_\alpha\left((x-a)^2 \le \frac{2}{\beta}(\log t + \log\varphi_\alpha(a) + \log c)\right).$$

Now choose a so that $\frac{2}{\beta}(\log t + \log\varphi_\alpha(a) + \log c) = 1$. We infer that

$$\nu_\alpha(f_a \ge t) \ge \nu_\alpha(x \in [a-1, a+1]) = \int_{a-1}^{a+1}\varphi_\alpha(x)dx \ge c'\varphi_\alpha(a)$$

where the last inequality holds for a large enough and follows after some approximation and algebra left to the reader (here c' is a constant that depends only α). But a has been chosen so that $\varphi_\alpha(a) = \frac{c''}{t}$ for some constant $c'' > 0$ depending only on α and β. Hence, $t\nu_\alpha(f_a \ge t) \ge c''$ which proves the proposition. □

Remark 4.2. *In [17], deviation bounds for log-convex densities ($\beta = 0$) under the exponential measure ($\alpha = 1$) were deduced from the Gaussian case using a simple pushforward argument. The same argument could be easily used to get deviation bounds for log-convex functions for other Gamma distributions.*

As already mentioned, the above result ($\beta > 0$) is due to weak tail of ν_α. Indeed, for such measures, we have $\int_x^\infty d\nu_\alpha \sim_\infty \varphi_\alpha(x)$, while for example for the standard Gaussian *measure,* $\int_x^\infty e^{-t^2/2}dt \sim_\infty e^{-x^2/2}/x$*, i.e. there is a gain of a factor* $1/x$ *with respect to the Gaussian density in this case.*

4.4. **The Talagrand Conjecture.** In this final section, we prove Talagrand's conjecture for Laguerre's semigroups, by means of the strategy of the supremum. Such a conjecture makes sense also in this setting since the Laguerre semigroups enjoy an hypercontractive property [25, 20].

Theorem 4.3. *Let* $\alpha > 0$ *and denote by* $(P_s^\alpha)_{s\geq 0}$ *the Laguerre semigroup reversible with respect to* ν_α*. Then, for any* $s > 0$*, there exists a constant* c *(that depends only on* s *and* α*) such that for all non-negative real functions* f *in* $\mathbb{L}^1((0,\infty),\nu_\alpha)$ *with* $\int f d\nu_\alpha = 1$*,*

$$\nu_\alpha(\{P_s^\alpha f \geq t\}) \leq \frac{c}{t\sqrt{\log t}}, \qquad t > 1.$$

Proof. Fix $s > 0$ and $t > 1$. We will use the strategy of the supremum. Namely, we first observe that

$$\sup_{f\geq 0, \int f d\nu_\alpha = 1} \nu_\alpha(\{P_s^\alpha f \geq t\}) \leq \nu_\alpha(\{\sup_{f\geq 0, \int f d\nu_\alpha = 1} P_s f \geq t\}).$$

Then, it is easy to see that, thanks to the kernel representation,

$$\sup_{f\geq 0, \int f d\nu_\alpha = 1} P_s^\alpha f(x) = \sup_{y>0} G_s^\alpha(x,y), \qquad x > 0.$$

Therefore we are left with an estimate on $G_s(x,y)$ (we look for an upper bound). The following asymptotics are know [1] to hold $I_\beta(x) \sim_\infty \frac{e^x}{\sqrt{2\pi x}}$ and $I_\beta(x) \sim_0 \frac{(x/2)^\beta}{\Gamma(\beta+1)}$. Up to a constant c that depends on α, we can safely assert that $I_{\alpha-1}(u) \leq ce^u/\sqrt{u}$, for $u \geq 1$ and $I_{\alpha-1}(u) \leq cu^{\alpha-1}$ for $u \leq 1$. In particular,

$$\begin{aligned}
\sup_{y>0} G_s^\alpha(x,y) &\leq c' x^{\frac{1-\alpha}{2}} e^{-\frac{x}{e^s-1}} \max\left(x^{\frac{\alpha-1}{2}} \sup_{0<y\leq y_x} e^{-\frac{y}{e^s-1}}; x^{-\frac{1}{4}} \sup_{y>y_x} y^{\frac{1-2\alpha}{4}} e^{-\frac{y}{e^s-1}+\frac{2\sqrt{xye^s}}{e^s-1}} \right) \\
&= c' x^{\frac{1-\alpha}{2}} e^{-\frac{x}{e^s-1}} \max\Big(x^{\frac{\alpha-1}{2}}; \\
&\qquad x^{-\frac{1}{4}} \exp\left\{ \frac{1}{e^s-1} \sup_{y>y_x} \frac{(1-2\alpha)(e^s-1)}{4} \log y - y + 2\sqrt{xye^s} \right\} \Big)
\end{aligned}$$

for some constant c' that depends on s and α and where y_x is such that $\frac{2\sqrt{xy_xe^s}}{e^s-1} = 1$, *i.e.* $y_x = \frac{(e^s-1)^2}{4xe^s}$. Hence, we need to bound from above

$$\sup_{y>y_x} \frac{(1-2\alpha)(e^s-1)}{4} \log y - y + 2\sqrt{xye^s} = \sup_{z>(e^s-1)/b} a\log z + bz - z^2$$

where we set $a = \frac{(1-2\alpha)(e^s-1)}{2}$ and $b = 2\sqrt{xe^s}$ (and used the change of variable $z = \sqrt{y}$, together with the fact that $\sqrt{y_x} = (e^s-1)/(2\sqrt{xe^s}) = (e^s-1)/b)$. Denote by $H(y) := a\log z + bz - z^2$. It is a tedious but easy exercise to prove that there exists a constant $c > 0$ than depends only on s and α, and $x_o > 0$ such that $\sup_{z>\sqrt{y_x}} H(z) \leq c + \frac{a}{2}\log x + xe^s$ for $x \geq x_o$ and $\sup_{z>\sqrt{y_x}} H(z) \leq -\frac{1}{cx}$ for $x \leq x_o$. Hence, after some algebra

$$\sup_{y>0} G_s^\alpha(x,y) \leq c'\left(1 + x^{\frac{1-2\alpha}{2}} e^x\right)$$

for some constant c' that depends only s and α. Denote by $F(x) := x + \frac{1-2\alpha}{2}\log x$, $x > 0$ and observe that F increasing for x large enough with inverse function we denote by F^{-1}

is also increasing. It is easy to see that $x \geq F^{-1}(x) \geq x - \frac{1-2\alpha}{2}\log x$ (for x large enough). Therefore, for t large enough

$$\begin{aligned}
\sup_{f\geq 0, \int f d\nu_\alpha = 1} \nu_\alpha(\{P_s^\alpha f \geq t\}) &\leq \nu_\alpha\left(\left\{x : F(x) \geq \log\left(\frac{t}{c'} - 1\right)\right\}\right) \\
&\leq \nu_\alpha\left(\left\{x : x \geq F^{-1}\left(\log\left(\frac{t}{c'} - 1\right)\right)\right\}\right) \\
&= \frac{1}{\Gamma(\alpha)} \int_{F^{-1}\left(\log\left(\frac{t}{c'}-1\right)\right)}^{\infty} x^{\alpha-1}e^{-x}dx \\
&\leq \kappa F^{-1}\left(\log\left(\frac{t}{c'} - 1\right)\right)^{\alpha-1} e^{-F^{-1}\left(\log\left(\frac{t}{c'}-1\right)\right)} \\
&\leq \kappa'(\log t)^{\alpha-1}e^{-\log(t)+\frac{1-2\alpha}{2}\log\log(t)} = \kappa' \frac{1}{t\sqrt{\log t}}
\end{aligned}$$

where we used that $\int_u^\infty x^{\alpha-1}e^{-x}dx \leq \kappa u^{\alpha-1}e^{-u}$ for u large enough and κ, κ' are constants that depends only on α and s. For t close to 1, the result follows from Markov's inequality (at the price of a possible bigger constant κ'). □

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