IMPERIAL

# DEPARTMENT OF MATHEMATICS

# GUIDE TO OPTIONAL MODULES

These notes should be read in conjunction with your student handbook. Some of the information may be subject to alteration.

Updated information will be posted on the Maths Central Blackboard site and your course blackboard page.

## MSc Pure Mathematics Overview

MSc students can take the following types of modules:

- The Pure Mathematics modules listed below which can also be taken by final year undergraduate MSci students.
- Modules (up to two total for the year) from other MSc programmes at Imperial College London, subject to approval of the host Department and the Programme Director. Students are not allowed to take modules from MSc Statistics, MSc Mathematics and Finance or MSc Machine Learning and Data Science. Students with exceptional reasons to take more than two modules outside the Pure Mathematics MSc programme (e.g., if needed for their project and agreed by their project supervisor) should contact the Programme Director, at whose discretion this restriction can be relaxed.

Students must take 8 modules (usually four in the first year and four in the second year for part-time students). The modules chosen should not overlap in a substantial way and together should form a coherent program. Students with a degree from Imperial College London will not be allowed to repeat a module they already attended as undergraduates. Please discuss your choice of modules with your Personal Tutor as soon as possible. In all but exceptional cases, full time students will be examined/ complete the coursework on four modules from the Autumn Term and four modules from the Spring Term.

### Advice on the choice of options

Students are advised to read these notes carefully and to discuss their option selections with their Personal Tutor.

It is anticipated that lecturers will give advice on suitable books at the start of each module. Students should contact the proposed lecturers if they desire any further details about module content in order to make their choice of course options.

You will not be committed to your choice of modules immediately; these will be confirmed after the second week of each term.  In exceptional circumstances students wishing to change registration after that point must seek approval from the Programme Director, but this should not happen after any assessed coursework is due (for the current module or the desired one).

### Module assessment and examinations

Most MATH7 modules are examined by one written examination of 2.5 hours in length.

Some of the modules may have an assessed coursework/progress test element, limited in most cases to 10% of overall module assessment. Some modules have a more substantial coursework component (for example, 25 percent) and others are assessed entirely by coursework. Details can be found in the tables below. Precise details of the number and nature of coursework assignments will be provided at the start of each module.

Students should bear in mind that single-term modules assessed by projects usually require extra time-commitment during that term. Students should note that, in principle, 1 ECTS represents around 25 hours of effort, and so a single 7.5 ECTS module represents 187.5 hours of effort: completing this in a single term is a substantial task. Thus, the Department allows students to take at most one such module in a term.

Note: Students who take modules which are wholly assessed by project will be deemed to be officially registered on the module through the submission of a specified number of pieces of assessed work for that module. Thus, once a certain point is reached in these modules, a student will be committed to completing it. In contrast, students only become committed to modules with summer examinations when they choose the modules for examinations in November.

# Pure Mathematics Module List

## ANALYSIS

| Module Code | Module Titles | Term | Lecturer | % exam | % CW |
|---|---|---|---|---|---|
| MATH70019 | Theory of Partial Differential Equations | 1 | Dr M. Sorella | 90 | 10 |
| MATH70028 | Probability Theory 1 | 1 | Dr Zhang | 90 | 10 |
| MATH70029 | Functional Analysis | 1 | Prof P. Germain | 90 | 10 |
| MATH70135 | Advanced Partial Differential Equations 1 | 1 | Prof G. Pavliotis | 90 | 10 |
| MATH70021 | Advanced Partial Differential Equations 2 | 2 | Prof M. Coti Zelati | 90 | 10 |
| MATH70030 | Fourier Analysis and Theory of Distributions | 2 | Dr I. Krasovsky | 90 | 10 |
| MATH70031 | Probability Theory 2 | 2 | Dr B. Dagallier | 90 | 10 |
| MATH70055 | Stochastic Calculus with Applications to non-Linear Filtering | 2 | Prof D. Crisan | 90 | 10 |

## GEOMETRY

| Module Code | Module Titles | Term | Lecturer | % exam | % CW |
|---|---|---|---|---|---|
| MATH70033 | Algebraic Curves | 1 | Dr S. Sivek | 90 | 10 |
| MATH70058 | Manifolds | 1 | Dr Y. Sun | 90 | 10 |
| MATH70032 | Geometry of Curves and Surfaces | 2 | Dr P. Cameron | 90 | 10 |
| MATH70034 | Algebraic Topology | 2 | Dr S. Sivek | 90 | 10 |
| MATH70056 | Algebraic Geometry | 2 | Dr. M. Booth | 90 | 10 |
| MATH70057 | Riemannian Geometry | 2 | Dr M. Guaraco | 90 | 10 |
| MATH70059 | Differential Topology | 2 | Dr M-A Lawn | 90 | 10 |
| MATH70060 | Complex Manifolds | 2 | Dr D. Parise | 90 | 10 |
| MATH70140 | Geometric Complex Analysis | 2 | Dr D. Cheraghi | 90 | 10 |

## ALGEBRA AND DISCRETE MATHEMATICS

| Module Code | Module Titles | Term | Lecturer | % exam | % CW |
|---|---|---|---|---|---|
| MATH70035 | Algebra 3 | 1 | Prof A. Corti | 90 | 10 |
| MATH70036 | Group Theory | 1 | Dr K. Kansal | 90 | 10 |
| MATH70037 | Galois Theory | 1 | Prof A. Corti | 90 | 10 |
| MATH70038 | Graph Theory | 1 | Dr M. Zordan | 90 | 10 |
| MATH70061 | Commutative Algebra | 1 | Prof P. Cascini | 90 | 10 |
| MATH70062 | Lie Algebras | 1 | Dr T. Gee | 90 | 10 |

| | | | | | |
|---|---|---|---|---|---|
| MATH70039 | Group Representation Theory | 2 | Dr N. Tamam | 90 | 10 |
| MATH70063 | Algebra 4 | 2 | Dr B. Briggs | 90 | 10 |
| MATH70132 | Mathematical Logic | 2 | Prof D. Evans | 90 | 10 |

**NUMBER THEORY**

| Module Code | Module Titles | Term | Lecturer | % exam | % CW |
|---|---|---|---|---|---|
| MATH70041 | Number Theory | 1 | Prof A. Caraiani | 90 | 10 |
| MATH70064 | Elliptic Curves | 1 | Prof Y. Lekili | 90 | 10 |
| MATH70042 | Algebraic Number Theory | 2 | Dr M. Pagano | 90 | 10 |

**FORMALISATION OF MATHEMATICS**

| Module Code | Module Titles | Term | Lecturer | % exam | % CW |
|---|---|---|---|---|---|
| MATH70040 | Formalising Mathematics | 2 | Dr B. Mehta | 0 | 100 |

In addition to these modules, students may take some modules outside of Pure Mathematics as explained under “Choice and Approval of Modules and Project" above. Amongst these we would like to highlight the Dynamical Systems modules offered in Applied Mathematics:

The descriptions for the below modules are available in the MSc Applied Mathematics module guide.

| Module Code | Module Titles | Term | Lecturer | % exam | % CW |
|---|---|---|---|---|---|
| MATH70007 | Dynamics of Learning and Iterated Games | 1 | Prof S. Van Strien | 40 (oral) | 60 |
| MATH70008 | Dynamical Systems | 1 | Prof J. Lamb | 90 | 10 |
| MATH70009 | Bifurcation Theory | 2 | Prof D. Turaev | 90 | 10 |
| MATH70053 | Random Dynamical Systems and Ergodic Theory: Seminar Course | 2 | Prof J. Lamb | 40 (oral) | 60 |

# Module descriptions

## MATH70019: Theory of Partial Differential Equations

### Brief Description

In this module, students are exposed to different phenomena which are modelled by partial differential equations. The course emphasizes the mathematical analysis of these models and briefly introduces some numerical methods.

### Learning Outcomes

On successful completion of this module, you will be able to:

- appreciate how to formally differentiate complicated finite dimensional functionals and simple infinite dimensional functionals;
- describe, select and use a variety of methods for solving partial differential equations;
- outline how various partial differential equations respect conservation laws;
- utilize energy methods to critically analyse the stability of solutions to PDEs;
- develop the general method of characteristics and derive the eikonal equation;
- justify the proper use of the calculus of variations in classical settings.

### Module content

The module is composed of the following sections:

1. Introduction to PDEs
    1.1. Basic Concepts
    1.2. Gauss Theorem
2. Method of Characteristics
    2.1. Linear and Quasilinear first order PDEs in two independent variables.
    2.2. Scalar Conservation Laws
    2.3. Hamilton-Jacobi Equations. General Method of Characteristics.
3. Diffusion
    3.1. Heat equation. Maximum principle
    3.2. Separation of variables. Fourier Series.
4. Waves
    4.1. The 1D wave equation
    4.2. 2D and 3D waves.
5. Laplace-Poisson equation
    5.1. Dirichlet and Neumann problems.
    5.2. Introduction to calculus of variations. The Dirichlet principle.
    5.3. Finite Element Method.
    5.4. Lagrangians and the minimum action principle.

## MATH70021: Advanced Partial Differential Equations 2

### Brief Description

The focus of the course is the theory of nonlinear partial differential equations (PDEs) and their modern treatment through analytical techniques. The emphasis is on methods (such as fixed point theorems, Fourier analysis) and how they apply to classical problems involving fluid mechanics and wave propagation.

**Learning Outcome**

On successful completion of this module you will be able to:

- appreciate the concepts of distribution (differentiation, convergence);
- manipulate the main properties of the Sobolev space H^m for integer m (inbeddings and compactness theorems, Poincaré inequality);
- derive the variational formulation of a specific elliptic boundary value problem and to provide the reasoning leading to the proof of the existence and uniqueness of the solution;
- develop the spectral theory of an elliptic boundary value problem;
- solve a parabolic boundary value problem using the spectral theory of the associated elliptic operator.
- interpret results from advanced textbooks and research papers on the theory of Partial Differential Equations;
- independently appraise and evaluate an advanced topic on Partial Differential Equations, namely the theory of nonlinear elliptic and parabolic equations on the whole space.

**Module Content**

An indicative list of topics is:

1. Fixed point theorems and applications: the contraction mapping principle, Brouwer and Schauder fixed-point theorems, semi linear parabolic equations.
2. Elements of Fourier analysis: the Fourier transform, the method of stationary phases, singular integrals.
3. Energy estimates and compactness: basic notions, spaces involving time, existence of solutions to nonlinear evolution equations.
4. Applications to equations arising in Mathematical Physics: the Euler and Navier-Stokes equations, dispersive equations, nonlinear heat equations, hyperbolic conservation laws.

## MATH70028: Probability Theory 1

**Brief Description**

This module rigorously develops mathematical probability theory using measure theory and applying many of the tools developed in the Year 2 module Lebesgue Theory and Integration (or an equivalent module). This module will give students a solid foundation for formulating and studying random phenomena using mathematics, laying the ground work for further studies in areas such as stochastic processes, stochastic calculus, and statistical mechanics.

**Learning Outcomes**

On successful completion of this module, you should be able to:

- Demonstrate knowledge of fundamental notions and structures of Probability Theory;
- Use the tools developed in a variety of applications;
- Demonstrate problem solving skills in probability theory
- Demonstrate additional competence in the subject through self-study of more advanced material
- Combine material from across the module to solve more advanced problems
- Communicate your knowledge of the area in a concise, accurate and coherent manner.

**Module Content**

An indicative list of topics is:

1. Probability spaces, measurability and random variables, convergence of random variables.
2. Independence, Weak Law of Large Numbers.
3. Borel-Cantelli Lemmas, Strong Law of Large Numbers.
4. Convergence in distribution, characteristic functions, Central Limit Theorem.
5. Conditional expectation and discrete time martingales.

## MATH70029: Functional Analysis

**Brief Description**

This module brings together ideas of continuity and linear algebra. It concerns vector spaces with a distance and involves linear maps. The vector spaces are often spaces of functions. It is an important requirement for further study of many areas of Mathematical Analysis including PDEs, Stochastic Analysis, Dynamical Systems and Quantum Mechanics.

**Learning Outcomes**

On successful completion of this module, you should be able to:

- Demonstrate knowledge of fundamental notions and structures of Functional Analysis by proving a range of results;
- Use the tools developed in a variety of applications;
- Demonstrate problem solving skills in functional analysis
- Demonstrate additional competence in the subject through self-study of more advanced material
- Synthesis topics from across the module to solve problems on more advanced applications
- Communicate your knowledge of the area in a concise, accurate and coherent manner.

**Module Content**

An indicative list of topics is:

1. Banach spaces: definition, Schauder basis.
2. Linear operators: boundedness, continuity. Dual spaces, Hahn-Banach lemma, Riesz representation theorem in Lp.
3. Baire category and UBP (Banach-Steinhaus), open mapping/closed graph theorem, closable operators
4. Hilbert spaces, the projection theorem and the Riesz representation theorem, orthogonal bases, Lax-Milgram.
5. Compactness: Riesz lemma. Compact operators, examples: Hilbert-Schmidt, Arzela-Ascoli.
6. Spectrum and resolvent: basics. Spectral Theory in Hilbert Spaces, symmetric, self-adjoint, Spectral theorem for compact self-adjoint operators on a Hilbert space (Riesz-Schauder)

## MATH70030: Fourier Analysis and Theory of Distributions

**Brief Description**

Fourier analysis is an important tool used in various branches of mathematics and beyond. The module provides a deeper understanding of it than what is briefly mentioned in general analysis courses. It also connects it to the theory of distributions. As a result of studying the module, students will understand the basics of the Fourier analysis and theory of distributions which will be sufficient for most branches of mathematics.

**Learning Outcomes**

On successful completion of this module, you will be able to:

- understand the issues of convergence for Fourier series,
- apply the Fourier and Laplace transforms,
- understand the motivation behind the notion of distribution,
- be in command of the basics of Fourier analysis and distribution theory sufficient for working in many areas of mathematics
- demonstrate competence with further advanced material in the area designated for self-study
- synthesize material from across the module to apply to advanced topics

**Module Content**

The module will assume familiarity with measure theory and functional analysis, especially L^p spaces and linear functionals.

Indicative content: Orthogonal systems in infinite-dimensional Euclidean spaces, Bessel inequality, Parseval equality, general Fourier series, trigonometric basis in L_2[-Pi,Pi], convergence of trigonometric Fourier series, Fejer's theorem and applications, Fourier transform and its properties, application to solution of differential equations, Plancherel theorem, Laplace transform, linear functionals, distributions, basic properties of distributions and applications, Fourier transform for distributions.

Those students who decide to do a PhD in a closer related area of analysis will be able to use the acquired basic knowledge and skills to relatively easily extend their knowledge to more sophisticated areas of the theory.

This Level 7 version of the module will involve extra self-study of extension material (in the form of a book chapter, additional notes or a research paper) applying or extending material from the above topics.

## MATH70031: Probability Theory 2

**Brief Description**

This module builds on Probability Theory 1, building a theory of continuous time stochastic processes (with Brownian Motion as a key example) along with introducing students to rigorous stochastic calculus. This module gives crucial background for students interested in stochastic differential equations, stochastic partial differential equations, and more generally stochastic analysis.

**Learning Outcome**

On successful completion of this module, you should be able to:

- demonstrate your understanding of the concepts and results associated with the elementary theory of Markov processes, including the proofs of a variety of results
- apply these concepts and results to tackle a range of problems, including previously unseen ones apply your understanding to develop proofs of unfamiliar results
- demonstrate additional competence in the subject through the study of more advanced material
- combine ideas from across the module to solve more advanced problems
- communicate your knowledge of the area in a concise, accurate and coherent manner

*Prerequisites: Probability Theory 1 (or an equivalent, rigorous measure-theoretic probability module. Some probability modules may not be sufficient background, see the exams and solutions for Probability Theory 1 if in doubt.) Functional Analysis is also very strongly recommended, but not strictly necessary if you are willing to spend significant effort catching up on that topic on your own.*

**Module Content**

1. Definition and construction of Brownian Motion, properties of sample Brownian paths, strong Markov property.
2. Continuous time local martingales
3. Stochastic integrals with Brownian motion, and general continuous local martingales
4. Itˆo's formula
5. Stochastic differential equations, martingale problems, applications to PDE.
6. Markov semigroups, functional inequalities, and convergence to equilibrium.

## MATH70032: Geometry of Curves and Surfaces

**Brief Description**

This module is an introduction to classical theory of differential geometry, where we discuss geometric features of curves and surfaces in (mostly) three dimensional Euclidean spaces.

**Learning Outcomes**

On successful completion of this module, you will be able to:

- identify regular curves and implement different re-parametrisations of curves in two and three dimensional spaces,
- learn about and calculate the geometric quantities of curvature and torsion of a regular curve,
- identify regular surfaces in 3 dimensional spaces using the notions of charts,
- analyse the regularity of maps from one surface into another surface, and also of functions on surfaces,
- use partitions to calculate the basic topological invariant of Euler characteristic,
- learn about the topological classification of compact surfaces, and identify them,
- calculate the first and second fundamental forms of a surface,
- learn about the existence and uniqueness of geodesics on general surfaces,
- link the Gaussian curvature to the local shape of a surface, and present different kinds of examples,
- analyse the global topological features of a surface by integrating local geometric features (Gauss-Bonnet and winding numbers)
- demonstrate competence with further advanced material in the area designated for self-study
- synthesis material from across the module to apply to advanced topics

**Module Content**

This module is an introduction to classical theory of differential geometry, where we discuss geometric features of curves and surfaces in (mostly) three dimensional Euclidean spaces. A curve, which is the trajectory of a particle moving in a smooth fashion, may twist in two manners described by the values called curvature and torsion. The twists of a surface in three dimensional space is naturally more involved. There are different notions of curvature: the Gaussian curvature and the mean curvature. The Guassian curvature describes the intrinsic geometry of the surface, and the mean curvature describes how it bends in space. We look at several examples of surfaces, and calculate their curvatures. We study the local shapes of surfaces based on their curvatures. For example, the Gaussian curvature of a sphere is strictly positive, which explains why any planar illustration of the countries distorts shapes. Remarkably, these local geometric notions can be combined to derive global information about the topology of the surface (for example the Gauss-Bonnet formula). This module starts with the basic real analysis taught in years 1 and 2, and leads into the more modern and general theory of manifolds.

An indicative list of sections and topics is:

1. Curves in two and three-dimensional spaces: re-parametrizations, curvature and torsion, Frenet-Serret formulae, curves are determined by curvature and torsion, winding number and the total curvature,
2. Surfaces: Charts, Tangent vectors, and tangent planes, Smooth maps from one surface into another surface, smooth functions on a surface, Normal vectors,
3. Curvature of a surface: the first and second fundamental forms, Christoffel symbols, normal curvature, Gaussian curvature, and mean curvature, Gauss's Theorema Egregium,
4. Area of a surface,
5. Geodesics on a surface: length-minimising curves, existence, non-existence and examples, geodesic curvature,
6. Gauss-Bonnet Theorem and applications
7. Topological classification of surfaces
8. Vector fields and the Poincare-Hopf Theorem

The module will assume familiarity with material in the second-year module Analysis II

This Level 7 version of the module will involve extra self-study of extension material (in the form of a book chapter, additional notes or a research paper) applying or extending material from the above topics.

## MATH70033: Algebraic Curves

### Brief Description

This module is meant as a first encounter with algebraic geometry, through the study of affine and projective plane curves over the field of complex numbers. We will also discuss some complex-analytic aspects of the theory (Riemann surfaces). Important results include the definition of local intersection multiplicities and Bézout's theorem, inflection points and the classification of plane cubics, linear systems of curves, and the degree-genus formula.

### Learning Outcomes

On successful completion of this module, you will be able to:

- solve geometric problems about affine and projective plane curves with algebraic techniques;
- determine the projectivizations of affine plane curves and the points at infinity;
- determine the tangent lines of plane curves at smooth and singular points;
- compute projective transformations and find convenient coordinate systems;
- compute intersection multiplicities using resultants and the axiomatic characterization;
- formulate, prove and apply Bézout's theorem;
- find inflection points of projective plane curves and use them to classify cubic curves;
- solve enumerative problems by means of the theory of linear systems;
- work with holomorphic charts to determine local and global properties of Riemann surfaces and morphisms;
- compute ramification degrees of morphisms of Riemann surfaces;
- formulate, prove and apply the degree-genus formula for smooth projective plane curves.

- demonstrate competence with further advanced material in the area designated for self-study
- synthesise material from across the module to apply to advanced topics

**Module Content**

An indicative list of sections and topics is:

1. Affine plane curves;
2. The geometry of projective spaces;
3. Projective plane curves;
4. Smooth and singular points, tangent lines;
5. Projective transformations and the classification of conics;
6. Intersection multiplicities (resultants and axiomatic characterization)
7. Bézout's theorem on intersections of projective plane curves;
8. the Legendre family of cubics, inflection points and the classification of non-degenerate smooth cubics;
9. linear systems of projective plane curves, projective duality and enumerative geometry;
10. Riemann surfaces;
11. local description of morphisms of Riemann surfaces (ramification);
12. classification of topological surfaces and genus (informal introduction);
13. Riemann-Hurwitz and the degree-genus formula.

Some related topics will appear in the problem sheets and the coursework (e.g., dual curves, group structure on smooth cubics).

This Level 7 version of the module will involve extra self-study of extension material (in the form of a book chapter, additional notes or a research paper) applying or extending material from the above topics.

## MATH70034: Algebraic Topology

**Brief Description**

This module gives a first introduction to algebraic topology. After some preliminary results on quotient spaces and CW-complexes, we discuss fundamental groups and the Galois correspondence for covering spaces. We then move on to homology theory and study simplicial and singular homology, as well as some applications like the Jordan curve theorem and invariance of domain. Throughout the module, we pay special attention to algebraic and categorical aspects.

**Learning Outcomes**

On successful completion of this module, you will be able to:

- define the basic invariants in algebraic topology and prove their main properties;
- use algebraic techniques to distinguish different homotopy types and classify topological objects;
- compute fundamental groups, simplicial homology groups and singular homology groups;
- apply the Galois correspondence to classify covering spaces of topological spaces;
- apply fundamental groups and homology groups to prove fundamental topological properties (Brouwer's fixed point theorem, Jordan's curve theorem, invariance of domain);
- formulate topological and algebraic constructions in a categorical language (universal properties); analyze the structure of quotients of topological spaces by covering space actions;
- represent groups geometrically by means of Cayley complexes

**Module Content**

An indicative list of sections and topics is:

Preliminaries:

1. Homotopy and homotopy type
2. Cell complexes
3. Operations on spaces The Fundamental Group:
4. Paths and Homotopy
5. Presentations of groups, amalgamated products and Van Kampen's Theorem
6. Covering Spaces
7. The Galois correspondence -Deck Transformations and Group Actions -Cayley complexes

Homology

8. Δ-complexes and simplicial homology
9. Singular homology
10. Homotopy invariance
11. Relative homology, exact sequences and excision
12. The equivalence of simplicial and singular homology
13. Mayer-Vietoris Sequences
14. Applications .

The main reference for this course is "Algebraic topology" by Hatcher.

This Level 7 version of the module will involve extra self-study of extension material (in the form of a book chapter, additional notes or a research paper) applying or extending material from the above topics.

*The module will assume familiarity with the material in the second-year modules: Groups and Rings, Analysis II*

## MATH70035: Algebra 3

### Brief Description

This course continues the study of commutative rings and introduces the notion of R-module, which is an analogue over rings of the notion of a vector space over a field. Using these ideals we prove fundamental results about various classes of rings, particularly polynomial rings in several variables.

### Learning Outcomes

On successful completion of this module, you will be able to:

- Understand the detailed theory of finite fields, their classification, and factorization of polynomials over finite fields
- Understand the theory of R-modules and their presentations
- Understand the classification of modules over Euclidean Domains, and how to use Smith Normal Form to determine the isomorphism class of such a module given a presentation
- Use this classification, in the case of K[T]-modules, to prove fundamental results in linear algebra
- Apply several different criteria for irreducibility of polynomials over various base rings
- Demonstrate competence with further advanced material in the area designated for self-study
- Synthesize material from across the module to apply to advanced topics

### Module Content

An indicative list of sections and topics is:

1. Chinese Remainder Theorem
2. Field Extensions and Finite Fields
3. R-modules

4. Free modules and presentations
5. Modules over Euclidean Domains
6. Noetherian rings
7. Gauss's Lemma and Factorization in polynomial rings
8. If R is a UFD, so is R[X]
9. Irreducible Polynomials and factorization of polynomials

This Level 7 version of the module will involve extra self-study of extension material (in the form of a book chapter, additional notes or a research paper) applying or extending material from the above topics.

## MATH70036: Group Theory

### Brief Description

This module builds on the Group Theory from the 1st year module Linear Algebra & Groups and the 2nd year module Groups and Rings. We start with a discussion of isomorphism theorems, and proceed to further example of groups and operations on them, including automorphism groups and semidirect products. Special attention is given to group actions and permutation groups: primitivity, multiple transitivity etc. Further we discuss solvable and nilpotent groups and their characterizations.

### Learning Outcomes

On successful completion of this module, you will be able to:

- identify features of, and develop arguments about, certain groups and classes of group;
- explain the principles of group actions, and work with elementary examples;
- construct and work with direct and semidirect products of groups;
- state the definition of and extra special group and construct small examples of them;
- construct certain classical series of doubly and triply transitive groups;
- state and prove the structure theorem for nilpotent groups;
- explain the principles of group actions, and work with elementary examples;
- determine the normal structure and calculate the automorphism groups of symmetric groups;
- work independently and with peers to articulate understanding of abstract concepts in algebra.
- demonstrate competence with further advanced material in the area designated for self-study
- Synthesize material from across the module to apply to advanced topics

### Module Content

An indicative list of sections and topics is:

1. Definition and basic properties of groups. Isomorphism Theorems. Sylow subgroups. Group actions, primitivity and multiple transitivity. Composition series. Nilpotent groups. Solvable groups. Symmetric groups.
2. Automorphism group of a group and semidirect products. Linear groups: centers and commutator subgroups, with small examples.
3. Further advanced material on these topics will be set for self-study.

## MATH70037: Galois Theory

### Brief Description

The formula for the solution to a quadratic equation is well-known. There are similar formulae for cubic and quartic equations but no formula is possible for quintics. The module explains why this happens.

**Learning Outcomes**

On successful completion of this module you should be able to:

- state, prove, and apply the fundamental theorem of Galois theory (aka the "Galois correspondence").
- work with simple examples such as cubic polynomials, cyclotomic polynomials, and polynomials over finite fields.
- compute Galois groups of splitting fields of cubic and bi-quadratic polynomials in arbitrary characteristic.
- state and apply the formulas for solving cubic and quartic equations, and to prove that there are no such formulas for equations of degree 5 or larger.
- compute Galois groups over the rational by the method of Frobenius elements.
- Demonstrate additional competence in the subject through the self-study of designated advanced material
- Combine topics from across the module to obtain more advanced results

**Module Content**

- Irreducible polynomials.
- Field extensions, degrees and the tower law.
- Extending embeddings.
- Normal field extensions, splitting fields, separable extensions.
- Groups of automorphisms, fixed fields.
- The fundamental theorem of Galois theory.
- Finite fields, cyclotomic extensions.
- Extensions of the rationals and Frobenius elements.
- The solubility of polynomials of degree at most 4 and the insolubility of quintic equations.

Material for self-study (in the form of a book chapter, additional notes or a research paper) applying or extending material from the above topics.

## MATH70038: Graph Theory

**Brief Description**

A graph is a structure consisting of vertices and edges. Graphs are used in many areas of Mathematics, and in other fields, to model sets with binary relations. In this module we study the elementary theory of graphs; we discuss matters such as connectivity, and criteria for the existence of Hamilton cycles. We treat Ramsey's Theorem in the context of graphs, with some of its consequences. We then discuss probabilistic methods in Graph Theory, and properties of random graphs.

**Learning Outcomes**

On successful completion of this module, you will be able to:

- Demonstrate facility with the terminology of graphs and simple graph constructions to analyse examples and prove results
- Explain the proofs of the theorems of König and Menger, and certain other related results. Apply these results to appropriate problems.
- State, prove and apply Turán's Theorem. Describe and apply certain results in the theory of Hamilton cycles, including Dirac's Theorem.

- Explain and reason about Ramsey's Theorem and related results in the context of graph colourings.
- Describe various models of random graphs and apply probabilistic arguments to situations in graph theory.
- Demonstrate competence with further advanced material in the area designated for self-study
- Synthesize material from across the module to apply to advanced topics

**Module Content**

An indicative list of sections and topics is:

1. Standard definitions and basic results about graphs. Common graph constructions: complete graphs, complete bipartite graphs, cycle graphs.
2. Matchings and König's Theorem. Connectivity and Menger's Theorem.
3. Extremal graph theory. The theorems of Mantel and Turán. Hamilton cycles, and conditions for their existence.
4. Ramsey Theory for graphs, with applications.
5. The Probabilistic Method and random graphs. Evolution of random graphs.

This Level 7 version of the module will involve extra self-study of extension material (in the form of a book chapter, additional notes or a research paper) applying or extending material from the above topics.

## MATH70039: Group Representation Theory

**Brief Description**

This module defines and begins the study of representations of groups, focusing on finite-dimensional complex representations of finite groups. These structures encode ways that groups can act as symmetries, which appear throughout mathematics (notably in algebra, number theory, and geometry, but also in analysis) as well as in physics and chemistry, among other places. We explain how to understand and classify these representations through characters, or traces. In the final unit we generalise the theory to finite-dimensional modules over rings, particularly semisimple algebras, whose theory retains many of the features of that of representations of groups.

**Learning Outcomes**

On successful completion of this module you will be able to:

- Recall and use basic definitions in group representations, their character theory, and modules over algebras, particularly finite-dimensional semisimple algebras;
- Explain and work with the features of complex representations of finite groups that allow one to simplify the theory (e.g., semisimplicity, character tables, etc.);
- Apply these results to classify representations of finite groups and semisimple algebras and compute the character tables of finite groups;
- perform basic constructions of representations of groups and to apply them to obtain all finite-dimensional representations of certain basic groups up to isomorphism;
- Explain the relationship between finite-dimensional irreducible representations of algebras and of finite- dimensional semisimple algebras, and the basic properties of their characters;
- Relate endomorphisms of representations to central elements in groups and semisimple algebras;
- Work independently and with peers to formulate and solve problems in algebra and geometry using tools of representation theory;
- Demonstrate ability to engage with more advanced material via self-study.
- Combine material from across the module to address more challenging problems

**Module Content**

1. Basic theory: definitions, Maschke's theorem, Schur's Lemma, classification and construction of representations of finite abelian groups, dihedral groups, and small symmetric and alternating groups;
2. Tensor products of representations and homomorphism spaces, the regular representation;
3. Character theory: behaviour under direct sums and tensor products, orthogonality relations, computation of character tables of certain groups;
4. Finite-dimensional modules over algebras: definition of finite-dimensional semisimple algebras and relationship of finite-dimensional modules of general algebras
5. to those of finite-dimensional semisimple algebras; constructions of projections via the center;
6. Other important examples of modules over algebras.
7. Further material, related to the above but of a more advanced character (consisting of a book chapter or research paper), will be designated for self-study.

## MATH70040: Formalising Mathematics

**Brief Description**

Computer theorem provers are mature enough now to tackle most undergraduate level mathematics, and also some much harder level mathematics. As these systems evolve, they will inevitably become useful as tools for researchers, and some believe that one day they will start proving interesting theorems by themselves. This project-assessed course is an introduction to the Lean theorem prover and during it we will learn how to formalise proofs of undergraduate and masters level theorems from across pure mathematics.

**Learning Outcomes**

On successful completion of this module you will be able to:

- understand the basics of how type theory can be used as a foundation for pure mathematics;
- understand how to "modularise" mathematical arguments, breaking them up into simple chunks, thus leading to clarity of understanding;
- understand how to "abstract" mathematical arguments, finding the correct generality in which statements should be made, thus again leading to clarity of understanding;
- state and prove many results from undergraduate and masters level pure mathematics modules in the Lean theorem prover;
- develop mathematical theories of your own in the Lean theorem prover;
- write formal proofs of theorems which other mathematicians can understand and follow
- understand how to turn more advanced mathematics into statements of dependent type theory.

**Module Content**

Note that the aim is to both learn the mathematics and to learn how to teach it to a computer. No experience in programming will be assumed. Lean is a functional programming language, so we will be picking up functional programming along the way. If you want to get a feeling for the kind of coding which will be involved, try playing the natural number game.

The following is an indicative list of areas where the mathematics could be drawn from:

1. Logic, functions, sets.
2. Lattice theory, complete lattices, Galois insertions and Galois connections. Examples in mathematics.

3. Groups and subgroups.
4. Closure operators in group theory and topology.
5. Filters as generalised subsets. Filters form a complete lattice.
6. Applications of filters to topology. New proofs of basic results in topology. Tychonoff's theorem.
7. Application of filters to analysis. New proofs of basic results in analysis.
8. What is cohomology? Group cohomology in low degree.

This level 7 version of the module will involve extension material and more advanced examples than the level 6 version.

## MATH70041: Number Theory

### Brief Description

The module is concerned with properties of natural numbers, and in particular of prime numbers, which can be proved by "elementary" methods (such as basic group theory ring theory).

### Learning Outcomes

On successful completion of this module, you will be able to:

- form arguments about and solve congruences, particular modulo primes, and apply this to the RSA algorithm;
- compute with quadratic residues;
- solve some particular Diophantine equations, including Pell's equation;
- compute continued fractions;
- construct transcendental numbers.
- explain and demonstrate mastery of such further material as is selected by the module leader for self-study.
- Combine material from across the module to solve more advanced problems

### Module Content

An indicative list of topics is:

1. Euclid's algorithm, unique factorization, linear congruences, Chinese Remainder Theorem.
2. The structure of $(Z/nZ)^\times$, including the Fermat-Euler theorem, Lagrange's theorem, the existence (and non-existence) of primitive roots.
3. Primality testing, factorization, and the RSA algorithm (including the basics of the Miller-Rabin test).
4. Quadratic reciprocity, Legendre symbols, Jacobi symbols.
5. Sums of 2 and 4 squares, using unique factorization in the Gaussian integers.
6. Pell's equation, existence of solutions via Dirichlet's theorem. Continued fractions, periodicity for quadratic irrationals, algorithm for solving Pell's equation via continued fractions.
7. Irrationality, Liouville's theorem, construction of a transcendental number.
8. Elementary results on primes in arithmetic progressions.

Other topics of the lecturer's choice as time permits, e.g. quadratic forms; Möbius inversion and Dirichlet Convolution; the Selberg sieve; particular examples of Diophantine equations.

Mastery material selected for further self-study, to be based on written material such as a textbook exerpt or research paper.

## MATH70042: Algebraic Number Theory

### Brief Description

An introduction to algebraic number theory using quadratic fields as the main source of examples. We will study rings of integers in finite extensions of the rational and discuss unique factorization and its failure, the decomposition of primes, the finiteness of the ideal class group, and Dirichlet's unit theorem.

### Learning Outcomes

On successful completion of this module you will be able to:

- explain how unique factorization domains, principal ideal domains and Euclidean domains are related.
- give examples of rings of integers in quadratic fields that are Euclidean domains and also counter-examples.
- define a Dedekind domain and explain why rings of integers in number fields are Dedekind domains.
- write a basis for the rings of integers in any given quadratic number field.
- explain what it means for a prime to be split, inert or ramified in an extension and, given a quadratic ring of integers and a prime, you will be able to say what happens to that prime.
- explain why the class group in a number field is finite and compute examples of class groups of quadratic number fields.
- state Dirichlet's unit theorem and to describe explicitly the group of units in a real or imaginary quadratic field.
- show mastery of more advanced material on these topics which will be set for self-study
- synthesize arguments from different parts of the module to solve more advanced problems.

### Module Content

An indicative list of topics is as follows.

We will review / introduce some background from ring theory, discuss unique factorization domains, principal ideal domains and Euclidean domains. We will study Gaussian and Eisenstein integers in detail and see several other examples of quadratic rings of integers. We will then discuss the structure theorem for finitely generated abelian groups, the notion of integral closure, Dedekind domains and study the ideal class group. We will prove that the ideal class group in a number field is finite and compute many examples. We will study the decomposition of primes in number fields and in quadratic fields in particular. We will end by discussing Dirichlet's unit theorem.

Furthermore advanced material (such as a book chapter or research paper) will be set by the module leader for independent study.

The module will assume familiarity will some topics in algebra, such as commutative rings and modules.

## MATH70055: Stochastic Calculus with Applications to non-Linear Filtering

### Brief Description

The module offers a bespoke introduction to stochastic calculus required to cover the classical theoretical results of nonlinear filtering. The first part of the module will equip the students with the necessary knowledge (e.g., Ito Calculus, Stochastic Integration by Parts, Girsanov's theorem) and skills (solving linear stochastic differential equation, analysing continuous martingales, etc) to handle a variety of applications. The focus will be on the use of stochastic calculus to the theory and numerical solution of nonlinear filtering.

**Learning Outcomes**

On successful completion of this module, you will be able to:

- understand the notion on Brownian motion and able to show that a stochastic process is a Brownian motion,
- Prove that a process is a martingale via Novikov's condition.
- Solve linear SDEs,
- Be able to check whether an SDE is well-posed.
- understand the mathematical framework of nonlinear filtering
- Deduce the filtering equations.
- Deduce the evolution equation of the mean and variance of the one-dimensional Kalman-Bucy filter,
- Show that the innovation process is a Brownian motion.
- Apply stochastic integration by parts.

**Module Content**

An indicative list of topics is:

1. Martingales on Continuous Time (Doob Meyer decomposition, L_p bounds, Brownian motion, exponential martingales, semi-martingales, local martingales, Novikov's condition)
2. Stochastic Calculus (Ito's isometry, chain rule, integration by parts)
3. Stochastic Differential Equations (well posedness, linear SDEs, the Ornstein-Uhlenbeck process, Girsanov's Theorem)
4. Stochastic Filtering (definition, mathematical model for the signal process and the observation process)
5. The Filtering Equations (well-posedness, the innovation process, the Kalman-Bucy filter)

*Prerequisites: Ordinary differential equations, partial differential equations, real analysis, probability theory.*

## MATH70056: Algebraic Geometry

**Brief Description**

Algebraic geometry is the study of the space of solutions to polynomial equations in several variables. In this module you will learn to use algebraic and geometric ideas together, studying some of the basic concepts from both perspectives and applying them to numerous examples.

**Learning Outcomes**

On successful completion of this module, you will be able to:

- Understand the dictionary between algebra and geometry that arises from zero loci of polynomials in n- dimensional space;
- Understand and compute irreducible and connected components of such zero loci;
- Understand the concepts of regular and rational maps and their algebraic and geometric meaning;
- Understand the projective space and the role it plays in compactifying zero loci of polynomials;
- Understand the notion of dimension of zero loci of polynomials and their behaviour under regular maps;
- Apply dimension theory and the Zariski topology in examples such as those coming from parameter spaces;
- Understand how to generalise these ideas to the setting of more general commutative rings via (maximal) spectra (Mastery)

**Module Content**

An indicative list of topics is:

1. Affine varieties, projective varieties. The Nullstellensatz.
2. Regular and rational maps between varieties. Completeness of projective varieties. Dimension. Parameter spaces.
3. Examples of algebraic varieties.
4. Spectrum and maximum spectrum (mastery)

*Prerequisites: Commutative Algebra*

## MATH70057: Riemannian Geometry

**Brief Description**

The main aim of this module is to understand geodesics and curvature and the relationship between them. Using these ideas we will show how local geometric conditions can lead to global topological constraints.

**Learning Outcomes**

On successful completion of this module, you will be able to:

- Understand the relevant structures required to make sense of differential topological notions, such as derivatives of smooth functions, and geometric notions, such as lengths and angles, on an abstract manifold.
- Define the Lie derivative and covariant derivative of a tensor field.
- Define geodesics and understand their length minimising properties.
- Define and interpret various measures of the curvature of a Riemannian manifold.
- Understand the effect of curvature on neighbouring geodesics.
- Prove the celebrated classical theorems of Bonnet--Myers and Cartan--Hadamard.

**Module Content**

An indicative list of topics is:

Topological and smooth manifolds, tangent and cotangent spaces, vector bundles, tensor bundles, Lie bracket, Lie derivative, Riemannian metrics, affine connections, the Levi-Civita connection, parallel transport, geodesics, Riemannian distance, the exponential map, completeness and the Hopf--Rinow Theorem, Riemann and Ricci curvature tensors, scalar curvature, sectional curvatures, submanifolds, the second fundamental form and the Gauss equation, Jacobi fields and the second variation of geodesics, the Bonnet--Myers and Cartan--Hadamard Theorems.

*Prerequisites: Geometry of Curves and Surfaces and Manifolds*

## MATH70058: Manifolds

**Brief Description**

The goal of this course is to introduce the theory of smooth manifolds. The class starts by defining smooth manifolds, submanifolds and tangent spaces. It will then develop more advanced topics like the theory of vector bundles, which will be used to introduce the notion of the tangent bundle, the cotangent bundle, vector fields and differential forms on a smooth manifold. This allows to define integration on an orientable manifold and then to prove Stokes' Theorem on a manifold with boundary.

**Learning Outcomes**

On successful completion of this module, you will be able to:

- Define smooth manifolds in an intrinsic way, by using the notion of charts, transition functions and smooth atlases.
- Determine sufficient conditions under which the level set of a smooth function is a submanifold.
- Study vector bundles on a manifold and determine necessary and sufficient condition for a vector bundle to be trivial.
- Study vector fields on a manifold and describing them locally, through the use of charts.
- Define the integration of differential forms on an orientable manifold.
- Prove Stokes' Theorem, which is one of the main tools used in differential topology.

**Module Content**

This module focuses on foundations as well as examples.

An indicative list of topics is:

Smooth manifolds, quotients, smooth maps, submanifolds, rank of a smooth map, tangent spaces, vector fields, vector bundles, differential forms, the exterior derivative, orientations, integration on manifolds (with boundary) and Stokes' Theorem.

## MATH70059: Differential Topology

**Brief Description**

In this module you will understand how geometry and topology interact on smooth manifolds. You will investigate different (co)homology theories, see how to relate them, and study how to use them to analyse the topology of a manifold.

**Learning Outcomes**

On successful completion of this module you will be able to:

- Apply the concepts of homology and cohomology, as well as central results such as Poincaré Duality and the De Rham theorem, to investigate manifolds.
- Use a Mayer-Vietoris argument to compute (co)homology groups.
- Describe the topology of a manifold by analysing the critical points of a Morse function and the gradient flow lines between them.
- Use and explain the equivalence between the different (co)homology theories introduced (De Rham, singular, Morse), for example how the CW complex of a manifold relates to the homology groups generated by the critical points of Morse functions on the manifold.
- Work independently and with peers to formulate and solve problems in geometry using tools of algebraic and differential topology.

Module Content

An indicative list of contents is:

1. De Rham Cohomology: Definition, Poincaré's Lemma, Mayer-Vietoris sequences, compactly supported de Rham cohomology, pairings and Poincaré Duality with applications, degree of a map, mapping degree theorem and examples.
2. Morse Theory: Introduction and basics, Fundamental Theorems of Morse Theory, the CW-structure associated to Morse-functions, stable and unstable manifolds, Morse-Smale functions, orientations, Morse homology and Morse Homology Theorem. Examples.

3. Singular Homology: Basic definitions, properties and examples, De Rham Theorem.

The module will assume familiarity with topics in Algebraic Topology and smooth manifolds. In particular students should be familiar with: Vector fields, differential forms (k-forms, exterior differential, closed and exact forms), integration on manifolds and Stokes' Theorem, basics of homological algebra (exact sequences, Snake Lemma).

## MATH70060: Complex Manifolds

### Brief Description

The goal of this course is to introduce the theory of almost complex manifolds and complex manifolds. Many important examples will be provided, such as Kähler manifolds and complex projective manifolds. After introducing some of the main tools, as the Hermitian metrics, the Chern connection and the co-homology of a complex manifold, the theory of Hodge decomposition for Kähler manifolds will be presented, together with many of its applications. The class will culminate with the Kodaira embedding theorem and with the main notions of Kodaira-Spencer deformation theory.

### Learning Outcomes

On successful completion of this module, you will be able to:

- Study many examples of complex and almost complex manifolds, such as Hopf manifolds, projective spaces, Kähler manifolds, and projective varieties.
- Introduce tools like Hermitian metrics, holomorphic vector bundles and Chern connections on a complex manifold.
- Study harmonic forms on a complex manifolds and then the Dalbout and the de Rham co-homology of a Kähler manifold, culminating with the Hodge decomposition theorem and several of its applications.
- Use holomorphic line bundles to study the Kodaira embedding theorem, which provides a characterisation of complex projective manifolds.
- Introduce the basic notions of the Kodaira-Spener deformation theory.

### Module Content

An indicative list of topics is:

Complex and almost complex manifolds, integrability. Examples such as the Hopf manifold, projective space, projective varieties. Hermitian metrics, Chern connection. Various equivalent formulations of the Kaehler condition. Hodge decomposition for Kaehler manifolds. Line bundles and Kodaira embedding. Statement of GAGA. Basic Kodaira-Spencer deformation theory.

*Prerequisite: Manifolds*

## MATH70061: Commutative Algebra

### Brief Description

This module is an introduction to commutative algebra which is the modern foundation of algebraic geometry and algebraic number theory. First we will cover such basic notations as prime and maximal ideals, the nilradical and the Jacobson radical. Then we study the very important construction of localisation both for rings and modules over them. We will apply these to a variety of results, for example primary decomposition of ideals and structure theorems for Artinian rings and discrete valuation rings.

**Learning Outcomes**

On successful completion of this module, you will be able to:

- define basic notions in commutative algebra and prove their main properties;
- use localisation to relate properties of rings, ideals, modules and morphisms between them with properties of their localisations;
- apply various chain conditions to prove properties of rings and modules satisfying these;
- use other standard arguments in commutative algebra;

**Module Content**

This module is an introduction to commutative algebra which is the modern foundation of algebraic geometry and algebraic number theory. First we will cover such basic notations as prime and maximal ideals, the nilradical and the Jacobson radical. Then we study the very important construction of localisation both for rings and modules over them. We will apply these to a variety of results, for example primary decomposition of ideals and structure theorems for Artinian rings and discrete valuation rings.

An indicative list of topics is:

Prime and maximal ideals, nilradical, Jacobson radical, localization. Modules. Primary decomposition of ideals. Applications to rings of regular functions of affine algebraic varieties. Artinian and Noetherian rings, discrete valuation rings, Dedekind domains. Krull dimension, transcendence degree. Completions and local rings. Graded rings and their Poincaré series.

After this module, you should be equipped to undertake a course in modern algebraic geometry.

## MATH70062: Lie Algebras

**Brief Description**

This module is an introduction to theory of complex Lie algebras and it culminates in the classification of finite dimensional semisimple complex Lie algebras in terms of root systems. It is completely self-contained, and only relies on a good understanding of linear algebra. However the proofs are quite intricate. It is also a good preliminary to the theory of Lie groups and algebraic groups, which study closely related objects, but the latter require much heavier machinery.

**Learning Outcomes**

On successful completion of this module, you will be able to:

- define basic notions of Lie algebras, such as ideals, derived series and lower central series,
- prove Engel's and Lie's theorem, and apply them in various contexts,
- prove and apply the additive Jordan decomposition theorem,
- define the Killing form and prove Chevalley's criteria,
- define Cartan and Borel subalgebras and prove their main properties,
- apply the above to complete the proof of the classification theorem of semisimple Lie algebras,
- construct explicitly the classical simple Lie algebras,
- work effectively with roots systems.

**Module Content**

An indicative list of topics is:

The semisimple complex Lie Algebras: root systems, Weyl groups, Dynkin diagrams, classification. Cartan and Borel subalgebras. Classification of irreducible representations.

## MATH70063: Algebra 4

### Brief Description

This module is a course of homological algebra. The main result is the existence of derived functors in the category of modules over an associative ring. We cover functors Ext and Tor in greater detail, particularly in the category of abelian groups. We define and study some basic properties of group cohomology.

### Learning Outcomes

On successful completion of this module, you will be able to:

- identify features of, and develop arguments about, rings, modules over rings and homomorphisms between them;
- understand the definition of the tensor product of module and use it in a variety of settings;
- define free, injective, projective, flat modules, state and prove their basic properties;
- state, apply, and explain the proof of the theorem about the existence of derived functors in the category of modules over a ring;
- understand and explain the relation between Ext and extensions of modules;
- compute functors Tor and Ext in specific situations, particularly in the category of abelian groups;
- understand and use for computation the explicit construction of the first and second cohomology groups, as well as the cohomology groups of a cyclic group.
- demonstrate capacity for independent study of an advanced topic to be specified by solving a range of problems.

### Module Content

An indicative list of sections and topics is:

1. Modules over rings: free, projective, injective, flat; tensor product
2. Functors Hom, Ext, Tor. General definition of a derived functor. Long exact sequences. Injective and projective resolutions. Homotopy.
3. Group cohomology via homogeneous and inhomogeneous cochains. The case of cyclic groups.
4. Mastery material for self-study, on advanced material relating to the topics above.

## MATH70064: Elliptic Curves

### Brief Description

An elliptic curve is an algebraic curve in two variables defined by an equation of the form $y^2=x^3+ax+b$. Elliptic curves play an important role in Number Theory, and have been central to many recent advances, such as the proof of Fermat's Last Theorem. In this course we study the theory of elliptic curves and their connections with Number Theory, Geometry and Algebra.

### Learning Outcomes

On successful completion of this module, you will be able to:

- solve equations in the p-adic numbers;
- find all rational points on plane conics;
- compute with the group law on an elliptic curve;
- compute the torsion subgroup of an elliptic curve over Q;
- compute the rank of an elliptic curve over Q;

- demonstrate mastery of further advanced material selected for self-study by applying it in a variety of problems.

**Module Content**

An indicative list of topics is:

1. The p-adic numbers. Curves of genus 0 over Q .
2. Cubic curves and curves of genus 1. The group law on a cubic curve.
3. Elliptic curves over p-adic fields and over Q. Torsion points and reduction mod p.
4. The weak Mordell-Weil theorem. Heights.
5. The (full) Mordell-Weil theorem.

## MATH70132: Mathematical Logic

**Brief Description**

The module is concerned with some of the foundational issues of mathematics: formal logic and set theory. In propositional and predicate logic, we analyse the way in which we reason formally about mathematical structures. In set theory, we will look at the ZFC axioms and use these to develop the notion of cardinality. Material on model theory will involve all of these topics.

**Learning Outcomes**

On successful completion of this module, you should be able to:

- Understand how the notion of truth is defined precisely in propositional and predicate logic and apply the definitions and accompanying results in a variety of contexts.
- Demonstrate understanding of formal systems for propositional and predicate logic by constructing examples of formal proofs and deductions, and by applying and deriving general results about these.
- Appreciate the expressive power of a first-order language (and its limitations) and compare structures via their first-order theories.
- Relate the semantic and syntactic aspects of formal logic, have an understanding of powerful general results such as the completeness and compactness theorems, and be able to apply these in a variety of ways.
- Use the ZFC axioms to justify constructionsin set theory, ranging from elementary applications, to constructions involving transfinite recursion, ordinals, cardinals and applications of these.
- Use general results to compute and compare cardinalities of infinite sets.
- Communicate your knowledge of the area in a concise, accurate and coherent manner.
- Combine your knowledge of predicate logic and set theory in the study of model theory and apply the results to deepen your understanding of theories of first-order structures.

**Module Content**

The module is concerned with some of the foundational issues of mathematics. In propositional and predicate logic, we analyse the way in which we reason formally about mathematical structures. In set theory, we will look at the ZFC axioms and use these to develop the notion of cardinality. These topics have applications to other areas of mathematics: formal logic has applications via model theory and ZFC provides an essential toolkit for handling infinite objects.

An indicative list of sections and topics is:

1. Propositional logic: Formulas and logical validity; a formal system; soundness and completeness.
2. Predicate logic: First-order languages and structures; satisfaction and truth of formulas; the formal system; Goedel's completeness theorem; the compactness theorem; the Loewenheim- Skolem theorem.
3. Set theory: The axioms of ZF set theory; ordinals; cardinality; the Axiom of Choice.
4. Model theory: Elementary substructures; the method of diagrams, the Tarski-Vaught test; the general Loewenheim- Skolem Theorems; Reduced products and ultraproducts; Los' theorem.

There are no formal prerequisites for the module although a level of mathematical understanding such as would be provided by a second year algebra or analysis module, together with an appetite for abstraction and proofs, will be assumed. We will use basic notions from algebra (groups, rings and vector spaces) in examples.

## MATH70135: Advanced Partial Differential Equations 1

### Brief Description

The focus of this course is on the concepts and techniques for solving partial differential equations (PDEs) that permeate various scientific disciplines. It is designed for a diverse audience in pure and applied mathematics, emphasizing rigor and the development of analytical proofs and techniques. The course places a strong emphasis on the theory of weak solutions of elliptic and parabolic equations and their regularity, involving distributions, Sobolev spaces, and the calculus of variations.

### Learning Outcomes

On successful completion of this module you should be able to:

- demonstrate an overview of a variety of partial differential equations, the behaviour of their solutions, and techniques to study them.
- understand and communicate some of the deep connections of PDEs to physics and geometry.
- state and prove well-posedness theorems for a variety of PDEs and explain their relevance.
- apply elliptic regularity theory in the theory of elliptic PDEs.
- apply suitable techniques to hyperbolic equations (wave equations).
- recognise and engage with current research in PDEs

### Module content

An indicative list of topics is:

1. Introduction, examples of PDEs that appear in applications. Elliptic, parabolic, hyperbolic PDEs. Course overview.
2. Distributions: definitions and examples, convergence and differentiability, support and convolution.
3. H¨older and Sobolev spaces: definitions and examples, approximation, traces, compactness (Rellich-Kondrachov Theorem), Sobolev inequalities (Gagliardo-Nirenberg-Sobolev, Morrey, Poincar´e), duality. Spaces involving time.
4. Elliptic PDEs: Basic existence-uniqueness theory: Strong/uniform ellipticity; weak formulation;
5. Lax-Milgram; energy estimates. Elliptic regularity theory: Difference quotients; interior and boundary regularity. Elements of calculus of variations.
6. Parabolic PDEs: initial boundary value problems, weak formulation, Galerkin method. Parabolic Regularity theory. Maximum principle, Harnack's inequality. Semigroups, Hille-Yosida theorem.

## MATH70140: Geometric Complex Analysis

### Brief Description

In this module we look at the subject of complex analysis from a more geometric point of view. We shall look at geometric notions associated with domains in the plane and their boundaries, and how they are transformed under holomorphic mappings. In turn, the behaviour of conformal maps is highly dependent on the shape of their domain of definition.

### Learning Outcomes

On successful completion of this module, you will be able to:

- identify features of, and develop arguments about, certain holomorphic maps;
- state, apply, and explain aspects of the Riemann Mapping Theorem for arbitrary simply connected plane domains;
- explain the automorphisms of the disk and the upper half plane;
- explain hyperbolic geometry, and basic notions of length, geodesics, isometries;
- apply the area theorem, and derive distortion estimates for arbitrary conformal mappings;
- acquire deeper understanding of holomorphic mappings through generalisation to quasi-conformal mappings;
- appreciate significance of and apply universal bounds in geometric function theory;
- explain the statement of the Beltrami-equation, and generalisation of the Riemann mapping theorem;
- work independently and with peers to understand abstract concepts in complex analysis.

### Indicative Module Content

Complex analysis is the study of the functions of complex numbers. It is employed in a wide range of topics, including dynamical systems, algebraic geometry, number theory, and quantum field theory, to name a few. On the other hand, as the separate real and imaginary parts of any analytic function satisfy the Laplace equation, complex analysis is widely employed in the study of two-dimensional problems in physics such as hydrodynamics, thermodynamics, Ferromagnetism, and percolations.

In this module we look at the subject of complex analysis from a more geometric point of view. We shall look at geometric notions associated with domains in the plane and their boundaries, and how they are transformed under holomorphic mappings. In turn, the behaviour of conformal maps is highly dependent on the shape of their domain of definition.

An indicative list of topics is:

1. Schwarz lemma and automorphisms of the disk,
2. Riemann sphere and rational maps,
3. Conformal geometry on the disk, Poincare metric, Isometries, Hyperbolic contractions,
4. Conformal Mappings, Conformal mappings of special domains, Normal families, Montel's theorem, General form of Cauchy integral formula, Riemann mapping theorem,
5. Growth and Distortion estimates, Area theorem,
6. Quasi-conformal maps and Beltrami equation, Linear distortion, Dilatation quotient, Absolute continuity on lines, Quasi-conformal mappings, Beltrami equation, application of MRMT,

*Prerequisites: It will be helpful if you have taken (or are taking) one or more of the following courses: Functional Analysis, Measure and Integration (or Lebesgue Measure and Integration), Fourier Analysis and Distributions.*