Imperial College London

DEPARTMENT OF MATHEMATICS

GUIDE TO OPTIONAL MODULES

FOURTH/FINAL YEAR (MSci)
2022-23

Notes and syllabus details on Fourth Year modules for students in their Fourth/Final Year

For degree codings:

G103	MATHEMATICS (BSc, MSci)
G104	MATHEMATICS WITH A YEAR ABROAD (MSci)

Joint MATHEMATICS AND COMPUTER SCIENCE programmes are administered by the Department of Computing.

These notes should be read in conjunction with your undergraduate student handbook and the programme specifications for your year. Some of the information may be subject to alteration.

Updated information will be posted on the Maths Central Blackboard site.

It is our current expectation that teaching and assessment will be delivered in person, with unrestricted room capacities. This is, of course, subject to change, where we receive advice from the College or the Government.

Dr Christopher Hallsworth, Mathematics DUGS
28 June 2022.

FOURTH YEAR OVERVIEW

The MSci Fourth Year is available to those on the G103 and G104 codings who perform to a satisfactory standard in their Third Year, here or abroad. There is considerable overlap with the taught postgraduate MSc programmes in Pure and Applied Mathematics, but the MSci is a separate degree.

The MSci programme is designed to provide a breadth and depth in mathematics to a level of attainment broadly equivalent to that of an MSc degree and takes place over three terms – Term 1 (also known as Autumn Term), Term 2 (also known as Spring Term) and Term 3 (also known as Summer Term).

Students choose six lectured modules from those made available to them in the Department and from certain modules elsewhere. Students also take the compulsory MSci project, which is equivalent to two lecture modules.

Most, but not all, of the MATH7 modules are also available in MATH6 form and 4th year students take the MATH7 version. Fourth Year examinations normally consist of 5 questions and are 2.5 hours long, whereas the corresponding exams for 3rd year students (if any) contain 4 questions in 2 hours.
Students may not take an MATH7 module if they have already taken the MATH6 version.

Lecturing will take place during Term 1 and Term 2. Each module will typically have three hours per week, which usually includes some classes. The normal expectation is that there should be a 'lecture'/'class' balance of about 5/1. The identification of particular class times within the timetabled periods is at the discretion of the lecturer, in consultation with the class and as appropriate for the module material.

ADVICE ON THE CHOICE OF OPTIONS

Students are advised to read these notes carefully and to discuss their option selections with their Personal Tutor. An `Options Fair' will take place after exams in the Summer Term, where staff will answer questions on all available options.

It is anticipated that lecturers will give advice on suitable books at the start of each module. Students should contact the proposed lecturers if they desire any further details about module content in order to make their choice of course options.

You will not be committed to your choice of most optional modules until the completion of your examination entry at the beginning of Term 2. The exception to this is that students do become committed to the completion of certain modules examined only by project at some stage during the module, as will be made clear by the lecturer at the start of the module.

MSci PROJECT

M4R ADVANCED RESEARCH PROJECT IN MATHEMATICS
Compulsory

Supervised by Various Academic Staff
Co-ordinator: Dr D. Helm
(Terms 1, 2 & 3)

A fundamental part of the MSci degree is a substantial compulsory project equivalent to <u>two</u> lecture modules. The main aim of this module is to give a deep understanding of a particular area/topic by means of a supervised project in some area of mathematics. The project may be theoretical and/or computational and the area/topic for each student is chosen in consultation with the Department.

The project provides an excellent *'apprenticeship in research'* and is therefore of particular value to students who are considering postgraduate study leading to a PhD.

Arrangements for this project will be set in motion after the Third Year examinations. **Students should approach potential supervisors in an area of interest before the end of their Third Year** and some preparatory work can be performed over the vacation between the Third and Fourth Years. Work on the project should continue throughout all three terms of the Fourth Year and submitted shortly after the Fourth Year examinations.

G104: For those on a Maths with a year abroad coding, the third year is spent abroad at another university. G104 students should ideally negotiate with possible M4R supervisors by e-mail during their abroad, but this is not always possible. On return to Imperial, students take the regular Year 4 MSci programme. On the rare occasion that a G104 student performs very poorly in their year away they may, at the discretion of the Senior Tutor, be transferred to the BSc G100 Mathematics Degree and take MATH6 modules in their Final Year.

EXTERNAL MODULES

Subject to the Department's approval, students may take a module given outside the Department, e.g. in the Departments of Physics or Computing. Students must obtain permission from the Director of Undergraduate Studies if they wish to consider such an option. Where this permission is granted, it is always on the understanding that it is at the student's own risk. Students must satisfy themselves that they are comfortable with the methods and timing of assessments, and that external modules have an appropriate ECTS value.

GRADUATION

Students graduating will receive an MSci degree that explicitly incorporates a BSc.

It is normally required that MSci students pass all course components in order to graduate. However, the Board of Examiners may compensate narrowly failed modules up to 15 ECTS in the Final Year of study.

The total of marks for examinations, assessed coursework, progress tests, assignments and projects, with the appropriate year weightings, is calculated and presented at the Examiners' Meeting (normally held at the end of June) for consideration by the Academic Staff and External Examiners. Degree classifications are determined according to the criteria given in the programme specification and borderline classification algorithm, which can be found on Maths Central. You may also wish to consult the programme specification:

https://www.imperial.ac.uk/mathematics/undergraduate/course-structure-and-content/

MARKS, YEAR TOTALS AND YEAR WEIGHTINGS

Within the Department each total module assessment is rescaled so that overall performances in different modules may be compared. The rescaling onto the scale 0 – 100 marks is such that 50 then corresponds to the lowest Pass Honours mark for a Masters level module and 70 corresponds to the lowest First Class performance.

Marks from the modules taken in the fourth year are combined into a year total expressed as a percentage.

The aggregate marks from each year will be combined with the following percentage weightings to produce an overall aggregate mark:

For G103

Year 1: 7.50 percent
Year 2: 20.00 percent
Year 3: 36.25 percent
Year 4: 36.25 percent,

For G104

Year 1: 7.50 percent
Year 2: 25.00 percent
Year 3: 25.00 percent
Year 4: 42.50 percent.

ECTS

To comply with the European 'Bologna Process', degree programmes are required to be rated via the ECTS (European Credit Transfer System) – which is based notionally on hour counts for elements within the degree. In principle, **1 ECTS should equate to around 25 hours of study (including examinations and private study).**

As in Third Year, each Fourth Year mathematics module has an ECTS value of 7.5 except for M4R which has an ECTS value of 15. You should check the ECTS value of any modules taken from other departments You must complete 60 ECTS over the year.

MODULE ASSESSMENT AND EXAMINATIONS

Most MATH7 modules are examined by one written examination of 2.5 hours in length. Written examinations for MATH6 modules are typically 2 hours in length.

Some of the modules may have an assessed coursework/progress test element, limited in most cases to 10% of overall module assessment. Some modules have a more substantial coursework component (for example, 25 percent) and others are assessed entirely by coursework. Details can be found in the tables below. Precise details of the number and nature of coursework assignments will be provided at the start of each module.

Students should bear in mind that single-term modules assessed by projects usually require extra time-commitment during that term. Students should note that, in principle, 1 ECTS represents around 25 hours of effort, and so a single 7.5 ECTS module represents 187.5 hours of effort: completing this in a single term is a substantial task. Thus, the Department generally advises that students should not take more than one such module in a term. Students wishing to take more than one such module in a term will be required to discuss this with the Senior Tutor.

The MSci project module is assessed by a written report as well as an oral presentation.

See module description for assessment of the module Topics in Advanced Statistics; note that your chosen component modules for this module will appear separately on your transcript.

Note: Students who take modules which are wholly assessed by project will be deemed to be officially registered on the module through the submission of a specified number of pieces of assessed work for that module. Thus, once a certain point is reached in these modules, a student will be committed to completing it. In contrast, students only become committed to modules with summer examinations when they enter for the examinations in February.

Students who do not obtain Passes in examinations at the first attempt may be expected to attend resit examinations. However, the Examinations Board has the power to compensate not-too-serious fails in final year modules and permit graduation.

Resit examinations are for Pass credit only – a maximum mark of the pass mark (50 percent for Masters level modules) will be credited. Once a Pass is achieved, no further attempts are permitted.

FOURTH YEAR MODULE LIST

Note that not all of the individual modules listed below are offered every session and the Department reserves the right to cancel a particular module if, for example, the number of students attending that module does not make it viable. Similarly, some modules are occasionally run as 'Reading/Seminar Courses'.

Modules marked below with a * are also available in MATH6 form for Third Year undergraduates students (who typically take a shorter examination). When a module is offered it is usually, but not always, available in both forms. **No student may take both the MATH6 and MATH7 forms of a module.**

All MATH6 and MATH7 modules except the MSci project are equally weighted and are worth 7.5 ECTS unless otherwise specified. The MSci project is double-weighted and is worth 15 ECTS. The module Topics in Advanced Statistics is weighted the same as a standard MATH7 module but is worth 10 ECTS and students require permission from DUGS to take this module.

In the tables below:

Column on % Exam – this indicates a written exam, unless otherwise indicated.

Column on % CW – this indicates any coursework that is completed for the module. This may include in-class tests, projects, or problem sets to be turned in.

The groupings of modules below have been organised to indicate some natural affinities and connections.

The indicated lecturers are provisional; TBC indicates 'to be confirmed'.

APPLIED MATHEMATICS/MATHEMATICAL PHYSICS/NUMERICAL ANALYSIS

FLUIDS

College Module Code	Dept. Module Code	Module Titles	Terms	Lecturer	% exam	% CW
MATH70001	M4A2*	Fluid Dynamics 1	1	Professor X. Wu	90	10
MATH70002	M4A10*	Fluid Dynamics 2	2	Professor J. Mestel	90	10
MATH70003	M4A28*	Introduction to Geophysical Fluid Dynamics	2	Dr P. Berloff	90	10
MATH70051	M4A32	Vortex Dynamics	2	Professor D. Crowdy	90	10
MATH70052	M4A30	Hydrodynamic Stability	2	Professor X. Wu	90	10

MATHEMATICAL METHODS

MATH70004	M4M7*	Asymptotic Methods	1	Dr G. Peng	90	10
MATH70005	M4XX* (new)	Optimisation	1	Dr D Kalise	90	10
MATH70006	M4M6*	Applied Complex Analysis	1	Dr S. Brzezicki	90	10

DYNAMICS

MATH70007	M4PA48*	Dynamics of Learning and Iterated Games	1	Professor S. van Strien	40 (Oral)	60
MATH70008	M4PA23*	Dynamical Systems	1	Professor J. Lamb	90	10
MATH70009	M4PA24*	Bifurcation Theory	2	Professor D. Turaev	90	10

MATH70053	M4PA40	Random Dynamical Systems and Ergodic Theory: Seminar Course	2	Professor J. Lamb	40 (oral)	60
MATH70010	M4PA16*	Geometric Mechanics	2	Professor D. Holm	90	10
MATH70011	M4A53*	Classical Dynamics	1	Dr C. Ford	90	10

FINANCE

MATH70012	M4F22*	Mathematical Finance: An Introduction to Option Pricing	2	Dr P. Siorpaes	90	10
MATH70130	M4XX (new)	Stochastic Differential Equations in Financial Modelling	1	Professor D. Brigo	90	10

BIOLOGY

MATH70014	M4A49*	Mathematical Biology	1	Dr E. Keaveny	90	10

MATHEMATICAL PHYSICS

MATH70015	M4A4*	Quantum Mechanics I	1	Dr E-M Graefe	90	10
MATH70016	M4A6*	Special Relativity and Electromagnetism	1	Dr G. Pruessner	90	10
MATH70017	M4A7*	Tensor Calculus and General Relativity	2	Dr C. Ford	90	10
MATH70018	M4A52*	Quantum Mechanics II	2	Dr R. Barnett	90	10

APPLIED PDEs, NUMERICAL ANALYSIS and COMPUTATION

MATH70054	M4A51	Introduction to Stochastic Differential Equations and Diffusion Processes	2	Professor G. Pavliotis	90	10
MATH70019	M4M3*	Theory of Partial Differential Equations	1	Dr E. Zatorska	90	10
MATH70020	M4M11*	Function Spaces and Applications	1	Professor P. Germain	90	10
MATH70021	M4M12*	Advanced Topics in Partial Differential equations	2	Dr A. Giorgini	90	10
MATH70022	M4A47*	Finite Elements: Numerical Analysis and Implementation	2	Professor C. Cotter & Dr D. Ham	50	50
MATH70023	M4N7*	Numerical Solution of Ordinary Differential Equations	1	Dr I. Shevchenko	0	100
MATH70024	M4N9*	Computational Linear Algebra	1	Professor C. Cotter	0	100
MATH70025	M4N10*	Computational Partial Differential Equations	2	Dr S. Mughal	0	100
MATH70026	M4A50*	Methods for Data Science	2	Dr P. Thomas and Dr B. Bravi	0	100
MATH70027	M4SC*	Scientific Computation	2	Dr P. Ray	0	100
MATH70134	M4XX (new)	Mathematical Foundations of Machine Learning	2	Dr A. Borovykh	90	10

PURE MATHEMATICS

College Module Code	Dept. Module Code	Module Titles	Terms	Lecturer	% exam	% CW

ANALYSIS

College Module Code	Module Codes	Module Titles	Terms	Lecturer	% exam	% CW
MATH70028	M4P6*	Probability Theory	2	Dr I. Krasovsky	90	10
MATH70029	M4P7*	Functional Analysis	1	Dr P. Rodriguez	90	10

MATH70030	M4P18*	Fourier Analysis and Theory of Distributions	2	Dr I. Krasovsky	90	10
MATH70135	M4P41	Analytic Methods in Partial Differential Equations	1	Dr A. Chandra	90	10
MATH70055	M4P67	Stochastic Calculus with Applications to non-Linear Filtering	2	Professor D. Crisan	90	10
MATH70031	M4P70*	Markov Processes	1	Professor N. Bingham	90	10

GEOMETRY

College Module Code	Module Codes	Module Titles	Terms	Lecturer	% exam	% CW
MATH70032	M4P5*	Geometry of Curves and Surfaces	2	Dr D. Cheraghi	90	10
MATH70033	M4P20*	Algebraic Curves	1	Dr J. Lai	90	10
MATH70034	M4P21*	Algebraic Topology	2	Profesor A. Skorobogatov	90	10
MATH70056	M4P33	Algebraic Geometry	2	Dr D. Helm	90	10
MATH70057	M4P51	Riemannian Geometry	2	Dr M. Guaraco	90	10
MATH70058	M4P52	Manifolds	1	Professor P. Cascini	90	10
MATH70059	M4P54	Differential Topology	2	Dr S. Lynch	90	10
MATH70060	M4P57	Complex Manifolds	2	Dr M. Matviichuk	90	10

ALGEBRA AND DISCRETE MATHEMATICS

MATH70035	M4P8*	Algebra 3	1	Dr J. Nicholson	90	10
MATH70036	M4P10*	Group Theory	1	Professor M. Liebeck	90	10
MATH70037	M4P11*	Galois Theory	2	Professor A Corti	90	10
MATH70038	M4P80*	Graph Theory	2	Professor A. Ivanov	90	10
MATH70039	M4P12*	Group Representation Theory	2	Dr T. Schedler	90	10
MATH70040	M4xx*	Formalising Mathematics	2	Professor K. Buzzard	0	100
MATH70061	M4P55	Commutative Algebra	1	Dr Y. Lekili	90	10
MATH70062	M4P46	Lie Algebras	1	Professor A. Ivanov	90	10
MATH70063	M4P63	Algebra 4	2	Dr S. Feyzbakhsh	90	10
MATH70132	M4P65*	Mathematical Logic	1	Professor D. Evans	90	10

NUMBER THEORY

MATH70041	M4P14*	Number Theory	1	Dr A. Pal	90	10
MATH70042	M4P15*	Algebraic Number Theory	2	Dr A. Pal	90	10
MATH70064	M4P32	Elliptic Curves	1	Dr J. Trias-Batle	90	10

STATISTICS

College Module Code	Module Codes	Module Titles	Terms	Lecturer	% exam	% CW
MATH70043	M4S1*	Statistical Theory	2	Dr K. Ray	90	10
MATH70044	M4S2*	Statistical Modelling 2	2	Dr C. Hallsworth	75	25
MATH70045	M4S4*	Applied Probability	1	Professor A. Veraart	90	10
MATH70046	M4S8*	Time Series Analysis	1	Dr E. Cohen	90	10
MATH70047	M4S9*	Stochastic Simulation	1	TBC	75	25
MATH70048	M4S14*	Survival Models	2	Professor A. Gandy	90	10
MATH70049	M4S20*	Introduction to Statistical Learning	2	Professor G. Nason	90	10

MATH70131	M4XX* (new)	Consumer Credit Risk Modelling	1	Dr A. Benchimol	75	25
	M4S18	**Topics in Advanced Statistics (choose one of each of the A/B options below; requires permission from DUGS)**	2			
MATH70092	M4S18A1 (5 ECTS)	Multivariate Analysis	2	TBC	90	10
MATH70091	M4S18A2 (5 ECTS)	Machine Learning (May not be taken with M34S20)	2	Dr S. Filippi	0	100
TBC	M4S18B1 (5 ECTS)	TBC	2	TBC	TBC	TBC
MATH70090	M4S18B2 (5 ECTS)	Bayesian Methods	2	Dr D. Mortlock	80	20

PROJECT (Compulsory)

College Module Code	Dept. Module Code	Module Titles	Terms	Lecturer	% exam	% CW
MATH70050	M4R	Research Project in Mathematics (MSci Project)	1, 2 + 3	Dr D. Helm	0	100

FOURTH YEAR MATHEMATICS SYLLABUSES

MATH70001 Fluid Dynamics 1

Brief Description

Fluid dynamics investigates motions of both liquids and gases. Being a major branch of continuum mechanics, it does not deal with individual molecules, but with an ‘averaged' motion of the medium (i.e. collections of molecules). The aim is to predict the velocity, pressure and temperature fields in flows arising in nature and engineering applications. In this module, the equations governing fluid flows are derived by applying fundamental physical laws to the continuum. This is followed by descriptions of various techniques to simplify and solve the equations with the purpose of describing the motion of fluids under different conditions.

Learning Outcomes

On successful completion of this module you will be able to

- state the underlying assumptions of the continuum hypothesis;

- compare and contrast the different frameworks that can be used to describe fluid motion and to identify the connections between them;

- derive exact solutions of the Navier-Stokes equations and justify the physical and mathematical assumptions made in obtaining them;

- perform simplifications arising under the assumption of inviscid flow which permit the integration of the Euler equations, leading to results such as Bernoulli's equation and Kelvin's circulation theorem;

- demonstrate a sound understanding of the method of conformal mappings and be able to use this method to analyse various two-dimensional inviscid flows;

- choose the appropriate conformal mapping to solve inviscid flow problems in complicated geometries;

- predict the shape of the flow streamlines for such problems.

Module Content

The module is composed of the following sections:

Introduction: The continuum hypothesis. Knudsen number. The notion of fluid particle. Kinematics of the flow field. Lagrangian and Eulerian frameworks. Streamlines and pathlines. Strain rate tensor. Vorticity and circulation. Helmholtz's first theorem. Streamfunction.

Governing Equations: Continuity equation. Stress tensor and symmetry, Constitutive relation. The Navier-Stokes equations.

Exact Solutions of the Navier-Stokes Equations: Couette and Poiseuille flows. The flow between two coaxial cylinders. The flow over an impulsively started plate. Diffusion of a potential vortex.

Inviscid Flow Theory: Integrals of motion. Kelvin's circulation theorem. Potential flows. Bernoulli's equation. Cauchy-Bernoulli integral for unsteady flows. Two-dimensional flows. Complex potential. Vortex, source, dipole and the flow past a circular cylinder. Adjoint mass. Conformal mapping. Joukovskii transformation. Flows past aerofoils. Lift force. The theory of separated flows. Kirchhoff and Chaplygin models.

MATH70002 Fluid Dynamics 2

Brief Description

In this module, we deal with a wide class of realistic problems by seeking asymptotic solutions of the governing Navier-Stokes equations in various limits. We shall start with the "slow, small or sticky" case, when the Reynolds number is low and we obtain the linear Stokes equations. Then we consider the lubrication limit, and show how a thin layer of fluid is able to exert enormous pressures and prevent moving solid bodies from touching. Next we shall consider the "fast and vast" limit of high Reynolds number, which is characteristic of most flows we encounter in everyday life. In the final part of the module we consider a mixture of advanced topics, including flight, bio-fluid-dynamics and an introduction to flow stability.

Learning Outcomes

On successful completion of this module you will be able to:

- simplify and solve the governing Navier-Stokes equations in situations where there is a short lengthscale in one of the coordinate directions;

- apply the general properties of low Reynolds number flows to predict the drag on slow-moving bodies, like a solid sphere or spherical bubble, and appreciate the causes of the 'Stokes paradox';

- analyse lubrication-like flows in thin layers;

- derive the boundary-layer equations and identify self-similar solutions for flows at large Reynolds number;

- determine stability criteria for various fundamental flows;

- model animal locomotion at low and high Reynolds numbers;

- interpret results from advanced fluid mechanics textbooks and research papers;

- independently appraise and evaluate some advanced topics in viscous fluid mechanics.

Module Content

The module is composed of the following sections:

I – Low-Reynolds-number flows

Dynamic Similarity. Properties of the Stokes equations. Uniqueness and minimal dissipation theorems. The analysis of the flow past a solid sphere and spherical bubble. Stokes paradox.

II – Lubrication Theory

Derivation of Reynolds' lubrication equation and examples. Hele-Shaw and thin film flows.

III – High-Reynolds-number flows; Boundary-layer theory

The notion of singular perturbations. Derivation of boundary-layer equations. Blasius flow, Falkner-Skan solutions and applications. Von Mises variables and their application to periodic boundary layers. Prandtl-Batchelor Theorem for flows with closed streamlines.

IV – Introduction to hydrodynamic stability

Importance of stability. Rayleigh-Taylor and Kelvin-Helmholtz instabilities. Circular flow stability criterion.

V – Swimming and Flight; Animal locomotion

Scallop theorem. Resistive Force Theory. Introduction to 3D-aerofoil theory. Flight strategies.

VI - Advanced Topics

Current research in areas such as convection and magnetohydrodynamics.

MATH70003 Introduction to Geophysical Fluid Dynamics

Brief Description

This is an advanced-level fluid-dynamics course with geophysical flavours. The lectures target upper-level undergraduate and graduate students interested in the mathematics of planet Earth, and in the variety of motions and phenomena occurring in planetary atmospheres and oceans. The lectures provide a mix of theory and applications.

Learning Outcomes

On successful completion of this module you will be able to:

- demonstrate a deep understanding of the foundations of geophysical fluid dynamics;

- model a broad range of natural phenomena associated with the atmosphere and ocean;

- appreciate the main concepts and terminology used in the field;

- derive the boundary layer equations for flow in a rotating frame and justify the relative importance of various terms in the equations of motion;

- describe, select appropriately and apply a range of methods and techniques for solving practical problems;

- independently appraise an advanced topic in geophysical fluid dynamics;

- evaluate results from research papers in the field of geophysical fluid dynamics.

Module Content

The module is composed of the following sections:

I - Introduction and basics;

II - Governing equations (continuity of mass, material tracer, momentum equations, equation of state, thermodynamic equation, spherical coordinates, basic approximations);

III - Geostrophic dynamics (shallow-water model, potential vorticity conservation law, Rossby number expansion, geostrophic and hydrostatic balances, ageostrophic continuity, vorticity equation);

IV - Quasigeostrophic theory (two-layer model, potential vorticity conservation, continuous stratification, planetary geostrophy);

V - Ekman layers (boundary-layer analysis, Ekman pumping);

VI - Rossby waves (general properties of waves, physical mechanism, energetics, reflections, mean-flow effect, two-layer and continuously stratified models);

VII - Hydrodynamic instabilities (barotropic and baroclinic instabilities, necessary conditions, physical mechanisms, energy conversions, Eady and Phillips models);

VIII - Ageostrophic motions (linearized shallow-water model, Poincare and Kelvin waves, equatorial waves, ENSO “delayed oscillator”, geostrophic adjustment, deep-water and stratified gravity waves);

IX - Transport phenomena (Stokes drift, turbulent diffusion);

X - Nonlinear dynamics and wave-mean flow interactions (closure problem and eddy parameterization, triad interactions, Reynolds decomposition, integrals of motion, enstrophy equations, classical 3D turbulence, 2D turbulence, transformed Eulerian mean, Eliassen-Palm flux).

MATH70051 Vortex Dynamics

Brief Description

This is an advanced module in applied mathematical methods applied to the subfield of fluid dynamics called vortex dynamics. The module will focus on the mathematical study of the dynamics of vorticity in an ideal fluid in two and three dimensions. The material will be pitched in such a way that it will be of interest to those specializing in fluid dynamics but can also be viewed as an application of various techniques in dynamical systems theory.

Learning Outcomes

On successful completion of this module, you will be able to:

- interpret the role of vorticity within a range of problems in fluid mechanics;

- derive and compare a range of vortex models, from the point vortex models to distributed models, including vortex patches;

- combine your knowledge of different branches of mathematics (e.g. vector calculus, complex analysis and the theories of Hamiltonian dynamical systems and partial differential equations) in order to describe the dynamics of vorticity;

- choose from an array of applied mathematical techniques to explicitly solve for vorticity distributions;

- appraise the role that vortex structures play in modelling physical systems.

Module Content

The module will cover the following topics:

- Eulerian description of fluid flows;

- Incompressible flows and streamfunctions;

- Vorticity, vortex lines and vortex tubes;

- Biot-Savart law;

- Euler's equations and the vorticity equation;

- Kelvin's circulation theorem;

- Bernoulli theorems;

- Point vortex model, complex potentials;

- Point vortex equilibria;

- Dynamics of point vortices;

- Vortex dynamics on a spherical surface;

- Vortex patch models;

- Vortex patch equilibria;

- Vortex patch dynamics and contour dynamics;

- Other distributed vortex models.

MATH70052 Hydrodynamic Stability

Brief Description

Fluid flows may exist in two distinct forms: the simple laminar state which exhibits a high degree of order and the turbulent state characterised by its complex chaotic behaviours in both time and space. The transition from a laminar state to turbulence is due to hydrodynamic instability, which refers to the phenomenon that small disturbances to a simple state amplify significantly thereby destroying the latter. This is of profound scientific and technological importance because of its relevance to mixing and transport in the atmosphere and oceans, drag and aerodynamic heating experienced by air/spacecrafts, jet noise, combustion in engines and even the operation of proposed nuclear fusion devices.

Learning Outcomes

On successful completion of this module you will be able to

- construct the basic concepts underpinning hydrodynamic stability theory;

- predict linear stability properties based on eigenvalue analysis;

- compare and contrast various different instability mechanisms;

- derive various theorems that help us decide whether a flow is stable or unstable;

- model the effects of nonlinearity within an asymptotic framework.

Module Content

Topics covered will be a selection from the following list.

Basic concepts of stability; linear and nonlinear stability, initial-value and eigenvalue problems, normal modes, dispersion relations, temporal/spatial instability.

Buoyancy driven instability: Rayleigh-Benard instability, formulation of the linearised stability problem, Rayleigh number, Rayleigh-Benard convection cells, discussion of the neutral stability properties.

Centrifugal instability: Taylor-Couette flow, formulation of the linear stability problem, Taylor number, Taylor vortices; inviscid approximation, Rayleigh's criterion; viscous theory and solutions, characterization of stability properties; boundary layers over concave walls, Görtler number, Görtler instability, Görtler vortices.

Inviscid/viscous shear instabilities of parallel flows: Inviscid/Rayleigh instability, Rayleigh equation, Rayleigh's inflection point theorem, Fjortoft's theorem, Howard's semi-circle theorem, solutions for special profiles, KelvinHelmholtz instability, general characteristics of instability, critical layer, singularity; Viscous/Tollmien-Schlichting instability, Orr-Sommerfeld (O-S) equation, Squire's theorem, numerical methods for solving the linear stability problem, discussion of instability properties.

Inviscid/viscous shear instabilities of (weakly) non-parallel flows: local-parallel-flow approximation and

application to free shear layers and boundary layers; non-parallel-flow effects, rational explanation of viscous instability mechanism, high-Reynolds-number asymptotic theory, multi-scale approach, parabolised stabilityeq uations; transition process and prediction (correlation); receptivity.

Nonlinear instability: limitations of linear theories, bifurcation and nonlinear evolution; weakly nonlinear theory, derivation of Stuart-Landau and Ginzburg-Landau equations; nonlinear critical-layer theory.

MATH70004 Asymptotic Methods

Brief Description

This advanced course presents a systematical introduction to asymptotic methods, which form one of the cornerstones of modern applied mathematics. The foundation of asymptotic approximations is laid down first. The key ideas and techniques for deriving asymptotic representations of integrals, and for constructing appropriate solutions to differential equations will be explained. The techniques introduced find wide applications in engineering and natural sciences.

Learning Outcomes

On successful completion of this module, you will be able to:

- appreciate the foundation upon which asymptotic approximations are based;

- describe a variety of asymptotic methods and for each method acquire a thorough understanding of the key ideas involved and their mathematical nature;

- demonstrate basic skills in applying each of these methods to solve classical problems;

- combine, modify and extend methods to unfamiliar problems, such as those that emerge from research topics or practical applications;

- outline how asymptotic methods can in principle be applied to a wide variety of problems;

- interpret results from advanced textbooks and research papers on asymptotic methods;

- construct advanced solution techniques by selecting an appropriate combination of different asymptotic methods to solve higher-dimensional problems.

Module Content

I - Asymptotic approximations (fundamentals)

Order notation. Diverging series, asymptotic expansions. Parameter expansions, overlap regions, distinguished limits and uniform approximations.Stokes phenomenon.

II - Introduction to perturbation methods

Asymptotic solution of algebraic equations with a small parameter. Regular vs. singular perturbations. Method of dominant balance. Local analysis of ordinary differential equations.

III - Asymptotic analysis of integrals

Method of integration by parts. Integrals of Laplace type: Laplace's method, Watson's Lemma. Integrals of Fourier type: method of stationary phase. Integral in the complex plane: method of steepest descent. Method of splitting the range of integration.

IV - Matched asymptotic expansion

Inner and outer expansions, matching principles, notions of 'boundary layer' and interior layer. Composite approximation. Application to relaxation oscillations.

V - Methods of multiple scales

WKB approximations including turning-point problems and eigenvalue quantisation. Secular terms and solvability conditions. Poincare-Lindstedt method for periodic solutions. Multiscale method for quasi-periodic solutions. Application to weakly perturbed oscillators, nonlinear resonance, parametric resonance.

VI - A selection of topics from the following: Stokes phenomenon, hyperasymptotics, expansions involving logarithmic terms, homogenisation.

MATH70005 Optimisation

Brief Description

This module is an introduction to the theory and practice of mathematical optimization and its many applications in mathematics, data science, and engineering. The module aims at endowing students with the necessary mathematical background and a thorough methodological toolbox to formulate optimization problems and developing an algorithmic approach to its solution.The module is structured into five parts: (i) formulation and classification of problems; (ii) unconstrained optimization; (iii) stochastic and nature-inspired optimization; (iv) convex optimization; (v) introduction to optimal control and dynamic optimization. The assessed coursework for this module involves a series of computational tasks.

Learning Outcomes

On successful completion of this module you will be able to

- formulate a mathematical optimization problem by identifying a suitable objective and constraints;

- identify the mathematical structure of an optimization problem and, based on this classification, choose an appropriate methodological approach;

- develop a mathematical and computational appreciation of convexity as a fundamental feature in optimization;

- implement different computational optimization algorithms such as gradient descent and related variants;

- analyse the results of a computational optimization method in terms of optimality guarantees, sensitivities, and performance.

- interpret the role played by optimization in its application to computational data science;

- design optimal control approaches relevant to tackling large-scale nonlinear problems.

Module Content

1. Mathematical preliminaries

2. Unconstrained optimization

3. Gradient descent methods

4. Linear and non-linear least squares problems

5. Stochastic gradient descent

6. Nature-inspired optimization

7. Convex sets and functions

8. Convex optimization problems and stationarity

9. KKT conditions

10. Duality

11. Introduction to dynamic optimization and optimal control.

This final topic is linked to the Mastery Material for MSci students which will involve the study of some of the following solution techniques:

- shooting and multiple shooting methods;

- the reduced gradient approach;

- two-point boundary value solvers for optimal control;

- dynamic Programming and the Hamilton-Jacobi PDE;

- the linear-quadratic regulator and the Riccati equation.

This will be examined by way of an extra question on the May examination paper.

MATH70006 Applied Complex Analysis

Brief Description

The aim of this module is to learn tools and techniques from complex analysis and the theory of orthogonal polynomials that can be used in mathematical physics. The course will focus on mathematical techniques, though will also discuss relevant physical applications, such as electrostatic potential theory. The course incorporates computational techniques in the lectures.

Learning Outcomes

On successful completion of this module, you will be able to:

- apply the technique of contour deformation for calculating integrals;

- appreciate the connection that exists between computational tools such as quadrature and orthogonal polynomials and complex analysis;

- evaluate singular integral equations with Cauchy and logarithmic kernels;

- use the Wiener-Hopf method to solve a class of integral equations;

- compute matrix functions using contour inegration;

- interpret results from advanced textbooks and research papers;

- independently appraise and evaluate an advanced topic in complex analysis.

Module Content

This module covers the following topics:

Revision of complex analysis: complex integration, Cauchy's theorem and residue calculus;

Singular integrals: Cauchy, Hilbert, and log kernel transforms;

Potential theory: Laplace's equation, electrostatic potentials, distribution of charges in a well;

Riemann–Hilbert problems: Plemelj formulae, additive and multiplicative Riemann–Hilbert problems;

Orthogonal polynomials: recurrence relationships, solving differential equations, calculating singular

integrals;

Integral equations: integral equations on the whole and half line, Fourier transforms, Laplace transforms;

Wiener–Hopf method: direct solution, solution via Riemann–Hilbert methods.

MATH70007 Dynamics of Learning and Iterated Games

Brief Description

Recently there has been considerable interest in modelling learning. The settings to which these models are applied is wide-ranging. Examples include optimization of strategies of populations in ecology and biology, iterated strategies of people in a competitive environment and learning models used by technology companies such as Google.

This module is aimed at discussing a number of such models in which learning evolves over time and which have a game theoretic background. The module will use tools from the theory of dynamical systems and will aim to be rigorous. Topics will include replicator systems, best response dynamics and fictitious games, reinforcement learning and no-regret learning.

Learning Outcomes

On successful completion of this module, you will be able to:

- analyse 2D replicators systems for one and two player games;

- work comfortably with the notions of Nash, Correlated Equilibrium, Course Correlated Equilibrium and Evolutionary Stable Strategies;

- explain the notion of reciprocity in relation to Iterated Prisoner Dilemma games;

- appreciate the connection between Reinforcement Learning and replicator systems;

- outline the idea behind no regret learning models and the Blackwell approachability theorem;

- derive the proofs behind the methods that are used in the final project;

- appraise and interpret results from advanced textbooks and research papers.

Module Content

The module will cover the following topics:

- Replicator systems;

- Rock-paper-scissor games;

- Iterated prisoner dilemma games;

- Best response dynamics;

- Two player games;

- Fictitious games as a learning model;

- Reinforcement learning;

- No regret learning.

MATH70008 Dynamical Systems

Brief Description

The theory of Dynamical Systems is an important area of mathematics which aims at describing objects whose state changes over time. For instance, the solar system comprising the sun and all planets is a dynamical system, and dynamical systems can be found in many other areas such as finance, physics, biology and social sciences. This course provides a rigorous treatment of the foundations of discrete-time dynamical systems.

Learning Outcomes

On successful completion of this module, you will be able to:

- demonstrate a familiarity with the basic concepts of topological dynamics;

- provide an outline of the ergodic theory of dynamical systems;

- appreciate the concept of symbolic dynamics through which topological and probabilistic dynamical properties can be understood;

- demonstrate an understanding of precise mathematical characterisations of chaotic dynamics;

- apply the above context in a number of one-dimensional settings, in particular in the context of piecewise affine expanding maps;

- independently appraise and evaluate advanced topological and probabilistic dynamical properties, beyond the foundations;

- independently develop and interpret examples in two and higher dimensions.

Module Content

The module covers the following topics:

- Introduction: orbits, periodic orbits and their local stability;

- Topological dynamics: invariant sets and limit sets, coding and sequence spaces, topological conjugacy, transitivity and mixing;

- Chaotic dynamics: sensitive dependence, topological entropy, topological Markov chains;

- Ergodic theory: sigma-algebras and measures, invariant measures, Poincaré recurrence, ergodicity and Birkhoff’s Ergodic Theorem, Markov measures and metric entropy;

- Additional reading material in line with M4 objectives.

MATH70009 Bifurcation Theory

Brief Description

This module serves as an introduction to bifurcation theory, concerning the study of how the behaviour of dynamical systems such as ODEs and maps changes when parameters are varied.The goal is to acquaint the students with the foundations of the theory, its main discoveries and the universal methods behind this theory that extend beyond its remit.

Learning Outcomes

On successful completion of this module, you will be able to:

- exploit basic dimension reduction methods (invariant manifold and invariant foliations);
- apply the method of normal forms;
- demonstrate a sound knowledge of the basics of stability theory;
- appreciate the role of control parameters and to construct bifurcation diagrams;
- describe the mathematical framework associated with classical local and global bifurcations;
- interpret results from advanced textbooks and research papers on bifurcation theory;
- independently appraise and evaluate the transition from periodic to quasiperiodic regimes and to chaos via destruction of quasiperiodicity.

Module Content

The following topics will be covered:

1) Bifurcations on a line and on a plane;

2) Centre manifold theorem; local bifurcations of equilibrium states;

3) Local bifurcations of periodic orbits – folds and cusps;

4) Homoclinic loops: cases with simple dynamics, Shilnikov chaos, Lorenz attractor;

5) Saddle-node bifurcations: destruction of a torus, intermittency, blue-sky catastrophe;

6) Routes to chaos and homoclinic tangency.

MATH70053 Random Dynamical Systems and Ergodic Theory: Seminar Course

Brief Description

This is a course on the theory and applications of random dynamical systems and ergodic theory. Random dynamical systems are deterministic dynamical systems driven by a random input. The goal will be to present a solid introduction to the subject and then to touch upon several more advanced developments in this field.

Learning Outcomes

On successful completion of this module, you will be able to:

- describe the fundamental concepts of random dynamical systems;
- summarize the ergodic theory of random dynamical systems;
- select and critically appraise relevant research papers and chapters of research monographs;
- combine the ideas contained in such papers to provide a written overview of the current state of affairs concerning a particular aspect of random dynamical systems theory;
- thoughtfully engage orally in discussions related to random dynamical systems.

Module Content

Introductory lectures include foundational material on:

- Invariant measures and ergodic theory

- Random (pullback) attractors

- Lyapunov exponents

- Random circle homeomorphisms

Further material is at a more advanced level, touching upon current frontline research. Students select material from research level articles or book chapters.

MATH70010 Geometric Mechanics

Brief Description

The course surveys a small section of the road to geometric mechanics (GM), by treating several examples in classical mechanics all in the same geometric framework. This framework is based on linear algebra, transformation theory, differential equations, variational calculus, and Lie group invariant variational principles, and their actions on functions and differential forms defined on manifolds.

Learning Outcomes

On successful completion of this module, you will be able to:

- describe motion on smooth manifolds using the following definitions: tangent bundle (TM), cotangent bundle (T*M, phase space), kinetic energy, Lagrangian, variational derivative, Hamilton's variational principle (LagHvp), Euler-Lagrange equations (EL eqns);

- derive and solve Euler-Lagrange equations for a variety of classical problems, including: oscillators, plane pendulum, charged particle motion, motion on a sphere, spherical pendulum, rotating rigid body, geodesic motion on the Lobachevsky half-plane, equations of motion for Newton's law of gravitation;

- prove and apply Noether's theorem about how Lie group symmetries of Hamilton's principle imply conservation laws in the GM framework for transformations of the matrix Lie groups of rotations, translations and scaling (in the context of the example problems);

- link fibre derivative, Legendre transformation, Hamilton's equation, Poisson brackets, and Noether quantities as momentum maps for Lie group actions;

- apply the Geometric Mechanics framework to rigid body motion and Newton's solution for the Kepler problem of planetary motion;

- demonstrate familiarity with the transformation theory for smooth invertible maps and the calculus of differential forms, including exterior derivative, wedge product, contraction with vector fields, and Cartan's formula for Lie derivative through worked exercises;

- relate Lie-Poisson brackets to coadjoint actions of matrix Lie groups on matrix Lie algebras;

- derive the equations for ideal fluid dynamics from Hamilton's principle with symmetry under smooth invertible maps.

Module Content

This module explains the principles of mechanics as a unified geometric framework, by using

Noether's theorem about Lie-group invariant variational principles to focus on Sophus Lie's momentum map concept as the key for applying Poincaré's geometric approach. The only prerequisites are linear algebra, vector calculus and some familiarity with the Euler-Lagrange variational principles and canonical Poisson brackets in classical mechanics at the beginning undergraduate level.

The course presents increasingly advanced ideas as a series of recurring thematic examples and worked exercises for classic problems of oscillation and rotation. For example, it treats the geometry of rigid-body motion from the series of viewpoints of Newton, Lagrange, Hamilton and Poincaré. This unfolding series of treatments of rigid body motion from several viewpoints reveals the framework of geometric mechanics and sets the stage for the introduction of the flows of Hamiltonian vector fields and their Lie-derivative actions on differential forms.

After learning a few useful methods of vector calculus in the language of differential forms, the student recognises the geometric structure of ideal fluid dynamics as something which has become familiar and natural; namely, geodesic motion for the kinetic energy metric on the manifold of smooth invertible maps (L2 norm of the Eulerian fluid velocity). Thus, the course reveals a geometric framework for variational dynamics with symmetry and develops the ideas in the framework from finite to infinite dimensions.

MATH70011 Classical Dynamics

Brief Description

Classical dynamics is developed through variational principles rather than Newtonian force laws. Lagrangian and Hamiltonian formulations are considered. The methods are applied to a variety of problems including pendulums, the Kepler problem, rigid bodies and motion of a charged particle in a magnetic field. The role of conserved quantities is emphasised. Advanced ideas including Hamilton-Jacobi theory, action-angle variables, adiabatic invariance and Hamiltonian Chaos are developed.

Learning Outcomes

On successful completion of this module, you will be able to:

- reformulate Newton's laws through variational principles;

- construct Lagrangians or Hamiltonians for dynamics problems in any coordinate system;

- solve the equations of motion for a wide variety of problems in dynamics;

- identify and exploit constants of the motion in solving dynamics problems;

- apply Lagrangian and Hamiltonian methods to problems in a variety of fields (e.g. Statistical Mechanics, Quantum Mechanics and Geometric Mechanics).

- independently appraise and evaluate two advanced topics from the following list: classical field theory, canonical perturbation theory, chaos.

- interpret results from advanced textbooks and research papers on two of the topics mentioned above.

Module Content

This module will cover the following topics:

1. Review of the Calculus of Variations.

2. Newtonian Mechanics: momentum, angular momentum, conservative forces.
3. Lagrangian Mechanics: Hamilton's Principle, Lagrangians for conservative and non-conservative systems, generalised coordinates and momenta, cyclic coordinates, Noether's theorem (conservation of angular momentum as an example).

4. Hamiltonian Mechanics: Phase Space, Hamilton's equations, Poisson brackets, canonical transformations, generating functions, Hamilton-Jacobi theory, action-angle variables, adiabatic invariance, integrability, application of Hamiltonian mechanics to rigid bodies.
5. Introduction to Hamiltonian Chaos.

MATH70012 Mathematical Finance: An Introduction to Option Pricing

Brief Description

The mathematical modelling of derivatives securities, initiated by Bachelier in 1900 and developed by Black, Scholes and Merton in the 1970s, focuses on the pricing and hedging of options, futures and other derivatives, using a probabilistic representation of market uncertainty. This module is a mathematical introduction to this theory, in a discrete-time setting. We will mostly focus on the no-arbitrage theory in market models described by

trees; eventually we will take the continuous-time limit of a binomial tree to obtain the celebrated Black-Scholes model and pricing formula.

Learning Outcomes

On successful completion of this module, you will be able to:

- appreciate the fundamental principles involved in pricing derivatives;

- describe and critically analyse simple market models and explore their qualitative properties;

- confidently perform calculations involving pricing and hedging in discrete market models;

- demonstrate a familiarity with some key concepts in modern probability theory and apply them to perform computations;

- outline a mathematical formulation describing the behaviour of a number of financial derivatives;

- construct dynamic programming techniques to solve problems where inter-temporal relations are important;

- appraise and critically evaluate one or more of the advanced topics listed below.

Module Content

The module will cover the following topics:

financial derivatives, arbitrage, no-arbitrage pricing, self-financing portfolios, non-anticipative trading strategies, hedging of derivatives, domination property, complete markets, ‘risk-neutral’ probabilities, the fundamental theorems of asset pricing, conditional probability and expectation, filtrations, Markov processes, martingales, change of measure.

Extra mastery component will include the following advanced topics: utility, optimal investment.

MATH70130 Stochastic Differential Equations in Financial Modelling

Brief Description

To deal with valuation, hedging and risk management of financial options, we briefly introduce stochastic differential equations using a Riemann-Stiltjes approach to stochastic integration. We introduce no-arbitrage theory in continuous time based on replicating portfolios, self-financing conditions and Ito's formula. We derive prices as risk neutral expectations. We derive the Black Scholes model and introduce volatility smile models. We illustrate valuation of different options and introduce risk measures like Value at Risk and Expected Shortfall, motivating them with the financial crises.

Learning Outcomes

On successful completion of this module you will be able to

- work comfortably with stochastic differential equations commonly encountered in finance

- explain what is meant by no-arbitrage markets and why no-arbitrage is important operationally;

- connect no-arbitrage by replication to the existence of a risk neutral measure;

- price and hedge several types of financial options with several SDE models;

- calculate risk measures such as Value at Risk and Expected Shortfall;

- write code to price options according to SDE models covered in the module.

- independently appraise and evaluate SDE models for financial products.

- adapt a range of numerical methods and apply them in a coherent manner to unfamiliar and open problems in finance.

Module Content

1. Recap of key tools from probability theory

2. Brownian motion

3. Ito and Stratonovich stochastic integrals

4. Ito and Stratonovich stochastic differential equations (SDEs)

5. No-arbitrage through replication

6. No arbitrage though risk neutral measure

7. Derivation of the Black Scholes formula

8. Introduction of a few volatility smile models

9. Pricing of several types of options

10. Introduction to crises and risk measures

11. The Barings collapse and the introduction of value at risk (VaR)

12. Problems of VaR and an alternative: expected shortfall (ES)

13. Numerical examples and problems with risk measures, including software code.

MATH70014 Mathematical Biology

Brief Description

Mathematical biology entails the use of mathematics to model biological phenomena in order to understand these systems, as well as predict their behaviour. It is an incredibly diverse field utilising the complete mathematical toolbox to ascertain insight into many areas of biology and medicine including population dynamics, physiology, epidemiology, cell biology, biochemical reactions, and neurology. This module aims to provide a foundational course in the subject area relying primarily on tools from applied dynamical systems, applied PDEs, asymptotic analysis and stochastic processes.

Learning Outcomes

On successful completion of this module you will be able to:

- translate biological phenomena into the language of mathematics;

- appreciate canonical problems in epidemiology, ecology, biochemistry and physiology;

- critically analyse sets of ordinary differential equations especially in the non-linear setting;

- critically analyse sets of partial differential equations especially when either travelling-wave solutions or pattern forming phenomena might emerge;

- utilise the concept of stochastic population processes for exact and approximate solutions;

- use the techniques of order-of-magnitude reasoning and dimensional analysis;

- interpret results from the research literature on Mathematical Biology and analyze how the syllabus content relates to this wider body of work;

- appraise and evaluate an advanced topic in Mathematical Biology from a selection of case studies.

Module Content

Examples and topics include:

1) One-dimensional systems: existence and uniqueness; fixed points and their stability; bifurcations; logistic growth; SIS epidemic model; spruce budworm model; law of mass action; Michaelis-Menten enzyme dynamics.

2) Multidimensional systems: existence, uniqueness, fixed point stability; two-dimensional systems; SIS model for two populations; genetic control systems; population competition models; predator-prey dynamics and the Lotka-Volterra model.

3) Oscillations and bifurcations: Poincaré-Bendixson Theorem; oscillations in predator-prey models; relaxation oscillators; Fitzhugh-Nagumo model; fixed point bifurcations; Hopf bifurcations and limit cycles.

4) Spatial dynamics: reaction-diffusion equations; Fisher-Kolmogorov equation; travelling waves in predator-prey systems; spatial SIS model; spread of rabies in a fox population; Turing instabilities; pattern formation in one and two dimensions.

5) Stochastic processes: continuous-time Markov chains; simple birth and death processes; stationary probability distributions; logistic growth process; branching processes and drug resistance; multivariate processes; stochastic enzyme dynamics; stochastic predator-prey dynamics.

Jupyter notebooks containing codes written in Python will be utilised throughout the course and a working knowledge of, or a willingness to learn and use Python, is expected.

MATH70015 Quantum Mechanics I

Brief Description

Quantum mechanics is one of the most successful theories in modern physics and has an exceptionally beautiful underlying mathematical structure. It provides the basis for many areas of contemporary physics, including atomic and molecular, condensed matter, high-energy particle physics, quantum information theory, and quantum cosmology, and has led to countless technological applications. This module aims to provide an introduction to quantum phenomena and their mathematical description. We will use tools and concepts from various areas of mathematics and physics, such as classical mechanics, linear algebra, probability theory, numerical methods, analysis and geometry.

Learning Outcomes

On successful completion of this module, you will be able to:

- appreciate Schrödinger's formulation of quantum mechanics, wave functions and wave equations;

- construct the mathematical framework of quantum mechanics, including the 4 postulates of quantum mechanics and the Dirac notation;

- solve the eigenvalue problem for basic one-dimensional quantum systems;

- exploit the method of stationary states to deduce the time-evolved quantum state from the initial state of a system;

- communicate fluently using the Dirac notation;

- interpret results from advanced quantum mechanics textbooks and research papers;

- independently appraise and evaluate an advanced (more contemporary) topic in quantum mechanics from those listed in the syllabus below.

Module Content

The module will cover the following topics:

- Hamiltonian dynamics;

- Schrödinger equation and wave functions;

- stationary states of one-dimensional systems;

- mathematical foundations of quantum mechanics;

- quantum dynamics;

- angular momentum.

A selection of topics among the following additional optional topics will be covered depending on students interests:

- approximation techniques;

- explicitly time-dependent systems;

- geometric phases;

- numerical techniques;

- many-particle systems;

- cold atoms;

- entanglement and quantum information.

MATH70016 Special Relativity and Electromagnetism

Brief Description

This module presents a beautiful mathematical description of a physical theory of great historical, theoretical and technological importance. It demonstrates how advances in modern theoretical physics are being made and gives a glimpse of how other theories (say quantum chromodynamics) proceed. This module does not follow the classical presentation of special relativity by following its historical development, but takes the field theoretic route of postulating an action and determining the consequences. The lectures follow closely the famous textbook on the classical theory of fields by Landau and Lifshitz.

Learning Outcomes

On successful completion of this module you will be able to

- demonstrate an understanding of the relation between space and time and apply Lorentz transforms;

- appreciate the structure of special relativity as derived from the principle of least action;

- determine relativistic particle trajectories;

- derive Maxwell’s equations from first principles and apply them to variety of interactions of charges and fields;

- critically analyse various solutions of the electromagnetic wave equations;

- describe electrostatic interactions and motion using Coulomb’s law;

- construct an expansion of electrostatic interactions in terms of multipoles.

Module Content

This course follows closely the following book: L.D. Landau and E.M. Lifschitz, Course on Theoretical Physics Volume 2: Classical Theory of Fields.

Special relativity: Einstein’s postulates, Lorentz transformation and its consequences, four vectors, dynamics of a particle, mass-energy equivalence, collisions, conserved quantities.

Electromagnetism: Magnetic and electric fields, their transformations and invariants, Maxwell’s equations, conserved quantities, wave equation.

MATH70017 Tensor Calculus and General Relativity

Brief Description

This module provides an introduction to General Relativity. Starting with the rather simple Mathematics of Special Relativity the goal is to provide you with the mathematical tools to formulate General Relativity. Some examples, including the Schwarzschild space-time are considered in detail.

Learning Outcomes

On successful completion of this module, you will be able to:

- appreciate the application of tensors in special relativity;

- demonstrate a working knowledge of tensor calculus;

- explain the concepts of parallel transport and curvature;

- formulate and solve the geodesic equation for a given space-time metric;

- derive Einstein's field equations and analyse Schwarzschild's solution;

- interpret results from advanced general relativity textbooks and research papers;

- appraise and critically evaluate two of the extensions and applications listed below.

Module Content

This module will cover the following topics:

1. Special Relativity

2. Tensors in Special Relativity

3. Tensors in General Coordinates Systems 4. Parallel Transport and Curvature

5. General Relativity

6. The Schwarzschild Spacetime

7. Variational Methods

8. Extensions and Applications (selected from gravitational waves, Einstein-Hilbert action, cosmology, Einstein-Cartan theory, differential geometry)

MATH70018 Quantum Mechanics II

Brief Description

Quantum mechanics (QM) is one of the most successful theories in modern physics and has an exceptionally beautiful underlying mathematical structure. Assuming some prior exposure to the subject (such as Quantum Mechanics I), this module aims to provide an intermediate/advanced treatment of quantum phenomena and their mathematical description. Quantum theory combines tools and concepts from various areas of mathematics and physics, such as classical mechanics, linear algebra, probability theory, numerical methods, analysis and geometry.

Learning Outcomes

On successful completion of this module, you will be able to:

- outline key aspects of quantum mechanics at the intermediate/advanced level;

- harness the power of symmetry in understanding quantum mechanics;

- describe many-particle quantum mechanical systems, and demonstrate familiarity with the formalism of second quantisation;

- solve complex quantum mechanical problems using the machinery introduced in this module;

- use the knowledge gained here as a solid foundation for a research project in quantum mechanics;

- interpret results from advanced quantum mechanics textbooks and research papers;

- appraise and evaluate a topic in quantum mechanics from the syllabus at an advanced level.

Module Content

This module will cover the following core topics:

- quantum mechanics in the momentum basis;

- the Heisenberg picture;

- the use of symmetry and general transformations in quantum mechanics;

- Elements of Quantum Computation;

- perturbation theory;

- adiabatic processes;

- second quantisation;

- introduction to many-particle systems;

- Fermi and Bose statistics.

Additional topics include: WKB theory, the Feynman path integral, quantum magnetism.

MATH70054 Introduction to Stochastic Differential Equations

Brief Description

This module provides an introduction to stochastic differential equations (SDEs), together with the necessary background material from stochastic analysis and the link between SDEs and partial differential equations. The course covers the following topics: elements of the theory of stochastic processes in continuous time, Brownian motion, construction of the Ito stochastic integral, existence and uniqueness theory for SDEs, methods for solving SDEs, connection between SDEs and Markov processes, the Fokker-Planck equation, ergodic theory for SDEs.

Learning Outcomes

On successful completion of this module, you will be able to:

- formulate the basics of the theory of stochastic processes in continuous time;

- appreciate the fundamental properties of Brownian motion;

- apply Ito's theory of stochastic integration;

- prove existence and uniqueness of solutions to stochastic differential equations under certain conditions;

- construct the link between stochastic differential equations and Markov processes;

- connect SDEs and the forward and backward (Fokker-Planck) partial differential equations;

- develop techniques for solving the Fokker-Planck equation;

- assemble tools from elementary Hilbert space theory to study the ergodic properties of SDEs.

Module Content

The module is composed of the following sections:

I - Introduction

II - Elements of probability theory and of stochastic processes in continuous time

III - Brownian motion and stochastic calculus

IV - Stochastic integrals

V - Stochastic differential equations

VI - Applications to partial differential equations

VII - Markov processes and invariant measures

MATH70019 Theory of Partial Differential Equations

Brief Description

In this module, students are exposed to different phenomena which are modelled by partial differential equations. The course emphasizes the mathematical analysis of these models and briefly introduces some numerical methods.

Learning Outcomes

On successful completion of this module you will be able to:

- appreciate how to formally differentiate complicated finite dimensional functionals and simple infinite dimensional functionals;
- describe, select and use a variety of methods for solving partial differential equations;
- outline how various partial differential equations respect conservation laws;
- utilize energy methods to critically analyse the stability of solutions to PDEs;
- develop the general method of characteristics and derive the eikonal equation;
- justify the proper use of the calculus of variations in classical settings.

Module Content

The module is composed of the following sections:

1. Introduction to PDEs

   1.1. Basic Concepts

   1.2. Gauss Theorem

2. Method of Characteristics

   2.1. Linear and Quasilinear first order PDEs in two independent variables.

   2.2. Scalar Conservation Laws

   2.3. Hamilton-Jacobi Equations. General Method of Characteristics.

3. Diffusion

   3.1. Heat equation. Maximum principle

   3.2. Separation of variables. Fourier Series.

4. Waves

   4.1. The 1D wave equation

   4.2. 2D and 3D waves.

5. Laplace-Poisson equation

   5.1. Dirichlet and Neumann problems.

   5.2. Introduction to calculus of variations. The Dirichlet principle.

   5.3 Finite Element Method.

   5.4 Lagrangians and the minimum action principle.

MATH70020 Function Spaces and Applications

Brief Description

The purpose of this course is to introduce the basic function spaces and to train the student in the basic methodologies needed to undertake the analysis of Partial Differential Equations and to prepare them for the course "Advanced topics in Partial Differential Equations'' where this framework will be applied. Most of the topics contained in the module do not require preliminary knowledge. However, knowledge of the material in the Y2 module on "Lebesgue Measure and Integration" (or a suitable equivalent) is recommended.

Learning Outcomes

On successful completion of this module, you will be able to:

- appreciate the main concepts of metric topology and integration theory (Fatou's lemma, monotone and dominated convergence theorems);

- manipulate concepts associated with Banach spaces (Cauchy sequence, completeness concept, bounded operators, continuous linear forms, dual space);

- apply the concept of uniform convergence of functions, and those related to spaces of differentiable functions;

- interpret the concept of convergence in Lebesgue spaces;

- manipulate convolutions and sequences of mollifiers to approximate continuous or Lebesgue integrable functions by infinitely differentiable functions with compact support;

- appreciate the notion of compactness and the difference between finite and infinite-dimensional normed vector spaces;

- interpret results from advanced textbooks and research papers;

- independently appraise and evaluate an advanced topic (namely the notions of weak and weak-star compactness in Banach spaces).

Module Content

The course will span the basic aspects of modern functional spaces: integration theory, Banach spaces, spaces of differentiable functions and of integrable functions, convolution and regularization, Hilbert spaces. The concepts of Distributions and Sobolev spaces will be taught in the follow-up course ''Advanced topics in Partial Differential Equations'' as they are tightly connected to the resolution of elliptic PDE's and the material taught in the present course is already significant.

In addition to the material below, this level 7 (Masters) version of the module will have additional extension material for self-study. This will require a deeper understanding of the subject than the corresponding level 6 (Bachelors) module. The extra material will relate to the concept of compactness in Banach spaces.

The syllabus of the course is as follows:

1) Review of metric topology and Lebesgue's integration theory

2) Normed vector spaces. Banach spaces. Continuous linear maps. Dual of a Banach space.

3) Examples of function spaces: continuously differentiable function spaces and Lebesgue spaces. Hölder and Minkowski's inequalities. Convolution and Mollification. Approximation of continuous or Lebesgue integrable functions by infinitely differentiable functions with compact support.

4) Compactness: Non- compactness of the unit ball in infinite-dimensional normed vector spaces. Criteria for compactness in space of continuous functions: the Ascoli theorem. Compact operators. Additional reading: weak and weak star topologies and Banach-Alaoglu's theorem

5) Hilbert spaces. The projection theorem. The Riesz representation theorem. The Lax-Milgram theorem. Hilbert bases and Parseval's identity. Application to Fourier series.

MATH70021 Advanced Topics in Partial Differential equations

Brief Description

This course develops the analysis of boundary value problems for elliptic and parabolic PDE's using the variational approach. It is a follow-up of 'Function spaces and applications' but is open to other students as well provided they have sufficient command of analysis. An introductory Partial Differential Equation course is not needed either, although certainly useful.

Learning Outcomes

On successful completion of this module you will be able to:

- appreciate the concepts of distribution (differentiation, convergence);

- manipulate the main properties of the Sobolev space H^m for integer m (inbeddings and compactness theorems, Poincaré inequality);

- derive the variational formulation of a specific elliptic boundary value problem and to provide the reasoning leading to the proof of the existence and uniqueness of the solution;

- develop the spectral theory of an elliptic boundary value problem;

- solve a parabolic boundary value problem using the spectral theory of the associated elliptic operator.

- interpret results from advanced textbooks and research papers on the theory of Partial Differential Equations;

- independently appraise and evaluate an advanced topic on Partial Differential Equations, namely the theory of nonlinear elliptic and parabolic equations on the whole space.

Module Content

The course consists of three parts. The first part (divided into two chapters) develops further tools needed for the study of boundary value problems, namely distributions and Sobolev spaces. The following two parts are devoted to elliptic and parabolic equations on bounded domains. They present the variational approach and spectral theory of elliptic operators as well as their use in the existence theory for parabolic problems.

The aim of the course is to expose the students to some important aspects of Partial Differential Equation theory, aspects that will be most useful to those who will further work with Partial Differential Equations be it on the theoretical side or on the numerical one.

The syllabus of the course is as follows:

1. Distributions. The space of test functions. Definition and examples of distributions. Differentiation. Convergence of distributions.

2. Sobolev spaces: The space H1. Density of smooth functions. Extension lemma. Trace theorem. The space H10. Poincare inequality. The Rellich-Kondrachov compactness theorem (without proof). Sobolev imbedding (in the simple case of an interval of R). The space Hm. Compactness and Sobolev imbedding for arbitrary dimension (statement without proof).

3. Linear elliptic boundary value problems: Dirichlet and Neumann boundary value problems via the Lax-Milgram theorem. Spectral theory. The maximum principle. Regularity (stated without proofs). Classical examples: elasticity system, Stokes system.

4. Linear parabolic initial-value problems: Bochner Spaces. Existence and uniqueness of weak solutions by the Galerkin method. Application to the incompressible Navier-Stokes equations.

5. Additional reading for the mastery question: fixed point methods for solving nonlinear elliptic and parabolic problems.

4. Linear parabolic initial-boundary value problems. Existence and uniqueness by spectral decomposition on the eigenbasis of the associated elliptic operator. Classical examples (Navier-Stokes equation).

5. Additional reading for the mastery question: the Poisson equation in the whole space. The representation formula. Its uniqueness via the Liouville theorem for harmonic functions. Behaviour at infinity.

MATH70022 Finite Elements: Numerical Analysis and Implementation

Brief Description

Finite element methods form a flexible class of techniques for numerical solution of PDEs that are both accurate and efficient. The finite element method is a core mathematical technique underpinning much of the development of simulation science. Applications are as diverse as the structural mechanics of buildings, the weather forecast, and pricing financial instruments. Finite element methods have a powerful mathematical abstraction based on the language of function spaces, inner products, norms and operators.

Learning Outcomes

On successful completion of this module, you will be able to:

- appreciate the core mathematical principles of the finite element method;

- employ the finite element method to formulate and analyse numerical solutions to linear elliptic PDEs;

- implement the finite element method on a computer;

- compare the application of various software engineering techniques to numerical mathematics;

- generalize the concept of a directional derivative;

- appraise and evaluate techniques for solving nonlinear PDEs using the finite element method.

Module Content

This module aims to develop a deep understanding of the finite element method by spanning both its analysis and implementation. In the analysis part of the module, students will employ the mathematical abstractions of the finite element method to analyse the existence, stability and accuracy of numerical solutions to PDEs. At the same time, in the implementation part of the module students will combine these abstractions with modern software engineering tools to create and understand a computer implementation of the finite element method.

This module is composed of the following sections:

I - Basic concepts: weak formulation of boundary value problems, Ritz-Galerkin approximation, error estimates, piecewise polynomial spaces, local estimates;

II - Efficient construction of finite element spaces in one dimension: 1D quadrature, global assembly of mass matrix and Laplace matrix;

III - Construction of a finite element space: Ciarlet's finite element, various element types, finite element interpolants;

IV - Construction of local bases for finite elements: efficient local assembly;

V - Sobolev Spaces: generalised derivatives, Sobolev norms and spaces, Sobolev's inequality;

VI - Numerical quadrature on simplices: employing the pullback to integrate on a reference element;

VII - Variational formulation of elliptic boundary value problems: Riesz representation theorem, symmetric and nonsymmetric variational problems, Lax-Milgram theorem, finite element approximation estimates;

VIII - Computational meshes: meshes as graphs of topological entities, discrete function spaces on meshes, local and global numbering;

IX - Global assembly for Poisson equation: implementation of boundary conditions, general approach for nonlinear elliptic PDEs;

X - Variational problems: Poisson's equation, variational approximation of Poisson's equation, elliptic regularity estimates, general second-order elliptic operators and their variational approximation;

XI - Residual form and the Gâteaux derivative;

XII - Newton solvers and convergence criteria.

MATH70023 Numerical Solution of Ordinary Differential Equations

Brief Description

The module is an introductory course in numerical methods for ordinary differential equations. The purpose of this module is to learn how to use the computer to find numerical solutions to ordinary differential equations as well as to provide you with theoretical knowledge and practical skills to lay the solid groundwork necessary to advance in scientific computing.

Learning Outcomes

On the successful completion of the module you will be able to

- use classical numerical methods for ordinary differential equations;

- analyse different properties of numerical methods (e.g. accuracy and stabilty);

- develop your own methods with prescribed properties;

- compare different methods with respect to accuracy, stability, computational and space complexity;

- create efficient numerical algorithms;

- construct numerical methods to solve boundary value problems for partial differential equations.

Module Content

This module will cover the following topics:

- Taylor series methods;

- Linear multi-step methods;

- Runge-Kutta methods;

- Adaptive step size control;

- Boundary value problems for ordinary differential equations;

- Introduction to the finite difference method and energy Inequalities method;

- Introduction to boundary value problems for partial differential equations.

MATH70024 Computational Linear Algebra

Brief Description

Linear systems of equations arise in countless applications and problems in mathematics, science and engineering. Often these systems are large and require a computer to solve. This course provides an overview of the algorithms used to solve linear systems and eigenvalue problems, in terms of their development, stability properties, and application.

Learning Outcomes

On successful completion of this module, you will be able to

- describe, select and use algorithms for QR decomposition of matrices;

- solve least-squares problems using QR decomposition;

- apply LU decomposition to solve linear systems;

- analyse and modify algorithms that take advantage of matrix structure;

- find numerical solutions to eigenvalue problems;

- critically analyse various iterative methods for solving linear systems.

- combine the techniques you have mastered in order to assess unseen algorithms;

- adapt the techniques to analyse related topics such as functions of matrices.

Module Content

The module will cover the following topics:

1) Direct methods:

Triangular and banded matrices, Gauss elimination, LU-decomposition, conditioning and finite-precision arithmetic, pivoting, Cholesky factorisation, QR factorisation and their numerical implementation.

2) Eigenvalue problems:

power method and variants, Jacobi's method, Householder reduction to tridiagonal form, eigenvalues of tridiagonal matrices, the QR method.

3) Iterative methods:

Krylov subspace methods: Lanczos method and Arnoldi iteration, conjugate gradient method, GMRES, preconditioning.

MATH70025 Computational Partial Differential Equations

Brief Description

This module will introduce a variety of computational approaches for solving partial differential equations, focusing mostly on finite difference methods, but also touching on finite volume and spectral methods. Students will gain experience implementing the methods and writing/modifying short programs in Matlab or another programming language of their choice. Applications will be drawn from problems arising in areas such as Mathematical Biology and Fluid Dynamics.

Learning Outcomes

On successful completion of this module, you will be able to:

- appreciate the physical and mathematical differences between different types of PDES;

- design suitable finite difference methods to solve each type of PDE;

- outline a theoretical approach to testing the stability of a given algorithm;

- determine the order of convergence of a given algorithm;

- demonstrate familiarity with the implementation and rationale of multigrid methods;

- develop finite-difference based software for use on research level problems;

- communicate your research findings as a poster, in a form suitable for presentation at a scientific conference.

Module Content

The module will cover the following topics:

1) Introduction to Finite Differences

2) Classificaton of PDEs

3) Explicit and Implicit methods for Parabolic PDEs

4) Iterative Methods for Elliptic PDEs. Jacobi, Gauss-Seidel, Overrelaxation

5) Multigrid Methods

6) Hyperbolic PDEs. Nonlinear Advection/Diffusion systems. Waves and PMLs

as well as various advanced practical topics from Fluid Dynamics, which will depend on the final project.

MATH70026 Methods for Data Science

Brief Description

This module provides an hands-on introduction to the methods of modern data science. Through interactive lectures, the student will be introduced to data visualisation and analysis as well as the fundamentals of machine learning.

Learning Outcomes

On successful completion of this module, you will be able to:

- Visualise and explore data using computational tools;

- Appreciate the fundamental concepts and challenges of learning from data;

- Analyse some commonly used learning methods;

- Compare learning methods and determine suitability for a given problem;

- Describe the principles and differences between supervised and unsupervised learning;

- Clearly and succinctly communicate the results of a data analysis or learning application;

- Appraise and evaluate new algorithms and computational methods presented in scientific and mathematical journals;

- Design and implement newly-developed algorithms and methods.

Module Content

The module is composed of the following sections:

- Introduction to computational tools for data analysis and visualisation;

- Introduction to exploratory data analysis;

- Mathematical challenges in learning from data: optimisation;

- Methods in Machine Learning: supervised and unsupervised; neural networks and deep learning; graph-based data learning;

- Machine learning in practice: application of commonly used methods to data science problems. Methods include: regressions, k-nearest neighbours, random forests, support vector machines, neural networks, principal component analysis, k-means, spectral clustering, manifold learning, network statistics, community detection;

- Current research questions in data analysis and machine learning and associated numerical methods.

MATH70027 Scientific Computation

Brief Description

This module introduces students to the analysis and implementation of efficient algorithms used to solve mathematical and computational problems connected to a broad range of scientific topics. Mathematical tools and concepts from linear algebra, calculus, numerical analysis, and statistics will be utilised to develop and analyse computational solutions to mathematical and scientific problems.The objectives are that by the end of the module all students should have a good familiarity with the essential elements of the Python programming language and be able to undertake programming tasks in a range of areas.

Learning Outcomes

On successful completion of this module you will be able to:

- analyse the performance of simple sorting and searching algorithms and implement them in Python;

- computationally analyse complex networks and dynamical processes of complex systems;

- effectively utilise important tools for data analysis such as discrete Fourier transforms;

- evaluate and implement numerical methods for mathematical optimisation and the solution of differential equations;

- assess the correctness and efficiency of simple data structures and algorithms on graphs and implement them in Python;

- independently appraise and evaluate a range of state-of-the art algorithms and computational methods;

- adapt a range of computational methods and apply them in a coherent manner to an open scientific problem.

Module Content

The module will cover the following topics:

1) Sorting and searching with scientific applications from fields such as bioinformatics;

2) Algorithms on graphs and basic data structures such as queues and hash tables;

3) Methods for data analysis using tools such as discrete Fourier transforms;

4) Analysis and use of common optimisation methods such as Simulated Annealing;

5) Numerical solution of differential equations arising in multiscale problems;

6) Computational analysis of complex systems.

MATH70134 Mathematical Foundations of Machine Learning

Brief Description

Machine learning techniques such as deep learning have recently achieved remarkable results in a very wide variety of applications such as image recognition, self-driving vehicles, partial differential equation solvers, trading strategies. However, how and why the recent (deep learning) models work is often still not fully understood. In this course we will begin with a general introduction into machine learning and continue to deep learning. We will focus on better some observed phenomena in deep learning aiming to gain insight into the impact of the optimization algorithms and network architecture through mathematical tools.

Learning Outcomes

On successful completion of this module you will be able to:
1) Demonstrate working familiarity with machine learning principles,
2) Design models using a variety of deep learning architectures
3) Implement neural network models in code
4) Select appropriate optimization algorithms to train deep learning models
5) Evaluate the ability of models to generalize by comparing their training and test performance
6) Independently evaluate new methodologies in deep learning

Module content

- The preliminaries: pre-processing: data cleaning, dimensionality reduction, clustering
- Regression (linear, Bayesian) and classification (a basic overview)
- Neural networks and a variety of architectures (fully-connected, convolutional)
- Generalisation and overfitting
- Training methods for neural networks and their impact on performance
- The role of noise
- Flat minima and escape times
- Links of neural networks to Gaussian processes
- Explainability in neural networks through reconstruction

MATH70028 Probability Theory

Brief Description

This module provides a rigorous approach to the fundamental properties of probability. It teaches fundamental notions and structures as well as tools relevant to modern probability theory and applications. The module is important for further study of probability theory and stochastic processes.

Learning Outcomes

On successful completion of this module, you should be able to:

- Demonstrate knowledge of fundamental notions and structures of Probability Theory;

- Use the tools developped in a variety of applications;

- Demonstrate problem solving skills in probability theory

-- Demonstrate additional competence in the subject through self-study of more advanced material

-- Combine material from across the module to solve more advanced problems

- Communicate your knowledge of the area in a concise, accurate and coherent manner.

Module Content

An indicative list of topics is:

Probability spaces. Random variables: (Bernoulli, Rademacher, Gaussian variables with in- tegration by parts formula). Probability Distributions.

Basic probability inequalities: Jensen, Tshebychev, Poincare* & Log-Sobolev Inequalities *. Tail of Distribution Estimates.

Convergence in probability, in p-th moment, almost everywhere. 0-1 Law.

Mutual Independence of Events/Random Variable and Vieta Formula. Product Probability Spaces. Conditional Expectations and Independence. Borel-Cantelli Lemmas.

Weak and Strong Laws of Large Numbers for Random Sequences and Series of Mutually Independent or Weakly Correlated Random Variables.

Applications : [Probabilistic proof of Weierstrass Theorem, Monte Carlo Method for Large Dimensional Integration, Macmillan's Theorem, Infinitely Often Events: Decadence and Re- currence of Human Civilisations, Normal Numbers...]

Weak Convergence & Characteristic Functions. Central Limit Theorem.

Infinite Product of Bernoulli measures versus Gaussian measure.

Birkhoff Ergodic Theorem.* Elements of Brownian motion.* Martingales.*

Topics denoted by * are more advanced and require self-study through directed reading.

MATH70029 Functional Analysis

Brief Description

This module brings together ideas of continuity and linear algebra. It concerns vector spaces with a distance, and involves linear maps. The vector spaces are often spaces of functions. It is an important requirement for further study of many areas of Mathematical Analysis including PDEs, Stochastic Analysis, Dynamical Systems and Quantum Mechanics.

Learning Outcomes

On successful completion of this module, you should be able to:

- Demonstrate knowledge of fundamental notions and structures of Functional Analysis by proving a range of results;

- Use the tools developped in a variety of applications;

- Demonstrate problem solving skills in functional analysis

-- Demonstrate additional competence in the subject through self-study of more advanced material

- Synthesise topics from across the module to solve problems on more advanced applications

- Communicate your knowledge of the area in a concise, accurate and coherent manner.

Module Content

An indicative list of topics is:

Metric Linear Spaces and basic examples of topological spaces with non metrisable topology.

Minkowski and Hoelder Inequality.

Existence of Hamel basis (axiom of choice 1st time).

Normed vector spaces

& example of not normed Frechet space (Schwartz test functions).

Banach spaces.

Classical Banach Spaces: l_p, c, c_0, $L_p(\mu)$, $C(\Omega)$, $C^{(m)}(\Omega)$.

Closed Subspaces, Completness, Separability and Compactness in Classical Spaces.

Schauder Basis.

Continuous linear maps.

Banach contraction mapping principle and applications to integral equations (Frdholm+Volterra).
Finite dimensional spaces.

The Hilbert space (orthonormal basis).

The Riesz-Fisher Theorem.

The Hahn-Banach Theorem. (Banach Limit.)

Dual spaces: Dual spaces of classical spaces. Reflexive Non-reflexive spaces.

Baire Cathegory Theorem (axiom of choice again).

Principle of Uniform Boundedness. (Application to Fourier Series).

Open Mapping and Closed Graph Theorems.

Compact operators.

Hermitian operators and the Spectral Theorem.

In addition to the above topics, this level 7 version of the module will also involve study of:

Gauge Norms and Orlicz spaces.

Weak topology (Banach-Alaoglu Thm)

Sobolev Spaces.

The module provides a general orientation in contemporary research problems in Mathematical Analysis includ- ing PDEs , Stochastic Analysis, Dynamical Systems and Quantum Mechanics.

MATH70030 Fourier Analysis and Theory of Distributions

Brief Description

Fourier analysis is an important tool used in various branches of mathematics and beyond. The module provides a deeper understanding of it than what is briefly mentioned in general analysis courses. It also connects it to the theory of distributions. As a result of studying the module, students will understand the basics of the Fourier analysis and theory of distributions which will be sufficient for most branches of mathematics.

Learning Outcomes

On successful completion of this module, you will be able to:

On successful completion of this module, you will be able to:

-understand the issues of convergence for Fourier series,

-apply the Fourier and Laplace transforms,

-understand the motivation behind the notion of distribution,

-be in command of the basics of Fourier analysis and distribution theory sufficient for working in many areas of mathematics

– demonstate competence with further advanced material in the area designated for self-study

-- synthesise material from across the module to apply to advanced topics

Module Content

The module will assume familiarity with measure theory and functional analysis, especially L^p spaces and linear functionals.

Indicative content: Orthogonal systems in infinite-dimensional Euclidean spaces, Bessel inequality, Parseval equality, general Fourier series, trigonometric basis in L_2[-Pi,Pi], convergence of trigonometric Fourier series, Fejer's theorem and applications, Fourier transform and its properties, application to solution of differential equations, Plancherel theorem, Laplace transform, linear functionals, distributions, basic properties of distributions and applications, Fourier transform for distributions.

Those students who decide to do a PhD in a closer related area of analysis will be able to use the acquired basic knowledge and skills to relatively easily extend their knowledge to more sophisticated areas of the theory.

This Level 7 version of the module will involve extra self-study of extension material (in the form of a book chapter, additional notes or a research paper) applying or extending material from the above topics.

MATH70135 Analytic Methods in Partial Differential Equations

Brief Description

This module introduces some of the partial differential equations (PDE) appearing in physics and geometry, as well as a number of classical techniques to study them analytically. One focus is on building up some intuition for the broad variety of PDEs, the phenomena they describe and for the mathematical techniques that have been developed to study them.

Learning Outcomes

On successful completion of this module you should:

-- have developed an intuition for a variety of partial differential equations. (behaviour of their solutions, techniques to study them).

-- understand some of the deep connections of PDE to physics and geometry.

-- be able to state and prove well-posedness theorems for a variety of PDE and understand their relevance.

-- be familiar with elliptic equations and elliptic regularity theory.

-- be familiar with hyperbolic equations (wave equations).

-- be able to attend a research talk in PDE

Module content

--Review of ODE Theory (Picard's Theorem, Gronwall's inequality).

--Theory of first order quasilinear PDE (Methods of Characteristics).

--Cauchy-Kovalevskaya Theorem (with sketch of the proof).

--Holmgren's uniqueness theorem (with proof via Cauchy-Kovalevskaya, examples).

--Laplace's equation (fundamental solution, regularity of harmonic functions, maximum principle, Green's function for a ball),

--General second order elliptic equations (Existence and Regularity Theory, Fredholm Alternative).

--Discussion of Schroedinger and Heat Equation (Schwartz space, Fourier techniques).

--Wave Equation (Energy estimate, domain of dependence, domain of influence, fundamental solution, solution via Fourier techniques, Duhamel's principle.)

It will be helpful if you have taken one or more of the following courses:

Functional Analysis, Measure and Integration (or Lebesgue Measure and Integration), Fourier Analysis and Distributions.

MATH70055 Stochastic Calculus with Applications to non-Linear Filtering

Brief Description

The module offers a bespoke introduction to stochastic calculus required to cover the classical theoretical results of nonlinear filtering. The first part of the module will equip the students with the necessary knowledge (e.g., Ito Calculus, Stochastic Integration by Parts, Girsanov's theorem) and skills (solving linear stochastic differential equation, analysing continuous martingales, etc) to handle a variety of applications. The focus will be on the use of stochastic calculus to the theory and numerical solution of nonlinear filtering.

Learning Outcomes

On successful completion of this module, you will be able to: a. understand the notion on Brownian motion and able to show that a stochastic process is a Brownian motion, b. Prove that a process is a martingale via Novikov's condition. c. Solve linear SDEs, d. Be able to check whetheran SDE is well-posed. d. understand the mathematical framework of nonlinear filtering e. Deduce the filtering equations. f. Deduce the evolution equation of the mean and variance of the one-dimensional Kalman-Bucy filter, g. Show that the innovation process is a Brownian motion. h. Apply stochastic integration by parts.

Module Content

An indicative list of topics is:

1. Martingales on Continuous Time (Doob Meyer decomposition, L_p bounds, Brownian motion, exponential martingales, semi-martingales, local martingales, Novikov's condition)

2. Stochastic Calculus (Ito's isometry, chain rule, integration by parts)

3. Stochastic Differential Equations (well posedness, linear SDEs, the Ornstein-Uhlenbeck process, Girsanov's Theorem)

4. Stochastic Filtering (definition, mathematical model for the signal process and the observation process)

5. The Filtering Equations (well-posedness, the innovation process, the Kalman-Bucy filter)

Prerequisites: Ordinary differential equations, partial differential equations, real analysis, probability theory.

MATH70031 Markov Processes

Brief Description

Markov processes are widely used to model random evolutions with the Markov property `given the present, the future is independent of the past'. The theory connects with many other subjects in mathematics and has vast applications.

Learning Outcomes

On successful completion of this module, you should be able to:

-- demonstrate your understanding of the concepts and results associated with the elementary theory of Markov processes, including the proofs of a variety of results

-- apply these concepts and results to tackle a range of problems, including previously unseen ones apply your understanding to develop proofs of unfamiliar results

- demonstrate additional competence in the subject through the study of more advanced material

- combine ideas from across the module to solve more advanced problems

- communicate your knowledge of the area in a concise, accurate and coherent manner.

Module Content

Markov processes are widely used to model random evolutions with the Markov property `given the present, the future is independent of the past'. The theory connects with many other subjects in mathematics and has vast applications. This course is an introduction to Markov processes. We aim to build intuitions and good foundations for further studies in stochastic analysis and in stochastic modelling.

The module is largely self-contained, but it is strongly advised that students should have taken the module Measure and Integration (or equivalent). A good knowledge of real analysis and metric spaces will be assumed.

The module is related to a number of modules in stochastic analysis, probability theory, dynamical systems and mathematical finance.

An indicative list of contents is:

1. Discrete time and finite state Markov chains : Chapman-Kolmogorov equations, irreducible, Perron-Frobenius theorem for stochastic matrices, recurrent and transient.

2. Discrete time Markov processes on general state space. Conditional expectations, Chapman-Kolmogorov equation, Feller property, strong Feller property, Kolmogorov's theorem, stopping times, strong Markov, stationary process, weak convergence and Prohorov's theorem, Existence of invariant measures : Krylov-Bogolubov method, Lyapunov method. Ergodicity by contraction method and Doeblin's criterion. Structures of invariant measures, ergodic theorems.

There will also be extra self-study of extension material (in the form of a book chapter, additional notes or a research paper) applying or extending material from the above topics.

MATH70032 Geometry of Curves and Surfaces

Brief Description

This module is an introduction to classical theory of differential geometry, where we disucss geometric features of curves and surfaces in (mostly) three dimensional Euclidean spaces.

Learning Outcomes

On successful completion of this module, you will be able to:

- identify regular curves and implement different re-parametrisations of curves in two and three dimensional spaces,

- learn about and calculate the geometric quantities of curvature and torsion of a regular curve,

- identify regular surfaces in 3 dimensional spaces using the notions of charts,

- analyse the regularity of maps from one surface into another surface, and also of functions on surfaces,

- use partitions to calculate the basic topological invariant of Euler characteristic,

- learn about the topological classification of compact surfaces, and identify them,

- calculate the first and second fundamental forms of a surface,

- learn about the existence and uniqueness of geodesics on general surfaces,

- link the Gaussian curvature to the local shape of a surface, and present different kinds of examples,

- analyse the global topological features of a surface by integrating local geometric features (Gauss-Bonnet and winding numbers)

– demonstate competence with further advanced material in the area designated for self-study

-- synthesise material from across the module to apply to advanced topics

Module Content

This module is an introduction to classical theory of differential geometry, where we disucss geometric features of curves and surfaces in (mostly) three dimensional Euclidean spaces. A curve, which is the trajectory of a particle moving in a smooth fashion, may twist in two manners described by the values called curvature and torsion. The twists of a surface in three dimensional space is naturally more involved. There are different notions of curvature: the Gaussian curvature and the mean curvature. The Guassian curvature describes the intrinsic geometry of the surface, and the mean curvature describes how it bends in space. We look at several examples of surfaces, and calculate their curvatures. We study the local shapes of surfaces based on their curvatures. For example, the Gaussian curvature of a sphere is strictly positive, which explains why any planar illustration of the countries distorts shapes. Remarkably, these local gometric notions can be combined to derive global information about the topology of the surface (for example the Gauss-Bonnet formula). This module starts with the basic real analysis taught in years 1 and 2, and leads into the more modern and general theory of manifolds.

An indicative list of sections and topics is:

• Curves in two and three-dimensional spaces: re-parametrizations, curvature and torsion, Frenet-Serret formulae, curves are determined by curvature and torsion, winding number and the total curvature,

• Surfaces: Charts, Tangent vectors, and tangent planes, Smooth maps from one surface into another surface, smooth functions on a surface, Normal vectors,

• Curvature of a surface: the first and second fundamental forms, Christoffel symbols, normal curvature, Gaussian curvature, and mean curvature, Gauss's Theorema Egregium,

• Area of a surface,

• Geodesics on a surface: length-minimising curves, existence, non-existence and examples, geodesic curvature,

• Gauss-Bonnet Theorem and applications

• Topological classification of surfaces

• Vector fields and the Poincare-Hopf Theorem

The module will assume familiarity with material in the second-year module Analysis II

This Level 7 version of the module will involve extra self-study of extension material (in the form of a book chapter, additional notes or a research paper) applying or extending material from the above topics.

MATH70033 Algebraic Curves

Brief Description

This module is meant as a first encounter with algebraic geometry, through the study of affine and projective plane curves over the field of complex numbers. We will also discuss some complex-analytic aspects of the theory (Riemann surfaces). Important results include the definition of local intersection multiplicities and Bézout's theorem, inflection points and the classification of plane cubics, linear systems of curves, and the degree-genus formula.

Learning Outcomes

On successful completion of this module, you will be able to:

- solve geometric problems about affine and projective plane curves with algebraic techniques;

- determine the projectivizations of affine plane curves and the points at infinity;

- determine the tangent lines of plane curves at smooth and singular points;

- compute projective transformations and find convenient coordinate systems;

- compute intersection multiplicities using resultants and the axiomatic characterization;

- formulate, prove and apply Bézout's theorem;

- find inflection points of projective plane curves and use them to classify cubic curves;

- solve enumerative problems by means of the theory of linear systems;

- work with holomorphic charts to determine local and global properties of Riemann surfaces and morphisms;

- compute ramification degrees of morphisms of Riemann surfaces;

- formulate, prove and apply the degree-genus formula for smooth projective plane curves.

– demonstate competence with further advanced material in the area designated for self-study

-- synthesise material from across the module to apply to advanced topics

Module Content

An indicative list of sections and topics is:

- Affine plane curves;

- The geometry of projective spaces;

- Projective plane curves;

- Smooth and singular points, tangent lines;

- Projective transformations and the classification of conics;

- Intersection multiplicities (resultants and axiomatic characterization)

- Bézout's theorem on intersections of projective plane curves;

- the Legendre family of cubics, inflection points and the classification of non-degenerate smooth cubics;

- linear systems of projective plane curves, projective duality and enumerative geometry;

- Riemann surfaces;

- local description of morphisms of Riemann surfaces (ramification);

- classification of topological surfaces and genus (informal introduction);

- Riemann-Hurwitz and the degree-genus formula.

Some related topics will appear in the problem sheets and the coursework (e.g., dual curves, group structure on smooth cubics).

This Level 7 version of the module will involve extra self-study of extension material (in the form of a book chapter, additional notes or a research paper) applying or extending material from the above topics.

MATH70034 Algebraic Topology

Brief Description

This module gives a first introduction to algebraic topology. After some preliminary results on quotient spaces and CW-complexes, we discuss fundamental groups and the Galois correspondence for covering spaces. We then move on to homology theory and study simplicial and singular homology, as well as some applications like the Jordan curve theorem and invariance of domain. Throughout the module, we pay special attention to algebraic and categorical aspects.

Learning Outcomes

On successful completion of this module, you will be able to:

- define the basic invariants in algebraic topology and prove their main properties;

- use algebraic techniques to distinguish different homotopy types and classify topological objects;

- compute fundamental groups, simplicial homology groups and singular homology groups;

- apply the Galois correspondence to classify covering spaces of topological spaces;

- apply fundamental groups and homology groups to prove fundamental topological properties (Brouwer's fixed point theorem, Jordan's curve theorem, invariance of domain);

- formulate topological and algebraic constructions in a categorical language (universal properties); analyze the structure of quotients of topological spaces by covering space actions;

- represent groups geometrically by means of Cayley complexes

Module Content

An indicative list of sections and topics is:

Preliminaries:

- Homotopy and homotopy type

- Cell complexes

- Operations on spaces

The Fundamental Group:

- Paths and Homotopy

- Presentations of groups, amalgamated products and Van Kampen's Theorem

- Covering Spaces

- The Galois correspondence -Deck Transformations and Group Actions -Cayley complexes

Homology

- Δ-complexes and simplicial homology

- Singular homology

- Homotopy invariance

- Relative homology, exact sequences and excision

- The equivalence of simplicial and singular homology

- Mayer-Vietoris Sequences

- Applications

The main reference for this course is "Algebraic topology" by Hatcher.

This Level 7 version of the module will involve extra self-study of extension material (in the form of a book chapter, additional notes or a research paper) applying or extending material from the above topics.

The module will assume familiarity with the material in the second-year modules: Groups and Rings, Analysis II

MATH70056 Algebraic Geometry

Brief Description

Algebraic geometry is the study of the space of solutions to polynomial equations in several variables. In this module you will learn to use algebraic and geometric ideas together, studying some of the basic concepts from both perspectives and applying them to numerous examples.

Learning Outcomes

On successful completion of this module, you will be able to:

1. Understand the dictionary between algebra and geometry that arises from zero loci of polynomials in n-dimensional space;

2. Understand and compute irreducible and connected components of such zero loci;

3. Understand the concepts of regular and rational maps and their algebraic and geometric meaning;

4. Understand the projective space and the role it plays in compactifying zero loci of polynomials;

5. Understand the notion of dimension of zero loci of polynomials and their behaviour under regular maps;

6. Apply dimension theory and the Zariski topology in examples such as those coming from parameter spaces;

7. Understand how to generalise these ideas to the setting of more general commutative rings via (maximal) spectra (Mastery)

Module Content

An indicative list of topics is:

Affine varieties, projective varieties. The Nullstellensatz.

Regular and rational maps between varieties. Completeness of projective varieties.

Dimension. Parameter spaces.

Examples of algebraic varieties.

Spectrum and maximum spectrum (mastery)

Prerequisites: Commutative Algebra

MATH70057 Riemannian Geometry

Brief Description

The main aim of this module is to understand geodesics and curvature and the relationship between them. Using these ideas we will show how local geometric conditions can lead to global topological constraints.

Learning Outcomes

On successful completion of this module, you will be able to:

- Understand the relevant structures required to make sense of differential topological notions, such as derivatives of smooth functions, and geometric notions, such as lengths and angles, on an abstract manifold.

- Define the Lie derivative and covariant derivative of a tensor field.

- Define geodesics and understand their length minimising properties.

- Define and interpret various measures of the curvature of a Riemannian manifold.

- Understand the effect of curvature on neighbouring geodesics.

- Prove the celebrated classical theorems of Bonnet--Myers and Cartan--Hadamard.

Module Content

An indicative list of topics is:

Topological and smooth manifolds, tangent and cotangent spaces, vector bundles, tensor bundles, Lie bracket, Lie derivative, Riemannian metrics, affine connections, the Levi-Civita connection, parallel transport, geodesics, Riemannian distance, the exponential map, completeness and the Hopf--Rinow Theorem, Riemann and Ricci curvature tensors, scalar curvature, sectional curvatures, submanifolds, the second fundamental form and the Gauss equation, Jacobi fields and the second variation of geodesics, the Bonnet--Myers and Cartan--Hadamard Theorems.

Prerequisites: Geometry of Curves and Surfaces and Manifolds

MATH70058 Manifolds

Brief Description

The goal of this course is to introduce the theory of smooth manifolds. The class starts by defining smooth manifolds, submanifolds and tangent spaces. It will then develop more advanced topics like the theory of vector bundles, which will be used to introduce the notion of the tangent bundle, the cotangent bundle, vector fields and differential forms on a smooth manifold. This allows to define integration on an orientable manifold and then to prove Stokes' Theorem on a manifold with boundary.

Learning Outcomes

On successful completion of this module, you will be able to:

- Define smooth manifolds in an intrinsic way, by using the notion of charts, transition functions and smooth atlases.

- Determine sufficient conditions under which the level set of a smooth function is a submanifold.

- Study vector bundles on a manifold and determine necessary and sufficient condition for a vector bundle to be trivial.

- Study vector fields on a manifold and describing them locally, through the use of charts.

- Define the integration of differential forms on an orientable manifold.

- Prove Stokes' Theorem, which is one of the main tools used in differential topology.

Module Content

This module focuses on foundations as well as examples.

An indicative list of topics is:

Smooth manifolds, quotients, smooth maps, submanifolds, rank of a smooth map, tangent spaces, vector fields, vector bundles, differential forms, the exterior derivative, orientations, integration on manifolds (with boundary) and Stokes' Theorem.

MATH70059 Differential Topology

Brief Description

In this module you will understand how geometry and topology interact on smooth manifolds. You will investigate different (co)homology theories, see how to relate them, and study how to use them

to analyse the topology of a manifold.

Learning Outcomes

On sccessful completeion of this module you will be able to:

- Apply the concepts of homology and cohomology, as well as central results such as Poincaré Duality and the De Rham theorem, to investigate manifolds.

- Use a Mayer-Vietoris argument to compute (co)homology groups.Describe the topology of a manifold by analysing the critical points of a Morse function and the gradient

flow lines between them.

- Use and explain the equivalence between the different (co)homology theories introduced (De Rham, singular, Morse), for example how the CW complex of a manifold relates to the homology groups generated by the critical points of Morse functions on the manifold.

- Work independently and with peers to formulate and solve problems in geometry using tools of algebraic and differential topology.

Module Content

An indicative list of contents is:

-De Rham Cohomology: Definition, Poincaré's Lemma, Mayer-Vietoris sequences, compactly supported de Rham cohomology, pairings and Poincaré Duality with applications, degree of a map, mapping degree theorem and examples.

-Morse Theory: Introduction and basics, Fundamental Theorems of Morse Theory, the CW-structure associated to Morse-functions, stable and unstable manifolds, Morse-Smale functions, orientations, Morse homology and Morse Homology Theorem. Examples.

-Singular Homology: Basic definitions, properties and examples, De Rham Theorem.

The module will assume familiarity with topics in Algebraic Topology and smooth manifolds. In particular students should be familiar with: Vector fields, differential forms (k-forms, exterior differential, closed and exact forms), integration on manifolds and Stokes' Theorem, basics of homological algebra (exact sequences, Snake Lemma).

MATH70060 Complex Manifolds

Brief Description

The goal of this course is to introduce the theory of almost complex manifolds and complex manifolds. Many important examples will be provided, such as Kähler manifolds and complex projective manifolds. After introducing some of the main tools, as the Hermitian metrics, the Chern connection and the co-homology of a complex manifold, the theory of Hodge decomposition for Kähler manifolds will be presented, together with many of its applications. The class will culminate with the Kodaira embedding theorem and with the main notions of Kodaira-Spencer deformation theory.

Learning Outcomes

On successful completion of this module, you will be able to:

- Study many examples of complex and almost complex manifolds, such as Hopf manifolds, projective spaces, Kähler manifolds, and projective varieties.

- Introduce tools like Hermitian metrics, holomorphic vector bundles and Chern connections on a complex manifold.

- Study harmonic forms on a complex manifolds and then the Dalbout and the de Rham co-homology of a Kähler manifold, culminating with the Hodge decomposition theorem and several of its applications.

- Use holomorphic line bundles to study the Kodaira embedding theorem, which provides a characterisation of complex projective manifolds.

- Introduce the basic notions of the Kodaira-Spener deformation theory.

Module Content

Prerequisite: Manifolds

An indicative list of topics is:

Complex and almost complex manifolds, integrability. Examples such as the Hopf manifold, projective space, projective varieties. Hermitian metrics, Chern connection. Various equivalent formulations of the Kaehler condition. Hodge decomposition for Kaehler manifolds. Line bundles and Kodaira embedding. Statement of GAGA. Basic Kodaira-Spencer deformation theory.

MATH70035 Algebra 3

Brief Description

This course continues the study of commutative rings and introduces the notion of R-module, which is an analogue over rings of the notion of a vector space over a field. Using these ideals we prove fundamental results about various classes of rings, particularly polynomial rings in several variables.

Learning Outcomes

On successful completion of this module, you will be able to:

- Understand the detailed theory of finite fields, their classification, and factorization of polynomials over finite fields

- Understand the theory of R-modules and their presentations

- Understand the classification of modules over Euclidean Domains, and how to use Smith Normal Form to determine the isomorphism class of such a module given a presentation

- Use this classification, in the case of K[T]-modules, to prove fundamental results in linear algebra

- Apply several different criteria for irreducibility of polynomials over various base rings

– demonstate competence with further advanced material in the area designated for self-study

-- synthesise material from across the module to apply to advanced topics

Module Content

An indicative list of sections and topics is:

- Chinese Remainder Theorem

- Field Extensions and Finite Fields

- R-modules

- Free modules and presentations

- Modules over Euclidean Domains

- Noetherian rings

- Gauss's Lemma and Factorization in polynomial rings

- If R is a UFD, so is R[X]

- Irreducible Polynomials and factorization of polynomials

This Level 7 version of the module will involve extra self-study of extension material (in the form of a book chapter, additional notes or a research paper) applying or extending material from the above topics.

MATH70036 Group Theory

Brief Description

This module builds on the Group Theory from the 1st year module Linear Algebra & Groups and the 2nd year module Groups and Rings. We start with a discussion of isomorphism theorems, and proceed to further example of groups and operations on them, including automorphism groups and semidirect products. Special attention is given to group actions and permutation groups: primitivity, multiple transitivity etc. Further we discuss solvable and nilpotent groups and their characterizations.

Learning Outcomes

On successful completion of this module, you will be able to:

– identify features of, and develop arguments about, certain groups and classes of group;

– explain the principles of group actions, and work with elementary examples;

– construct and work with direct and semidirect products of groups;

– state the definition of and extraspecial group and construct small examples of them;

– construct certain classical series of doubly and triply transitive groups;

– state and prove the structure theorem for nilpotent groups;

– explain the principles of group actions, and work with elementary examples;

– determine the normal structure and calculate the automorphism groups of symmetric groups;

– work independently and with peers to articulate understanding of abstract concepts in algebra.

– demonstate competence with further advanced material in the area designated for self-study

-- synthesise material from across the module to apply to advanced topics

Module Content

An indicative list of sections and topics is:

Definition and basic properties of groups. Isomorphism Theorems. Sylow subgroups. Group actions, primitivity and multiple transitivity. Composition series. Nilpotent groups. Solvable groups. Symmetric groups. Automorphism group of a group and semidirect products. Linear groups: centres and commutator subgroups, with small examples.

Further advanced material on these topics will be set for self-study.

MATH70037 Galois Theory

Brief Description

The formula for the solution to a quadratic equation is well-•known. There are similar formulae for cubic and quartic equations but no formula is possible for quintics. The module explains why this happens.

Learning Outcomes

On successful completion of this module you should be able to:

-- state, prove, and apply the fundamental theorem of Galois theory (aka the "Galois correspondence").

-- work with simple examples such as cubic polynomials, cyclotomic polynomials, and polynomials over finite fields.

-- compute Galois groups of splitting fields of cubic and bi-quadratic polynomials in arbitrary characteristic.

-- state and apply the formulas for solving cubic and quartic equations, and to prove that there are no such formulas for equations of degree 5 or larger.

- compute Galois groups over the rationals by the method of Frobenius elements.

-- Demonstrate additional competence in the subject through the self-study of designated advanced material

-- Combine topics from across the module to obtain more advanced results

Module Content

Familiarity with the following topics from Algebra 3

will be assumed: irreducible polynomials and factorization of polynomial;

Gauss's Lemma and factorization in polynomial rings.

An indicative list of topics is:

Field extensions, degrees and the tower law

Splitting fields, normal extensions, separable extensions

Automorphisms, fixed fields and the fundamental theorem

Examples: cubic and biquadratic extensions, finite fields

Extensions of the rationals and Frobenius elements

Cyclotomic extensions

Kummer theory and the insolubility of quintic equations

Material for self-study (in the form of a book chapter, additional notes or a research paper) applying or extending material from the above topics.

MATH70038 Graph Theory

Brief Description

A graph is a structure consisting of vertices and edges. Graphs are used in many areas of Mathematics, and in other fields, to model sets with binary relations. In this module we study the elementary theory of graphs; we discuss matters such as connectivity, and criteria for the existence of Hamilton cycles. We treat Ramsey's Theorem in the context of graphs, with some of its consequences. We then discuss probabilistic methods in Graph Theory, and properties of random graphs.

Learning Outcomes

On successful completion of this module, you will be able to:

- Demonstrate facility with the terminology of graphs and simple graph constructions to analyse examples and prove results

- Explain the proofs of the theorems of König and Menger, and certain other related results. Apply these results to appropriate problems.

- State, prove and apply Turán's Theorem. Describe and apply certain results in the theory of Hamilton cycles, including Dirac's Theorem.

- Explain and reason about Ramsey's Theorem and related results in the context of graph colourings.

- Describe various models of random graphs and apply probabilistic arguments to situations in graph theory.

– demonstate competence with further advanced material in the area designated for self-study

-- synthesise material from across the module to apply to advanced topics

Module Content

An indicative list of sections and topics is:

Standard definitions and basic results about graphs. Common graph constructions: complete graphs, complete bipartite graphs, cycle graphs.

Matchings and König's Theorem. Connectivity and Menger's Theorem.

Extremal graph theory. The theorems of Mantel and Turán. Hamilton cycles, and conditions for their existence.

Ramsey Theory for graphs, with applications.

The Probabilistic Method and random graphs. Evolution of random graphs.

This Level 7 version of the module will involve extra self-study of extension material (in the form of a book chapter, additional notes or a research paper) applying or extending material from the above topics.

MATH70039 Group Representation Theory

Brief Description

This module defines and begins the study of representations of groups, focusing on finite-dimensional complex representations of finite groups. These structures encode ways that groups can act as symmetries, which appear throughout mathematics (notably in algebra, number theory, and geometry, but also in analysis) as well as in physics and chemistry, among other places. We explain how to understand and classify these representations through characters, or traces. In the final unit we generalise the theory to finite-dimensional modules over rings, particularly semisimple algebras, whose theory retains many of the features of that of representations of groups.

Learning Outcomes

On successful completion of this module you will be able to:

- Recall and use basic definitions in group representations, their character theory, and modules over algebras, particularly finite-dimensional semisimple algebras;

- Explain and work with the features of complex representations of finite groups that allow one to simplify the theory (e.g., semisimplicity, character tables, etc.);

- Apply these results to classify representations of finite groups and semisimple algebras and compute the character tables of finite groups;

- perform basic constructions of representations of groups and to apply them to obtain all finite-dimensional representations of certain basic groups up to isomorphism;

- Explain the relationship between finite-dimensional irreducible representations of algebras and of finite-dimensional semisimple algebras, and the basic properties of their characters;

- Relate endomorphisms of representations to central elements in groups and semisimple algebras;

- Work independently and with peers to formulate and solve problems in algebra and geometry using tools of representation theory;

- Demonstrate abilility to engage with more advanced material via self-study.

-- Combine material from across the module to address more challenging problems

Module Content

- Basic theory: definitions, Maschke's theorem, Schur's Lemma, classification and construction of representations of finite abelian groups, dihedral groups, and small symmetric and alternating groups;

- Tensor products of representations and homomorphism spaces, the regular representation;

- Character theory: behaviour under direct sums and tensor products, orthogonality relations, computation of character tables of certain groups;

- Finite-dimensional modules over algebras: definition of finite-dimensional semisimple algebras and relationship of finite-dimensional modules of general algebras

to those of finite-dimensional semisimple algebras; constructions of projections via the center;

- Other important examples of modules over algebras.

- Further material, related to the above but of a more advanced character (consisting of a book chapter or research paper), will be designated for self-study.

MATH70132 Mathematical Logic

Brief Description

The module is concerned with some of the foundational issues of mathematics: formal logic and set theory. In propositional and predicate logic, we analyse the way in which we reason formally about mathematical structures. In set theory, we will look at the ZFC axioms and use these to develop the notion of cardinality. Material on model theory will involve all of these topics.

Learning Outcomes

On successful completion of this module, you should be able to:

- Understand how the notion of truth is defined precisely in propositional and predicate logic and apply the definitions and accompanying results in a variety of contexts.

- Demonstrate understanding of formal systems for propositional and predicate logic by constructing examples of formal proofs and deductions, and by applying and deriving general results about these.

- Appreciate the expressive power of a first-order language (and its limitations) and compare structures via their first-order theories.

- Relate the semantic and syntactic aspects of formal logic, have an understanding of powerful general results such as the completeness and compactness theorems, and be able to apply these in a variety of ways.

- Use the ZFC axioms to justify constructions in set theory, ranging from elementary applications, to constructions involving transfinite recursion, ordinals, cardinals and applications of these.

- Use general results to compute and compare cardinalities of infinite sets.

- Communicate your knowledge of the area in a concise, accurate and coherent manner.

- Combine your knowledge of predicate logic and set theory in the study of model theory and apply the results to deepen your understanding of theories of first-order structures.

Module Content

The module is concerned with some of the foundational issues of mathematics. In propositional and predicate logic, we analyse the way in which we reason formally about mathematical structures. In set theory, we will look at the ZFC axioms and use these to develop the notion of cardinality. These topics have applications to other areas of mathematics: formal logic has applications via model theory and ZFC provides an essential toolkit for handling infinite objects.

An indicative list of sections and topics is:

Propositional logic: Formulas and logical validity; a formal system; soundness and completeness.

Predicate logic: First-order languages and structures; satisfaction and truth of formulas; the formal system; Goedel's completeness theorem; the compactness theorem; the Loewenheim- Skolem theorem.

Set theory: The axioms of ZF set theory; ordinals; cardinality; the Axiom of Choice.

Model theory: Elementary substructures; the method of diagrams, the Tarski-Vaught test; the general Loewenheim- Skolem Theorems; Reduced products and ultraproducts; Los' theorem.

There are no formal prerequisites for the module although a level of mathematical understanding such as would be provided by a second year algebra or analysis module, together with an appetite for abstraction and proofs, will be assumed. We will use basic notions from algebra (groups, rings and vector spaces) in examples.

MATH70040 Formalising Mathematics

Brief Description

Computer theorem provers are mature enough now to tackle most undergraduate level mathematics, and also some much harder level mathematics. As these systems evolve, they will inevitably become useful as tools for researchers, and some believe that one day they will start proving interesting theorems by themselves. This project-assessed course is an introduction to the Lean theorem prover and during it we will learn how to formalise proofs of undergraduate and masters level theorems from across pure mathematics.

Learning Outcomes

On successful completion of this module you will be able to

- understand the basics of how type theory can be used as a foundation for pure mathematics;

- understand how to "modularise" mathematical arguments, breaking them up into simple chunks, thus leading to clarity of understanding;

- understand how to "abstract" mathematical arguments, finding the correct generality in which statements should be made, thus again leading to clarity of understanding;

- state and prove many results from undergraduate and masters level pure mathematics modules in the Lean theorem prover;

- develop mathematical theories of your own in the Lean theorem prover;

- write formal proofs of theorems which other mathematicians can understand and follow

- understand how to turn more advanced mathematics into statements of dependent type theory.

Module Content

Note that the aim is to both learn the mathematics and to learn how to teach it to a computer. No experience in programming will be assumed. Lean is a functional programming langauge, so we will be picking up functional programming along the way. If you want to get a feeling for the kind of coding which will be involved, try playing the natural number game.

The following is an indicative list of areas where the mathematics could be drawn from:

*) Logic, functions, sets.

*) Lattice theory, complete lattices, Galois insertions and Galois connections. Examples in mathematics.

*) Groups and subgroups.

*) Closure operators in group theory and topology.

*) Filters as generalised subsets. Filters form a complete lattice.

*) Applications of filters to topology. New proofs of basic results in topology. Tychonoff's theorem.

*) Application of filters to analysis. New proofs of basic results in analysis.

*) What is cohomology? Group cohomology in low degree.

This level 7 version of the module will involve extension material and more advanced examples than the level 6 version.

MATH70061 Commutative Algebra

Brief Description

This module is an introduction to commutative algebra which is the modern foundation of algebraic geometry and algebraic number theory. First we will cover such basic notations as prime and maximal ideals, the nilradical and the Jacobson radical. Then we study the very important construction of localisation both for rings and modules over them. We will apply these to a variety of results, for example primary decomposition of ideals and structure theorems for Artinian rings and discrete valuation rings.

Learning Outcomes

On successful completion of this module, you will be able to:

- define basic notions in commutative algebra and prove their main properties;

- use localisation to relate properties of rings, ideals, modules and morphisms between them with properties of their localisations;

- apply various chain conditions to prove properties of rings and modules satisfying these;

- use other standard arguments in commutative algebra;

Module Content

This module is an introduction to commutative algebra which is the modern foundation of algebraic geometry and algebraic number theory. First we will cover such basic notations as prime and maximal ideals, the nilradical and the Jacobson radical. Then we study the very important construction of localisation both for rings and modules over them. We will apply these to a variety of results, for example primary decomposition of ideals and structure theorems for Artinian rings and discrete valuation rings.

An indicative list of topics is:

Prime and maximal ideals, nilradical, Jacobson radical, localization. Modules. Primary decomposition of ideals. Applications to rings of regular functions of affine algebraic varieties. Artinian and Noetherian rings, discrete valuation rings, Dedekind domains. Krull dimension, transcendence degree. Completions and local rings. Graded rings and their Poincaré series.

After this module, you should be equipped to undertake a course in modern algebraic geometry.

MATH70062 Lie Algebras

Brief Description

This module is an introduction to theory of complex Lie algebras and it culminates in the classification of finite dimensional semisimple complex Lie algebras in terms of root systems. It is completely self-contained, and only relies on a good understanding of linear algebra. However the proofs are quite intricate. It is also a good preliminary to the theory of Lie groups and algebraic groups, which study closely related objects, but the latter require much heavier machinery.

Learning Outcomes

On successful completion of this module, you will be able to:

- define basic notions of Lie algebras, such as ideals, derived series and lower central series,

- prove Engel's and Lie's theorem, and apply them in various contexts,

- prove and apply the additive Jordan decomposition theorem,

- define the Killing form and prove Chevalley's criteria,

- define Cartan and Borel subalgebras and prove their main properties,

- apply the above to complete the proof of the classification theorem of semisimple Lie algebras,

- construct explicitly the classical simple Lie algebras,

- work effectively with roots systems.

Module Content

An indicative list of topics is:

The semisimple complex Lie Algebras: root systems, Weyl groups, Dynkin diagrams, classification. Cartan and Borel subalgebras. Classification of irreducible representations.

MATH70063 Algebra 4

Brief Description

This module is a course of homological algebra. The main result is the existence of derived functors in the category of modules over an associative ring. We cover functors Ext and Tor in greater detail, particularly in the category of abelian groups. We define and study some basic properties of group cohomology.

Learning Outcomes

On successful completion of this module, you will be able to:

– identify features of, and develop arguments about, rings, modules over rings and homomorphisms between them;

– understand the definition of the tensor product of module and use it in a variety of settingss;

– define free, injective, projective, flat modules, state and prove their basic properties;

– state, apply, and explain the proof of the theorem about the existence of derived functors in the category of modules over a ring;

– understand and explain the relation between Ext and extensions of modules;

– compute functors Tor and Ext in specific situations, particularly in the category of abelian groups;

– understand and use for computation the explicit construction of the first and second cohomology groups, as well as the cohomology groups of a cyclic group.

– demonstrate capacity for independent study of an advanced topic to be specified by solving a range of problems.

Module Content

An indicative list of sections and topics is:

Modules over rings: free, projective, injective, flat; tensor product

Functors Hom, Ext, Tor. General definition of a derived functor. Long exact sequences. Injective and projective resolutions. Homotopy.

Group cohomology via homogeneous and inhomogeneous cochains. The case of cyclic groups.

Mastery material for self-study, on advanced material relating to the topics above.

MATH70041 Number Theory

Brief Description

Brief description of module (at most 600 characters): The module is

concerned with properties of natural numbers, and in particular of prime

numbers, which can be proved by "elementary" methods (such as basic group

theory ring theory).

Learning Outcomes

On successful completion of this module, you will be able to:

– form arguments about and solve congruences, particular modulo primes, and apply

this to the RSA algorithm;

- compute with quadratic residues;

- solve some particular Diophantine equations, including Pell's

equation;

- compute continued fractions;

- construct transcendental numbers.

- explain and demonstrate mastery of such further material as is selected by the module leader for self-study.

-- Combine material from across the module to solve more advanced problems

Module Content

An indicative list of topics is:

Euclid's algorithm, unique factorization, linear congruences, Chinese

Remainder Theorem.

The structure of (Z/nZ)×, including the Fermat-Euler theorem,

Lagrange's theorem, the existence (and non-existence) of primitive

roots.

Primality testing, factorization, and the RSA algorithm (including the basics of the

Miller-Rabin test).

Quadratic reciprocity, Legendre symbols, Jacobi symbols.

Sums of 2 and 4 squares, using unique factorization in the Gaussian

integers.

Pell's equation, existence of solutions via Dirichlet's theorem.

Continued fractions, periodicity for quadratic irrationals, algorithm

for solving Pell's equation via continued fractions.

Irrationality, Liouville's theorem, construction of a transcendental

number.

Elementary results on primes in arithmetic progressions.

Other topics of the lecturer's choice as time permits, e.g. quadratic forms; Möbius inversion and

Dirichlet Convolution; the Selberg sieve; particular examples of Diophantine equations.

Mastery material selected for further self-study, to be based on written material such as a textbook exerpt or research paper.

MATH70042 Algebraic Number Theory

Brief Description

An introduction to algebraic number theory using quadratic fields as the main source of examples. We will study rings of integers in finite extensions of the rational and discuss unique factorization and its failure, the decomposition of primes, the finiteness of the ideal class group, and Dirichlet's unit theorem.

Learning Outcomes

On successful completion of this module you will be able to:

-- explain how unique factorization domains, principal ideal domains and Euclidean domains are related.

-- give examples of rings of integers in quadratic fields that are Euclidean domains and also counter-examples.
- define a Dedekind domain and explain why rings of integers in number fields are Dedekind domains.

-- write a basis for the rings of integers in any given quadratic number field.

-- explain what it means for a prime to be split, inert or ramified in an extension and, given a quadratic ring of integers and a prime, you will be able to say what happens to that prime.

-- explain why the class group in a number field is finite and compute examples of class groups of quadratic number fields.

-- state Dirichlet's unit theorem and to describe explicitly the group of units in a real or imaginary quadratic field.

-- show mastery of more advanced material on these topics which will be set for self-study

-- synthesise arguments from different parts of the module to solve more advanced problems.

Module Content

An indicative list of topics is as follows.

We will review / introduce some background from ring theory, discuss unique factorization domains, principal ideal domains and Euclidean domains. We will study Gaussian and Eisenstein integers in detail and see several other examples of quadratic rings of integers. We will then discuss the structure theorem for finitely generated abelian groups, the notion of integral closure, Dedekind domains and study the ideal class group. We will prove that the ideal class group in a number field is finite and compute many examples. We will study the decomposition of primes in number fields and in quadratic fields in particular. We will end by discussing Dirichlet's unit theorem.

Further more advanced material (such as a book chapter or research paper) will be set by the module leader for independent study.

The module will assume familiarity will some topics in algebra, such as commutative rings and modules.

MATH70064 Elliptic Curves

Brief Description

An elliptic curve is an algebraic curve in two variables defined by an equation of the form $y^2=x^3+ax+b$. Elliptic curves play an important role in Number Theory, and have been central to many recent advances, such as the proof of Fermat's Last Theorem. In this course we study the theory of elliptic curves and their connections with Number Theory, Geometry and Algebra.

Learning Outcomes

On successful completion of this module, you will be able to:

– solve equations in the p-adic numbers;

- find all rational points on plane conics;

- compute with the group law on an elliptic curve;

- compute the torsion subgroup of an elliptic curve over Q;

- compute the rank of an elliptic curve over Q;

- demonstrate mastery of further advanced material selected for self-study by applying it in a variety of problems.

Module Content

An indicative list of topics is:

The p-adic numbers.

Curves of genus 0 over Q .

Cubic curves and curves of genus 1.

The group law on a cubic curve.

Elliptic curves over p-adic fields and over Q.

Torsion points and reduction mod p.

The weak Mordell-Weil theorem.

Heights.

The (full) Mordell-Weil theorem.

MATH70043 Statistical Theory

Brief Description

This module seeks to provide a more unified perspective of the core statistical problems introduced in earlier modules by developing a general mathematical theory for parametric statistical models. We will deal with the criteria and theoretical results necessary to develop and evaluate statistical procedures for point estimation, hypothesis testing and confidence intervals. We will consider several approaches, including maximum likelihood estimation and Bayesian approaches, and study when they are provably optimal.

Learning Outcomes

On sucessful completion of this module, you will be able to-

-Apply key results related to optimal statistical procedures

- Evaluate and compare estimators using their sampling properties

- Explain what it means for statistics to be sufficient and complete

- Explain the Rao-Blackwell theorem and how it can be used to improve an estimator

- Use elementary ideas from decision theory to evaluate statistical procedures

- Explain the Bayesian approch to estimation

- Demonstrate an integrated understanding of the concepts of this module by independent study of related material.

Module Content

An indicative list of sections and topics is:

- Theories of point estimation and hypothesis testing
- Exponential families
- Sufficiency and the Rao-Blackwell theorem
- The Cramer-Rao lower bound
- Maximum likelihood estimation and its asymptotic theory
- Bayesian estimation
- Decision theory
- Completeness and minimum variance unbiased estimators
- The Neyman-Pearson lemma and likelihood ratio tests
- Independent study of extension material (in the form of a book chapter, additional notes or a research paper) applying or extending material from the above topics.

MATH70044 Statistical Modelling 2

Brief Description

This module builds on earlier statistical modelling work, and we begin with a review of the linear model in generality. We extend the linear model in two different directions: firstly, using generalized linear models (GLMs) to handle non-normally distributed errors, and secondly, using random effects to handle correlated observations. The module has an applied focus, relying on statistical computation in R to analyze a variety of data. We will discuss practical techniques for comparing models and for identifying when models fit that fit poorly.

Learning Outcomes

On successful completion of this module, you will be able to:

- Use R to fit linear models.

- Select appropriate generalized linear models for modelling data with with non-normal error distributions, e.g. poisson, binomial, gamma.

- Use models to give predictions, together with an associated confidence interval.

- Identify models that fit poorly, and suggest improved models.

- Model data with correlated observations using linear mixed models.

- Explain the properties of different estimators of random effects variances, such as maximum likelihood and REML.

- Interpret output from R using plain language.

- Demonstrate an integrated understanding of the concepts of this module by independent study of related material.

Module Content

An indicative list of sections and topics is:

- Linear models: Least squares, normal equations, Gauss-Markov theorem, goodness of fit, diagnostics. Studentized residuals. Confidence intervals and prediction intervals.

- Generalized linear models: Specification, estimation (iterated reweighted least squares), inference, diagnostics.

- Mixed models: Specification, estimation (ANOVA, maximum likelihood, REML) and inference (parametric bootstrap) for variance components.

- Independent study of extension material (in the form of a book chapter, additional notes or a research paper) applying or extending material from the above topics.

MATH70045 Applied Probability

Brief Description

This module introduces stochastic processes and their applications. The theory of different kinds of processes will be described and illustrated with applications in several areas. The groundwork will be laid for further deep work, especially in such areas as genetics, engineering and finance.

Learning Outcomes

"On successful completion of this module, you will be able to:

- Use the Poisson process to model random arrivals.

- Extend the Poisson process to accommodate common depatures from the Poisson assumptions.

- Work with Markov chains in continous time.

- Determine the long-term behaviour of a continuous-time Markov chain.

- Determine whether states are recurrent or transient.

- Solve differential and difference equations to determine quantities of interest for stochastic processes.

- Explain basic properties of Brownian motion."

- Demonstrate an integrated understanding of the concepts of this module by independent study of related material.

Module Content

An indicative list of sections and topics is:

- Review of probability and discrete time Markov Chains

- Random walks

- Poisson processes and their properties: superposition, thinning of Poisson processes; Non-homogeneous, compound, and doubly stochastic Poisson processes.

- Autocorrelation functions.

- Probability generating functions.

- General continuous-time Markov chains: generator, forward and backward equations, holding times, stationarity, long-term behaviour, jump chain, explosion; birth and death processes, reversibility, recurrence/transience.

- Differential and difference equations and pgfs. Embedded processes. Time to extinction.

- Queues.

- Brownian motion and its properties.

- Independent study of extension material (in the form of a book chapter, additional notes or a research paper) applying or extending material from the above topics.

MATH70046 Time Series Analysis

Brief Description

A time series is a series of data points indexed and evolving in time. They are prevalent in many areas of modern life, including science, engineering, business, economics, and finance. This module is a self-contained introduction to the analysis of time series. Weight is given to both the time domain and frequency domain viewpoints, and important structural features (e.g. stationarity, reversibility) are treated rigorously. Attention is given to estimation and prediction (forecasting), and useful computational algorithms and approaches are introduced.

Learning Outcomes

On successful completion of this module, you will be able to:

- Appreciate that time series should be considered observations from an underlying stochastic process.

- Define what it means for a time series to be stationary.

- Identify autocorrelation within time series data.

- Work with standard models of time series.

- Appreciate that time series can exhibit trend and seasonality and know how to adjust for these.

- Determine the spectral representation of stationary time series and use the spectral density function to provide an alternative viewpoint of second-order structure.

- Derive and implement estimators of mean, correlation and spectral properties.

- Extend time series models, the notion of stationarity, and frequency domain representations to multivariate time series.

- Derive forecasts from standard time series models and quantify their uncertainty.

- Demonstrate an integrated understanding of the concepts of this module by independent study of related material.

Module Content

An indicative list of sections and topics is:

- Discrete time stochastic processes and examples.

- Autocovariance, autocorrelation, stationarity.

- Trend removal and seasonal adjustment.

- AR, MA and ARMA processes, characteristic polynomials, general linear process, invertibility, directionality and reversibility.

- Spectral representation, aliasing, linear filtering.

- Estimation of mean and autocovariance sequence, the periodogram, tapering for bias reduction.

- Parametric model fitting.

- Forecasting.

- Independent study of extension material (in the form of a book chapter, additional notes or a research paper) applying or extending material from the above topics.

MATH70047 Stochastic Simulation

Brief Description

Computational techniques have become an important element of modern statistics. Computation particularly underpins simulation methods, which are widely applied when studying models of complex systems, e.g. in biology and in finance. This module provides an up-to-date view of such simulation methods, covering areas from basic random variate generation to advanced MCMC (Markov Chain Monte Carlo) methods. The implementation of stochastic simulation algorithms will be carried out in R, a language that is widely used for statistical computing and well suited to scientific programming more generally.

Learning Outcomes

On successful completion of this module, you will be able to:

- Use algorithms for efficient generation of pseudo-random numbers.

- Evaluate intractable definite integrals using Use Monte Carlo methods.

- Implement MCMC algorithms to draw samples from intractable distributions.

- Assess the performance of MCMC algorithms using suitable diagnostic procedures.

- Explain how sequential Monte Carlo methods can be used to understand time-structured problems.

- Demonstrate an integrated understanding of the concepts of this module by independent study of related material.

Module Content

An indicative list of sections and topics is:

- Pseudo-random number generators.

- Generalized methods for random variate generation.

- Monte Carlo integration.

- Variance reduction techniques.

- Markov chain Monte Carlo methods (including Metropolis-Hastings and Gibbs samplers).

- Monitoring and optimisation of MCMC methods.

- Introduction to sequential Monte Carlo methods.

- Independent study of extension material (in the form of a book chapter, additional notes or a research paper) applying or extending material from the above topics.

MATH70048 Survival Models

Brief Description

Survival models are fundamental to actuarial work, as well as being a key concept in medical statistics.

This module will introduce the ideas, placing particular emphasis on actuarial applications.

Learning Outcomes

On successful completion of this module, you will be able to:

- Appreciate survival analysis models as a framework that deals with lifetimes and censored observations;

- Use several methods to define event time distributions and the relation between these definitions;

- Describe, select and use methods for fitting parametric, semi-parametric and non-parametric survival analysis models, including regression models and multi state models.

- Explain the counting process approach to survival analysis and its benefits;

- Use and critically analyse methods occurring in actuarial applications, such as methods for the construction and use of life tables.

- Demonstrate an integrated understanding of the concepts of this module by independent study of related material.

Module Content

An indicative list of sections and topics is:

- Concepts of survival models,

- Right and left censored data and randomly censored data.

- Estimation procedures for lifetime distributions:

- Empirical survival functions,

- Kaplan-Meier estimates,

- Cox model.

- Statistical models of transfers between multiple states,

- Maximum likelihood estimators.

- Counting process models.

- Actuarial Applications:

- Life table data and expectation of life,

- Binomial model of mortality,

- The Poisson model,

- Estimation of transition intensities that depend on age,

- Graduation and testing of crude and smoothed estimates for consistency.

- Independent study of extension material (in the form of a book chapter, additional notes or a research paper) applying or extending material from the above topics.

MATH70049 Introduction to Statistical Learning

Brief Description

This module provides an introduction to methods of statistical learning. That is, using statistical and artificial intelligence (AI) methods to learn from data, often when the data set is large, complex or of high dimension. We will consider both supervised and unsupervised learning. For the former we use a training set of data to learn patterns within data and then use our knowledge of those patterns to devise methods for predicting the outcomes of those patterns for new data. For the latter, there is not (usually) an outcome measure, but we seek to learn about the patterns within the data set itself. The methods in this module are of immense interest in academic and business circles and underpins much of the modern data tech industry to achieve tasks such as suggesting movies you might like to watch given information on those that you have watched, and from people with similar viewing patterns, developing machine methods for identifying cancer and seeking new ways to understand the economy. It is strongly recommended that students will have already passed the module Statistical Modelling I, or similar.

Learning Outcomes

On successful completion of this module, you will be able to:

Appreciate the structure and likely distribution(s) of various forms of data and which kinds of methods might be suitable for their analysis.

Understand the concept of multivariate data and situations where supervised and unsupervised learning might be employed.

Know and deploy a variety of important statistical and AI methods, using the R language to solve data related problems.

Understand the limitations of methods and how their application can be checked and or corrected.

Module Content

An indicative list of sections and topics is:Linear Models for Regression (variable selection, Lasso, Ridge Regression); Linear methods for classification (Logistic Regression); Basis expansions (piecewise polynomials and splines; smoothing splines, wavelets); Kernel regression (local linear and local polynomial); Additive models and Trees (boosting); Projection pursuit regression and Neural Networks; Support Vector Machines; Cluster Analysis and Multidimensional Scaling; Random Forests; Data Ethics. A Mastery topic for self-study directed by the Module Leader, on advanced material related to the topics above.

MATH70092 Multivariate Analysis

Indicative Module Content

Multivariate Analysis is concerned with the theory and analysis of data that has more than one outcome variable at a time, a situation that is ubiquitous across all areas of science. Multiple uses of univariate statistical analysis is insufficient in this settings where interdependency between the multiple random variables are of influence and interest. In this module we look at some of the key ideas associated with multivariate analysis. Topics covered include: multivariate notation, the covariance matrix, multivariate characteristic functions, a detailed treatment of the multivariate normal distribution including the maximum likelihood estimators for mean and covariance, the Wishart distribution, Hotelling's T^2 statistic, likelihood ratio tests, principle component analysis, ordinary, partial and multiple correlation, multivariate discriminant analysis.

MATH70091 Machine Learning (May not be taken with Introduction to Statistical Learning)

Indicative Module Content

This module will provide an introduction to Bayesian statistical pattern recognition and machine learning. The lectures will focus on a variety of useful techniques including methods for feature extraction, dimensionality reduction, data clustering and pattern classification. State-of-art approaches such as Gaussian processes and exact and approximate inference methods will be introduced. Real-world applications will illustrate how the techniques are applied to real data sets. **This half-module may not be taken with M3/4S20 Introduction to Statistical Learning.**

MATH70090 Bayesian Methods

Indicative Module Content

In this module we will develop tools for designing, fitting, validating, and comparing the highly structured Bayesian models that are so quickly transforming how scientists, researchers, and statisticians approach their data. This will include: motivation of Bayesian methods, basic Bayesian tools, comparisons with likelihood methods; standard single-parameter models, conjugate, informative, non-informative, flat, invariant, and Jeffries prior distributions, summarizing posterior distributions, and the posterior as an average of the prior and data; multi-parameter models including Gaussian models and Gaussian linear regression, semi-conjugate prior distributions, evaluating an estimator, and nuisance parameters; hierarchical and multilevel models, finite mixture models, the two-level Gaussian model, shrinkage; model checking, selection, and improvement techniques, posterior predictive checks, Bayes factors, comparisons with significance tests and p-values.