PLUS!

PLUS is an informal lunchtime meeting where students can come and talk to each other and to some of the staff about recreational mathematics puzzles. There are occasional meetings throughout the year—this term they will be on Thursdays, in weeks 3, 5, 7, 9 and 11, from 1200 to 1300, in the Clore. The next one will hence be on **Thursday 22/10/15** from 12-1.

1. (a) January 1st 2016 will fall on a Friday. With what frequency does January 1st fall on that day of the week?

(b) How often does the day on which we commemorate Isaac Newton's birth fall on a Wednesday?

2. If $x + y + z = 0$, show that

$$\left(\frac{x^2 + y^2 + z^2}{2}\right) \cdot \left(\frac{x^5 + y^5 + z^5}{5}\right) = \left(\frac{x^7 + y^7 + z^7}{7}\right).$$

3. Let $d(n)$ be the number of perfect squares that are divisors of the number n, so that for example $d(20) = 2$, $d(36) = 4, \ldots$. What is the average value of $d(n)$? More precisely, set

$$Ave(n) = \frac{d(1) + d(2) + d(3) + \cdots + d(n)}{n}.$$

Find the limit of $Ave(n)$ as n tends to infinity – that is the mean number of square factors of any number.

4. It is rumoured that Archimedes prowled around the fairgrounds of Syracuse with a prop called the 'Magic Globe', a perfect sphere with a map of the world on. He had written in secret the latitudes and longitudes of two surface locations on a folded slip of paper. "Try your luck!" he barked to the crowd of onlookers. "Choose your favourite point on the globe, and if the angles of the triangle formed by your point and my two points are all acute, then I win – in all other cases, you win!". The triangle sides are simple line segments, with no fancy spherical triangles. Show how by choosing the angular separation of his surface locations optimally and paying out at evens on the bet, this was very profitable for Archimedes. His untimely death put paid to his intention to relaunch the prop as the 'Golden Globe' - which name was to be something of an in-joke for his more knowledgeable followers...

5. Determine all triples (a, b, c) of positive integers such that each of the numbers

$$ab - c, bc - a, ca - b$$

is a power of 2.

6. Determine all functions $f : \mathbb{R} \to \mathbb{R}$ ($\mathbb{R}$ the reals) such that

$$f(x + f(x + y)) + f(xy) = x + f(x + y) + yf(x)$$

for all real numbers x and y.

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