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Symmetrization inequalities on one-dimensional integer lattice

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Symmetrization inequalities on $\mathbb{R}^d$

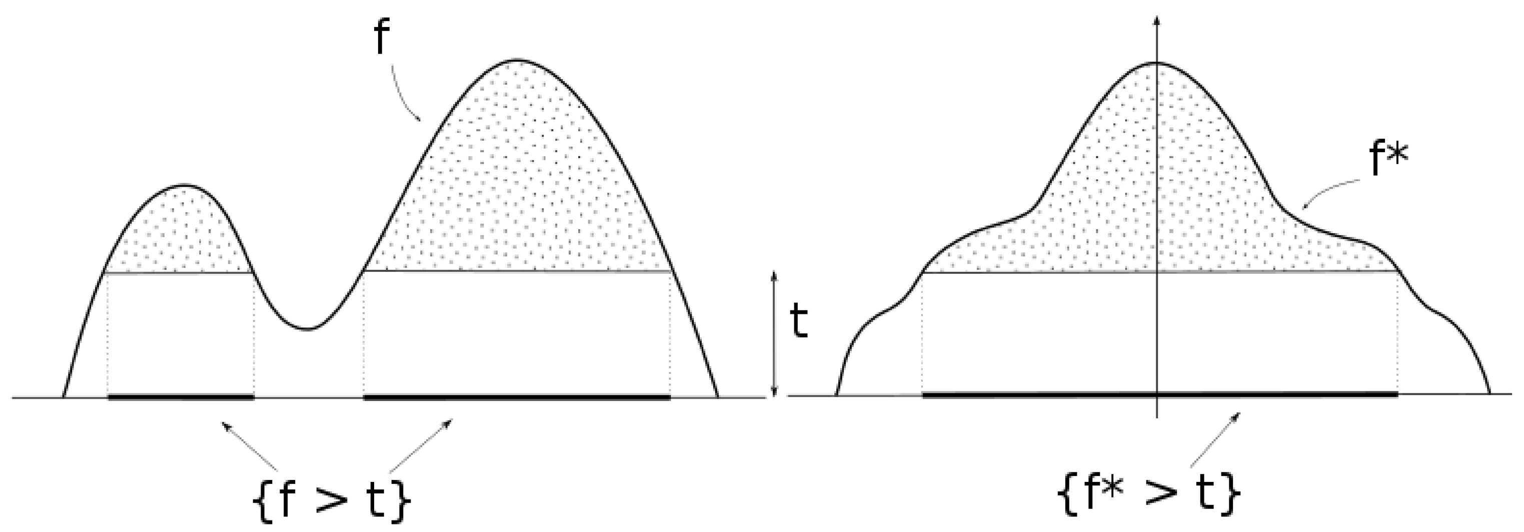
We consider the famous *Polya-Szegö inequality* [10]:

$$\int_{\mathbb{R}^d} |\nabla u|^2 dx \geq \int_{\mathbb{R}^d} |\nabla u^*|^2 dx, \tag{1}$$

where u^* is the *symmetric-decreasing rearrangement* of the function u, defined by

$$u^*(x) := \int_0^\infty \chi_{\{|u|>t\}^*}(x) dt, \tag{2}$$

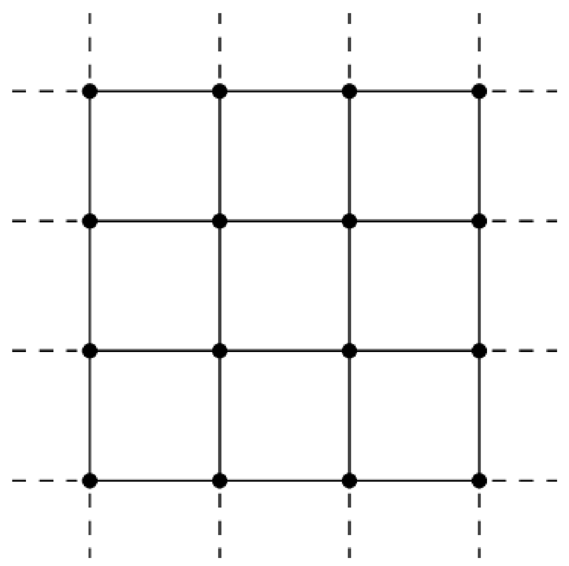
where, given a set $\Omega \subset \mathbb{R}^d$, Ω^* is the ball centered at origin whose measure is same as the measure of Ω.



In Euclidean spaces, symmetrization inequalities including (1) are a standard and powerful tool to establish the symmetry of optimizers of many variational problems. In the past, they have also been successfully used to provide the sharp constants in various important functional inequalities(Sobolev inequality, Cafferreli-Kohn-Nirenberg inequality, etc) [1], [11].

Open Problem

Find a labeling $\eta : \mathbb{Z}^2 \to \mathbb{N}$ of the 2-D lattice $\mathbb{Z}^2$:

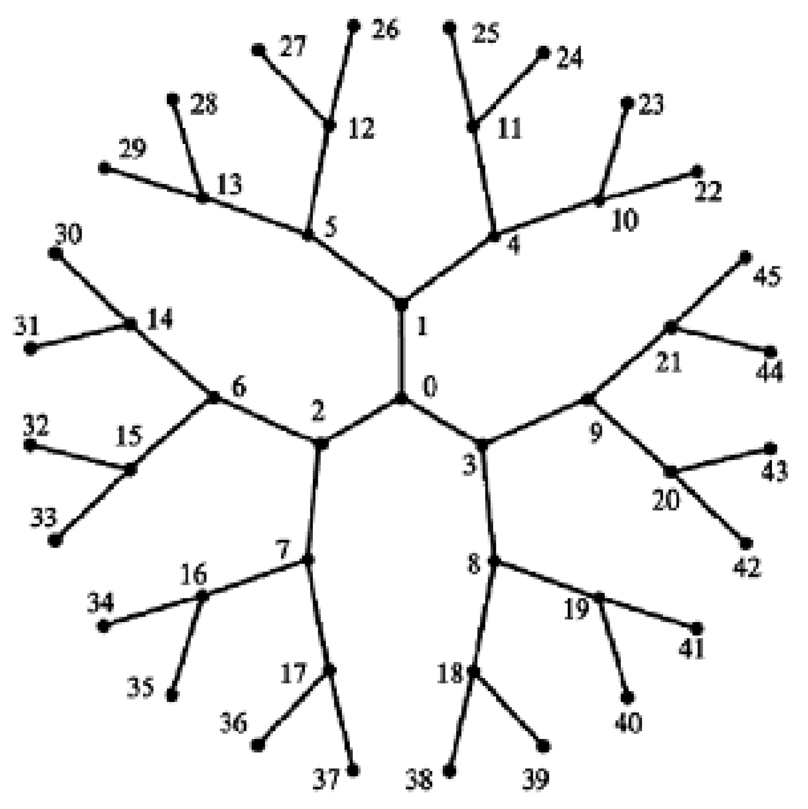


such that we have $\sum_{x\sim y} |u(x) - u(y)|^2 \geq \sum_{x\sim y} |u^*(x) - u^*(y)|^2$, where u^* is the decreasing rearrangement of u w.r.t. the labeling η.

Symmetrization inequalities on graphs

The oldest results on the topic dates back to the early 20th century. These results are compiled in chapter X of the book by Hardy, Littlewood and Polya [7].

In late 90's, Alexander R. Pruss [13] studied symmetrization inequalities on regular trees, and used them to prove Faber-Krahn inequality.



They proved the following Polya-Szegö inequality

$$\sum_{x\sim y} |u(x) - u(y)|^2 \geq \sum_{x\sim y} |u^*(x) - u^*(y)|^2, \tag{3}$$

where u^* is defined as a decreasing rearrangement of the function u with respect to the labeling given in the diagram above.

Applications of Open Problem

The open problem posed can have the following potential applications:

1. It implies isoperimetric inequality on the lattice by taking u to be a characteristic function, see [8] for a survey on isoperimetric problems on graphs.
2. It can be used to prove Faber-Krahn inequality on the lattice, see [14] for the existing results in this direction.
3. It can be used to prove the Sobolev inequality on the lattice, see [9] for the existence of the optimizer of Sobolev inequality on the lattice.
4. It can be used to prove the Hardy's inequality on the lattice, see [4] for the asymptotic behaviour of the sharp constant in Hardy's inequality.

Results

In [5], we studied the symmetrization inequalities on integers. We proved the following main results:

1. Let $w : \mathbb{Z}^+ \to \mathbb{R}$ be a non-negative increasing function. Let $u :^+ \to \mathbb{R}$ be a function. Then
$$\sum_{n\in^+} |u(n) - u(n+1)|^p w(n) \geq \sum_{n\in^+} |\widetilde{u}(n) - \widetilde{u}(n+1)|^p w(n), \tag{4}$$
for all $p \geq 1$, where $\widetilde{u}(0)$ is the largest value of u, $\widetilde{u}(1)$ is the second largest value of u and so on. Moreover, if $w(n) > 0$, if u produces equality in (4), then $|u| = \widetilde{u}$. In particular, $|u|$ is a decreasing function.
2. Let $u \in l^2(\mathbb{Z})$, then we have
$$\sum_{n\in\mathbb{Z}} |u(n) - u(n-1)|^2 \geq \sum_{n\in\mathbb{Z}} |u^*(n) - u^*(n-1)|^2, \tag{5}$$
for some radially decreasing function u^*.

As an application, we applied inequality (4) to prove *weighted Hardy's inequality*[3], [2]:

$$\sum_{n=1}^\infty |u(n) - u(n-1)|^2 n^\alpha \geq (\alpha-1)^2/4 \sum_{n=1}^\infty |u|^2 n^{\alpha-2}, \tag{6}$$

for $1 < \alpha \leq 2$.

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