The randomised Heston model

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Introduction

- Over the past two decades, a number of volatility models have been proposed to try to understand the dynamics of the implied volatility.
- Continuous stochastic volatility models (driven by Brownian motion) effectively fit the market smiles for larger maturities. Problem: the small-maturity smile is much more flattened compared with the market data, which has the well-observed 'small-time explosion' feature [3].
- To model this feature, researchers suggest refinements to existing models, including:
  - i. Adding jumps (exponential Lévy models for example). Drawback: an explosion rate ($|t\log t|$) larger than market observations; calibration complexity.
  - ii. Introducing fractional Brownian motion. Drawback: increase of computational cost.

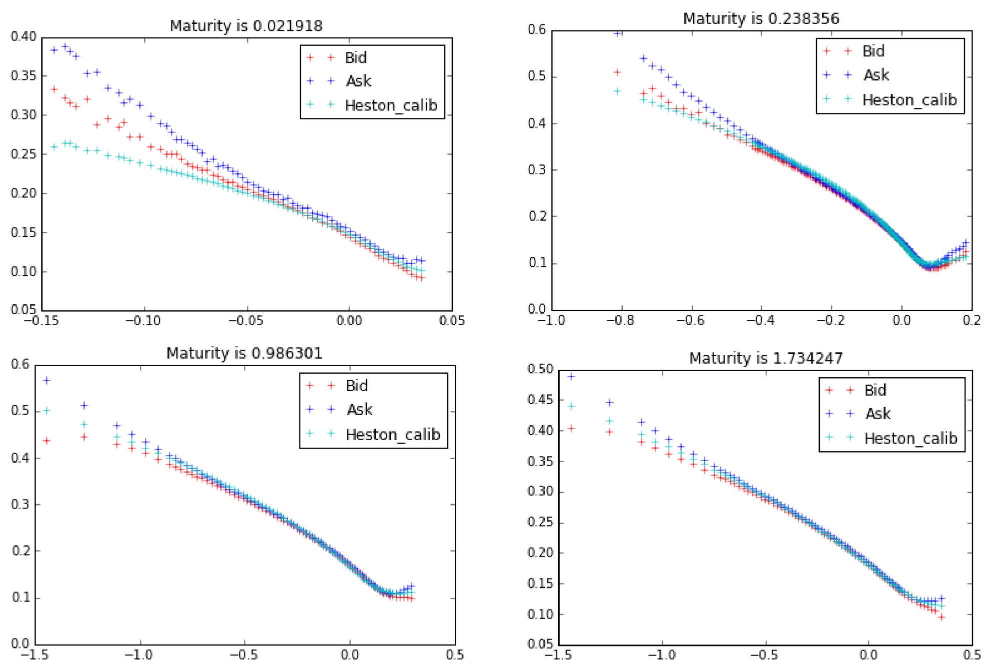

Figure 1 Calibrate standard Heston to SPXW data on Sep 20th, 2016. The Heston fits well except for the first graph.

Our method Initial randomisation: assume that the starting point of the variance process is a random variable, denoted by $\mathcal{V}$.

Goal

- Capture the small-time explosion feature of the implied volatility (denoted by $\sigma_t(x)$);
- Derive small- and large-time asymptotic results of the option price and the implied volatility.

Model description

Assume that the log-price process X (with zero interest rates) satisfies:

$$
\begin{aligned}
\mathrm{d}X_t &= -\frac{1}{2}V_t\mathrm{d}t + \sqrt{V_t}\left(\rho\,\mathrm{d}W_t^{(1)} + \overline{\rho}\,\mathrm{d}W_t^{(2)}\right), & X_0 = 0,\\
\mathrm{d}V_t &= \kappa(\theta - V_t)\mathrm{d}t + \xi\sqrt{V_t}\mathrm{d}W_t^{(1)}, & V_0 \overset{(\mathrm{Law})}{=} \mathcal{V},
\end{aligned}
\tag{1}
$$

where $\rho\in[-1,1]$, $\overline{\rho} := \sqrt{1-\rho^2}$, and κ,θ,ξ are strictly positive. The random variable $\mathcal{V}$ satisfies:

- $\mathcal{V}\perp(\mathscr{F}_t)_{t\geq0}$;
- $\mathcal{V}$ is supported on $(\mathfrak{v}_-,\mathfrak{v}_+)$ with $0\leq\mathfrak{v}_-<\mathfrak{v}_+\leq\infty$;
- $\mathfrak{m} := \sup\left\{u\in\mathbb{R} : \mathbb{E}(\mathrm{e}^{u\mathcal{V}})<\infty\right\}$ is in $(0,\infty]$.

Remark 0.1.

- Compared with the standard Heston, for any $t>0$, the variance of V_t is increased by $\mathrm{e}^{-2\kappa t}\mathbb{V}(\mathcal{V})$.
- The variable $\mathcal{V}$ can be categorised into three classes:
  - i. Bounded support: $\mathfrak{v}_+$ is finite (then $\mathfrak{m}$ is infinite). Example: uniform distribution;
  - ii. 'Thin-tail': $\mathfrak{m}=\mathfrak{v}_+=\infty$. Examples: folded-Gaussian distribution, Rayleigh distribution;
  - iii. 'Fat-tail': $\mathfrak{m}<\mathfrak{v}_+=\infty$. Examples: Gamma distribution, non-central χ-squared distribution.

Notation For a process $(Y_t)_{t\geq0}$ satisfying a large deviations principle as $t\downarrow0$ with speed $g(t)$ and good rate function Λ_Y^* we denote $Y\sim\mathrm{LDP}_0(g(t),\Lambda_Y^*)$.

Main results

Define $\Lambda(u) := \xi^{-1}u(\overline{\rho}\cot(\xi\overline{\rho}u/2)-\rho)^{-1}$, where u is in (u_-,u_+) with constants $u_\pm$ precisely defined in [2] satisfying $u_-<0<u_+$.

Case 1: bounded support

Theorem 0.2. *If $\mathfrak{v}_+<\infty$, define $\Lambda^*_{\mathfrak{v}_+}(x) := \sup_{u\in(u_-,u_+)}\{xu-\mathfrak{v}_+\Lambda(u)\} = xu^*(x)-\mathfrak{v}_+\Lambda(u^*(x))$. then $X\sim\mathrm{LDP}_0(t,\Lambda^*_{\mathfrak{v}_+})$. Moreover, for any $x\neq0$,*

$$\lim_{t\downarrow0}\sigma_t^2(x) = \frac{x^2}{2\Lambda^*_{\mathfrak{v}_+}(x)}.$$

Uniform randomisation: full asymptotics

Theorem 0.3. *If $\mathcal{V}\overset{(Law)}{=}\mathcal{U}(\mathfrak{v}_-,\mathfrak{v}_+)$, then for $x\neq0$,*

$$
\begin{aligned}
&\mathbb{E}\left(\mathrm{e}^{X_t}-\mathrm{e}^x\right)^+ = (1-\mathrm{e}^x)^+\\
&+\exp\left(-\frac{\Lambda^*_{\mathfrak{v}_+}(x)}{t}\right)\frac{A(x)t^{5/2}(1+\mathcal{O}(t))}{\sqrt{2\pi}(\mathfrak{v}_+-\mathfrak{v}_-)\Lambda(u^*(x))},
\end{aligned}
$$

$A(\cdot)$ is the same as in the standard Heston [2].

Case 2: thin-tail randomisation

Theorem 0.4. *If $\mathfrak{m}=\mathfrak{v}_+=\infty$, assume that $\mathcal{V}$ has the density f satisfying $-\log f(v)\sim l_1v^{l_2}$ with $(l_1,l_2)\in\mathbb{R}_+^*\times(1,\infty)$, as $v\uparrow\infty$. Then $X\sim\mathrm{LDP}_0(t^\gamma,\underline{\Lambda}^*)$ with $\underline{\Lambda}^*(x) := Cx^{2\gamma}$, $\gamma=\frac{l_2}{1+l_2}$ and $C=(2l_1l_2)^{\frac{1}{1+l_2}}(1+l_2)/(2l_2)$. For any $x\neq0$,*

$$\lim_{t\downarrow0}t^{1-\gamma}\sigma_t^2(x) = \frac{1}{2C}x^{2(1-\gamma)}.$$

For the fat-tail case more assumptions on the mgf of $\mathcal{V}$ are needed (see [4] for detail). These assumptions are mild enough to include various common distributions, and an explosion rate of $\sqrt{t}$ is captured:

Case 3: fat-tail randomisation

Theorem 0.5. *Define $\Lambda^*(x) := \sqrt{2\mathfrak{m}}|x|$ for $x\in\mathbb{R}$. If $\mathfrak{m}<\mathfrak{v}_+=\infty$, then $X\sim\mathrm{LDP}_0(\sqrt{t},\Lambda^*)$, and for any $x\neq0$,*

$$\lim_{t\downarrow0}t^{1/2}\sigma_t^2(x) = \frac{|x|}{2\sqrt{2\mathfrak{m}}}.$$

Proof. large deviations techniques, results of regular variations, Fourier Transform method. □

Large-time asymptotics Same as the standard Heston, see [1, 4] for detail.

Numerical examples

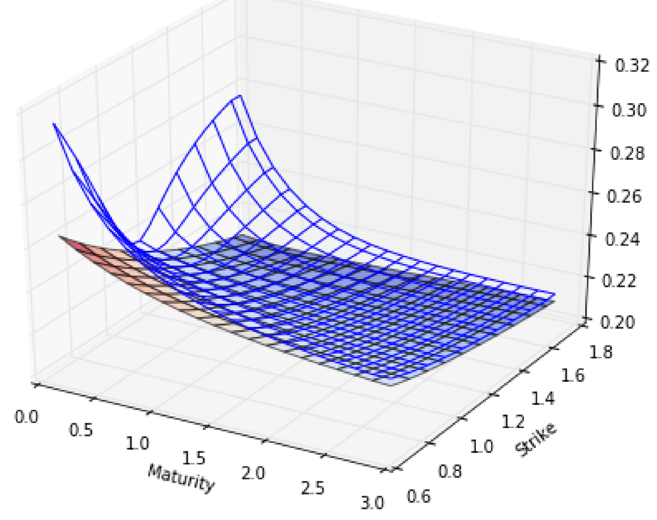

Figure 2 Volatility surfaces of standard (coloured) and randomised Heston, $\mathcal{V}\overset{(Law)}{=}\mathcal{U}(0,0.135)$. Theorem 0.2 suggests the absence of an explosion factor; however, the small-time volatility smile is still much steeper compared with the standard Heston.

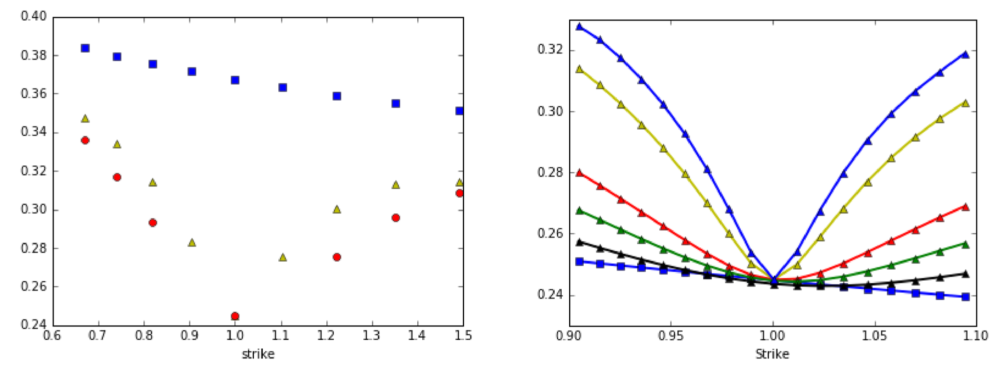

Figure 3 Uniform randomisation with $(\mathfrak{v}_-,\mathfrak{v}_+) = (0,0.135)$. On the left: true implied volatility (triangles) versus leading (squares)- and second (circles)-order asymptotics of the implied volatility, derived from Theorem 0.3. Time to maturity is half-month. On the right: the at-the-money curvature increases as the time to maturity tends to zero.

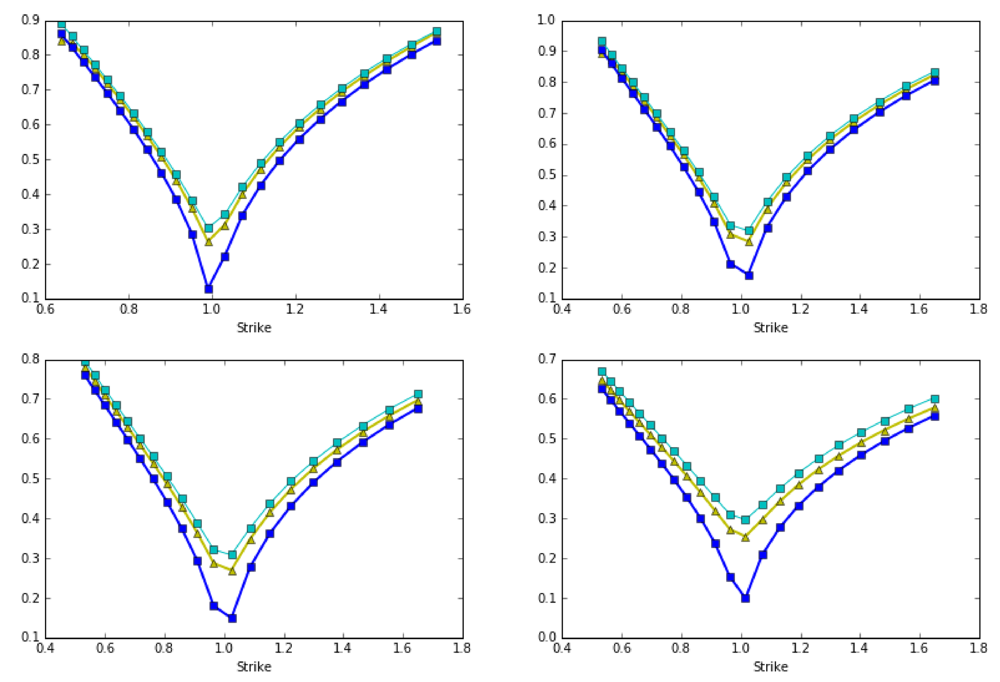

Figure 4 The Gamma randomisation $\Gamma(0.4,3.868)$. Blue and cyan squares are first- and second- order asymptotics. Triangles are the true implied volatility. Time to maturity is 3 days, one week, two weeks and one month.

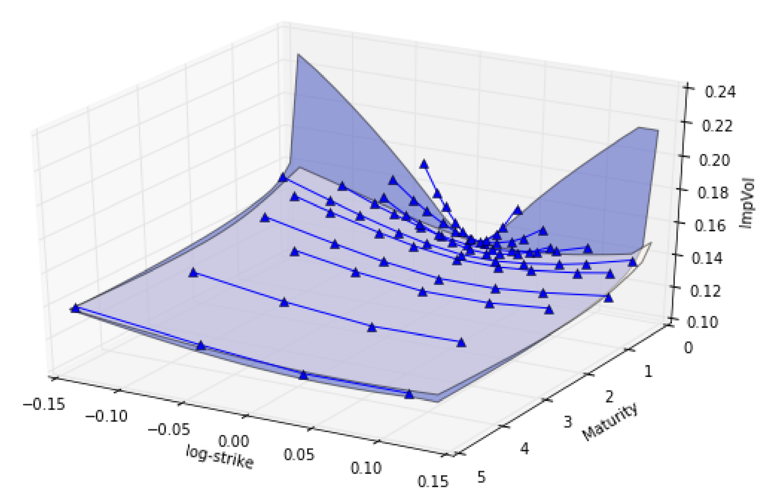

Figure 5 Calibrate standard and randomised Heston to the USD/JPY FX data on Jan 20th, 2017. Blue triangles are market data. The table below summarises the RMSEs of two different schemes ($\times10^{-3}$).

Maturity	standard	randomised
$\leq$ 1 month	11.356	5.568
$\leq$ 1 year	8.260	5.239
$\leq$ 5 years	6.894	4.655

Conclusion

- The randomised Heston model provides a much steeper small-time volatility smile that is in accordance with market observations; the effect fades away as the maturity increases.
- Any explosion rate t^γ with $\gamma\in[0,1/2]$ can be achieved by a suitable choice of $\mathcal{V}$.

References

[1] M. Forde and A. Jacquier. The large-maturity smile for the Heston model. *Finance and Stochastics*, 15 (4): 755-780, 2011.

[2] M. Forde, A. Jacquier and R. Lee. The small-time smile and term structure of implied volatility under the Heston model. *SIAM Journal on Financial Mathematics*, 3(1): 690-708, 2012.

[3] J. Gatheral. The volatility surface: A practitioner's guide. Wiley New York, 2006.

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