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Schottky's Storms

Applications of classical function theory to vortex dynamics.

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What connection can the abstract picture in the background have with solutions to problems of physical interest in fluid dynamics?.....

The problem:

Many fluid flows in the physical world exhibit **vorticity** – a localized rotation, or "swirling" of the fluid. Examples of vortices include hurricanes and ocean eddies.

A problem of physical interest is the effect on vortex motion of **solid boundaries** in the flow domain. For example the motion of ocean eddies near a coastline or around islands. Figure 1 shows such an example.

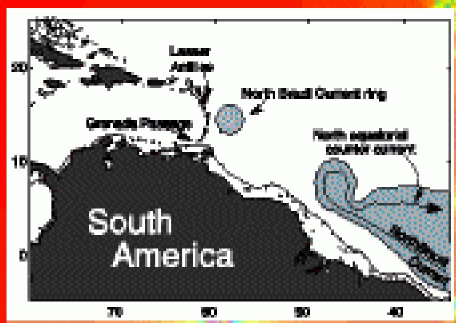

Figure 1: Currents off the north Brazil coast in the vicinity of the Lesser Antilles islands. Simmons & Nof [1]

In the 150 years since Helmholtz began the study of vortex dynamics the general problem of vortex motion in a domain with an **arbitrary number of boundaries** has remained unresolved. Here we present an overview of our solution to this problem.

Our results use function theory developed over 100 years ago by Schottky and Klein (pictured).

Schottky Klein

Our approach:

It can be shown (Lin [2]) that the equations of motion of a system of vortices in a general multiply connected domain can be written down in terms of **a function G** (G is in fact a special kind of Green's function.) The problem remains to determine G.

We have constructed an explicit analytical formula for G in a particular type of multiply connected domain, namely one whose boundaries are all circles. Such a domain is referred to as a **circular domain**. Then, using a further result of Lin [2] concerning **conformal mappings**, we can in fact determine vortex motion in **any** multiply connected domain.

Our results:

Our explicit formula for G for an arbitrary circular domain is shown below. It is constructed in terms of a special function called the **Schottky-Klein prime function**. Schottky and Klein introduced this function following consideration of geometric ideas of iterated reflections in circles. These very abstract ideas also lead to the generation of pictures such as that in the background.

$$G(\zeta;\alpha) = -\frac{1}{4\pi}\log\left|\frac{\omega(\zeta;\alpha)\overline{\omega}(\zeta^{-1},\alpha^{-1})}{\omega(\zeta;\overline{\alpha}^{-1})\overline{\omega}(\zeta^{-1},\overline{\alpha})}\right|$$

where α denotes the (time-varying) position of the vortex, and where we define the function $\omega(\zeta,\alpha)$ as

$$\omega(\zeta,\alpha) = (\zeta-\alpha)\prod_{\theta_i\in\Theta}\frac{(\theta_i(\zeta)-\alpha)(\theta_i(\alpha)-\zeta)}{(\theta_i(\zeta)-\zeta)(\theta_i(\alpha)-\alpha)}$$

where each map θ_i in the group Θ is a composition of reflections in the circular boundaries of the associated circular domain.

Below are plots of the trajectories of a single vortex in various flow domains. All the domains are constructed by a conformal mapping from a circular domain. In fact the function G is also relevant to the construction of conformal mappings to multiply connected slit domains like those in Figures 4 and 5. Such domains can be used to model chains of islands like that shown in Figure 3.

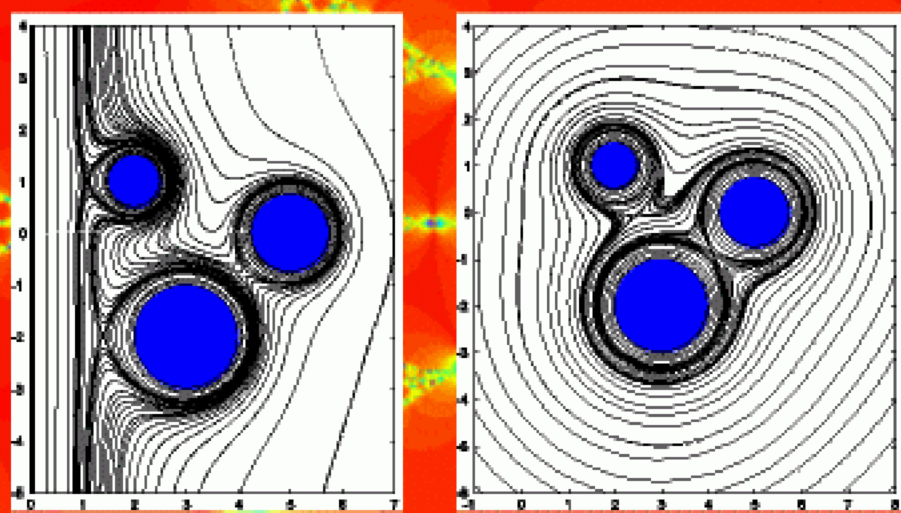

Figure 2: Circular islands near a coastline (left) and in an unbounded ocean (right). The lines show the possible trajectories of a single vortex.

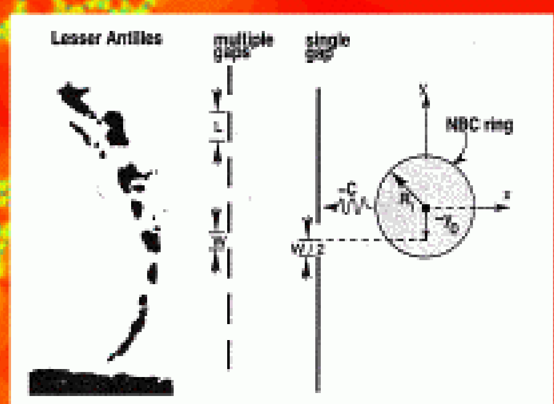

Figure 3: The North Brazil Current (NBC) ring in the vicinity of the Lesser Antilles near the Brazil coast [1]. Chains of islands like these may be modelled by slit domains such as those in Figures 4 and 5.

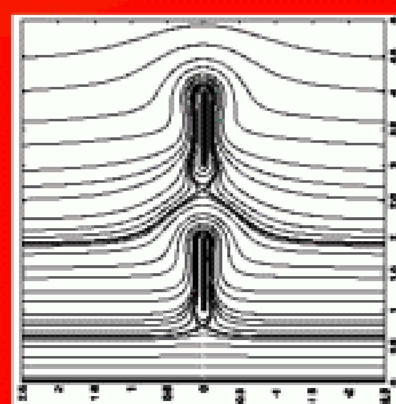

Figure 4: Vortex trajectories in a slit domain modelling a chain of islands near a coastline.

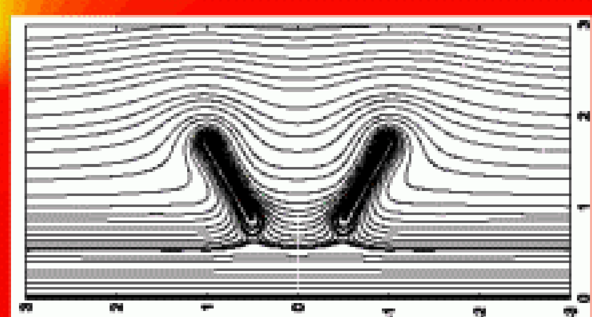

Figure 5: Vortex trajectories in another example of a slit domain.

References:

[1]: Simmons & Nof, "The squeezing of eddies through gaps", J. Phys. Ocean. (2002).

[2]: Lin, "On the motion of vortices in two dimensions – I", Proc. Nat. Acad. Sci. (1941).