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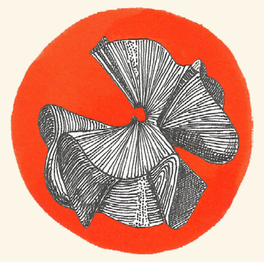
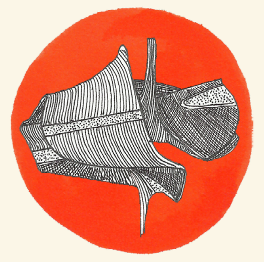

Fanosearch

[Classifying Fano Varieties]

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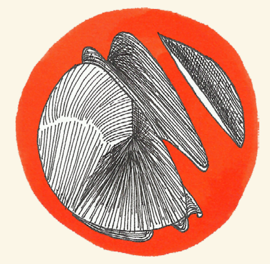

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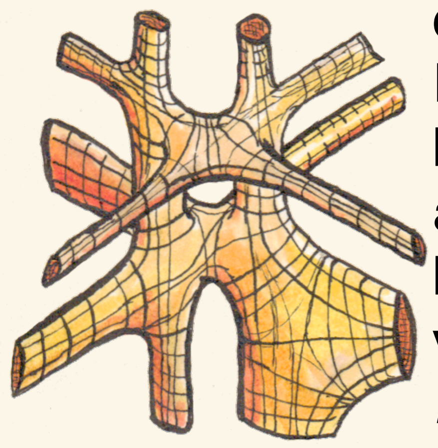
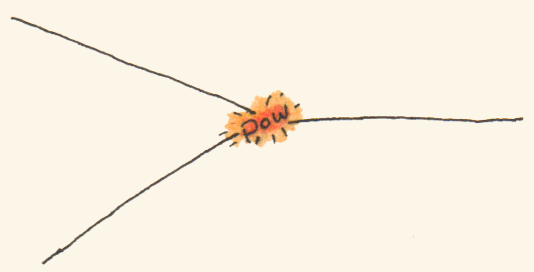
Fano varieties are a special class of complex manifolds which occur naturally in algebraic geometry. In particular, Fano varieties are among the basic building blocks of Mori's Minimal Model Program, which aims to classify all algebraic varieties up to birational equivalence (an algebro-geometric version of surgery in algebraic topology). Thus *Fano varieties play a fundamental rôle in algebraic geometry*.



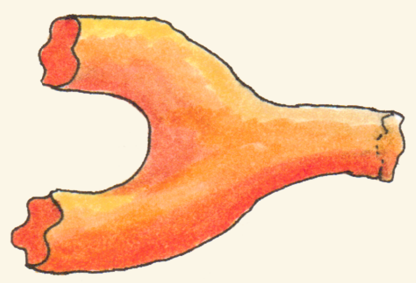
Kollár-Miyaoka-Mori showed that in any given dimension the number of Fano varieties is finite. Therefore, it is natural to ask the question: **how many Fano varieties are there in dimension n, and can we describe them all explicitly?**

A natural invariant of a Fano variety is its quantum period. This is a formal power series obtained by studying the quantum cohomology of the Fano variety.

Quantum cohomology, like ordinary cohomology, is the study of subspaces and their intersection. Unlike ordinary cohomology, the classical intersection product is replaced with the quantum intersection product which is modified by the inclusion of quantum corrections. The quantum corrections are determined by Gromov–Witten invariants, which count curves linking the subvarieties. Much as quantum mechanics considers all possible paths an object may take between two points, the quantum corrections are determined by all possible ways of joining the subvarieties.



Classical Intersection



Quantum Intersection

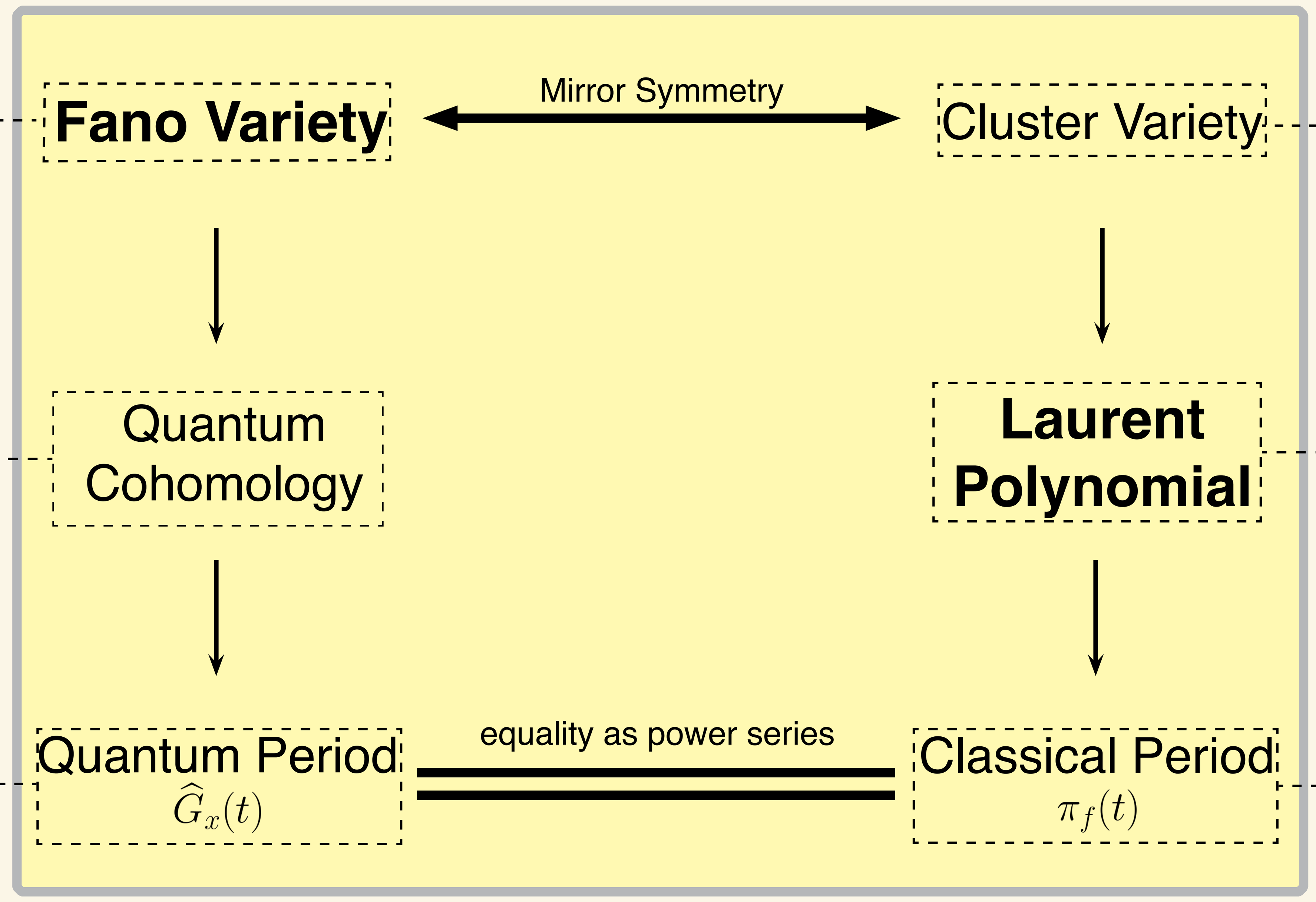
Quantum cohomology replaces the strict intersection of classical particle physics with the 'fuzzy', stringy intersection coming from string theory.

Quantum cohomology allows us to associate a system of partial differential equations to a given Fano variety. These equations can be solved using a formal power series of the form: $\widehat{G}_x(t) = \sum (m!) p_m t^m$. This is the **quantum period** of the Fano variety.

Fano varieties in dimensions 1, 2 and 3 are already classified. The Fanosearch programme of Coates-Corti-Galkin-Kasprzyk aims to classify Fano varieties in dimension 4. Since Fano varieties are hard to find explicitly, the approach is to obtain an indirect classification using Mirror Symmetry. This predicts that every Fano is mirror dual to another object, which is to be determined. The classification problem translates into the following questions:

- What is the mirror dual of a Fano variety?
- Can we obtain a classification of these objects in dimension 4?
- How do we reconstruct a Fano from its mirror dual?

The strategy is to classify Fano varieties by classifying their mirror duals.



Mirror Symmetry is a circle of ideas motivated by observations in string theory. It relates geometric structures on a given space with different structures on another space, called the mirror dual. Many properties of an object can be determined by studying its mirror dual, and in some cases the mirror dual may be easier to work with than the object we first started with.

It is conjectured for Fano varieties that the mirror dual is a cluster variety. A Fano variety is mirror dual to a cluster variety if the quantum period of the Fano variety is equal, as a power series, to the classical period of the cluster variety.

Cluster varieties can be studied using explicit combinatorial and computer techniques. Thus, a classification of cluster varieties of dimension 4 appears to be within reach. Such a classification would, by mirror symmetry, imply a classification of 4-dimensional Fano varieties.

Laurent Polynomials are the fundamental object in this classification programme. A Laurent polynomial f is defined to be an element of the ring $\mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$. They are similar to ordinary polynomials except that we also allow a finite number of terms with negative exponents.

$$1 + x_1 + 2ix_2 + \frac{x_3}{x_2} + \frac{1}{x_1 x_2 x_3}$$

Laurent polynomials can be viewed as functions from $(\mathbb{C}^*)^n$ to $\mathbb{C}$.

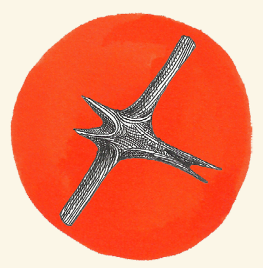
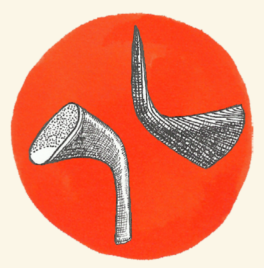
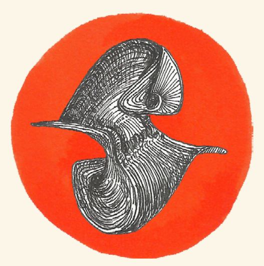
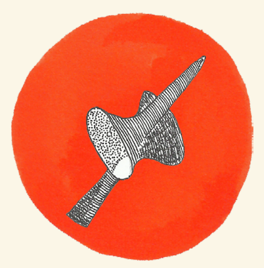
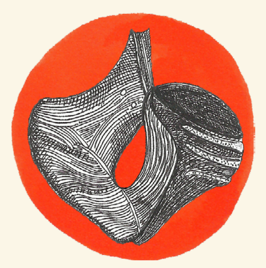
A Laurent polynomial determines an infinite power series called its **classical period.** This power series coincides with the following complex integral on its radius of convergence:

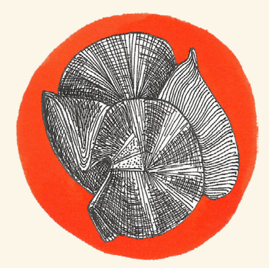
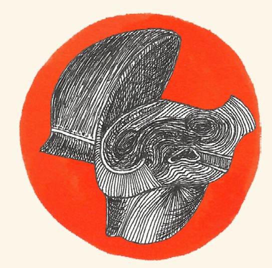
$$\pi_f(t) := \left(\frac{1}{2\pi i}\right)^n \int_{|x_n|=1} \cdots \int_{|x_1|=1} \frac{1}{1 - t.f(x_1, .., x_n)} \frac{dx_1}{x_1} \cdots \frac{dx_n}{x_n}$$

The initial mirror-symmetric conjecture was that the mirror dual to a Fano variety should be a pair $((\mathbb{C}^*)^n, f)$, where f is a Laurent polynomial whose classical period coincides with the quantum period of the Fano.
Since there are finitely many Fano varieties, one expects a finite number of mirror duals. However, there exist infinitely many Laurent polynomials with the same classical period. The conjecture is refined by considering cluster varieties.

Cluster varieties are objects constructed by gluing together (possibly infinitely many) copies of $(\mathbb{C}^*)^n$. Each local $(\mathbb{C}^*)^n$ on a cluster variety also carries the data of a Laurent polynomial $f \in \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$. These local Laurent polynomials are related to one another by a process called mutation.
As $\pi_f(t)$ is preserved by mutations, all local Laurent polynomials on a cluster variety have the same classical period. Thus we can unambiguously define the classical period of a cluster variety as the classical period of any of its local Laurent polynomials.

Conclusion: The motto of Fanosearch is: Fano varieties can be classified by classifying their mirror duals. The power of this approach is illustrated in a recent paper by Coates-Corti-Galkin-Kasprzyk, in which techniques developed to classify Fano 4-folds are applied to the 2- and 3-dimensional case. The authors reproduce the classification of Fano varieties in both 2- and 3-dimensions. These results provide a consistency check for the conjectures and techniques developed so far, and provide hope that a full classification of Fano 4-folds will be obtained in the near future.

References:
arxiv.org/abs/1303.3288
arxiv.org/abs/1212.1785
arxiv.org/abs/1302.5089