# Canonical Substitution Tilings

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## Substitution Rules and Tilings

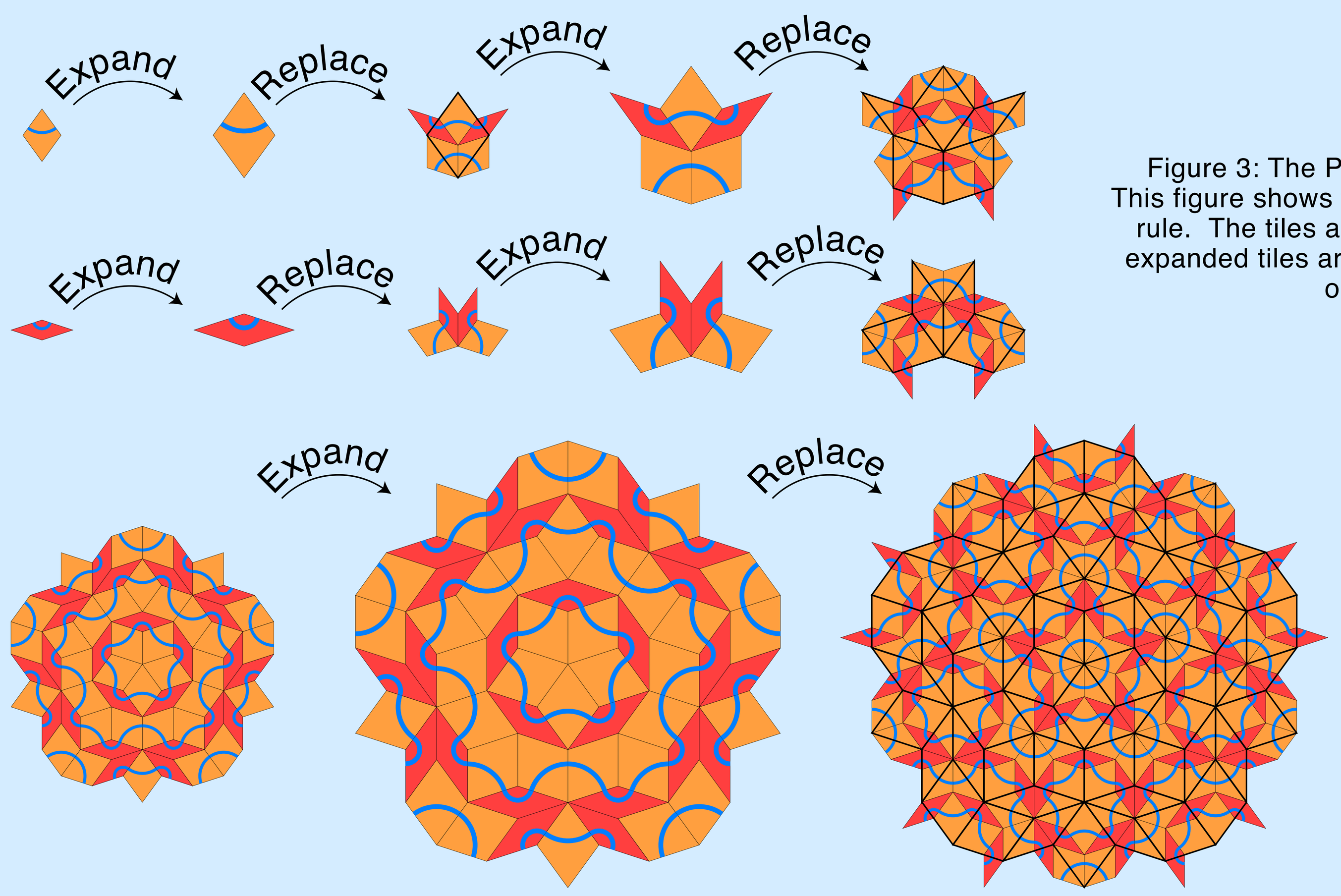


Figure 3: The Penrose Substitution Rule. This figure shows the action of the substitution rule. The tiles are first expanded, then the expanded tiles are replaced by copies of the original tiles.

Substitution rules are an important method way of constructing non-periodic tilings. An example is the Penrose substitution rule shown in Figure 3 above. A substitution rule has two parts. First a tiling, or patch of tiles is expanded by multiplication by a uniform constant. For each prototile there is then a replacement rule, which replaces the expanded tile with a patch of the original tiles. We require that the replacement rule act in such a way that no holes are created in the tiling and does not give overlapping tiles.

Iterating a substitution rule on the original tiles gives arbitarily large patches of tiles. For every substitution rule we may therefore consider full tilings of the plane which can be generated by that substitution rule. These tilings are non-periodic under certain conditions. For example if the scaling is an irrational number (as is the case for the tilings in this poster) then the tiling is non-periodic.

## Tilings of the Plane

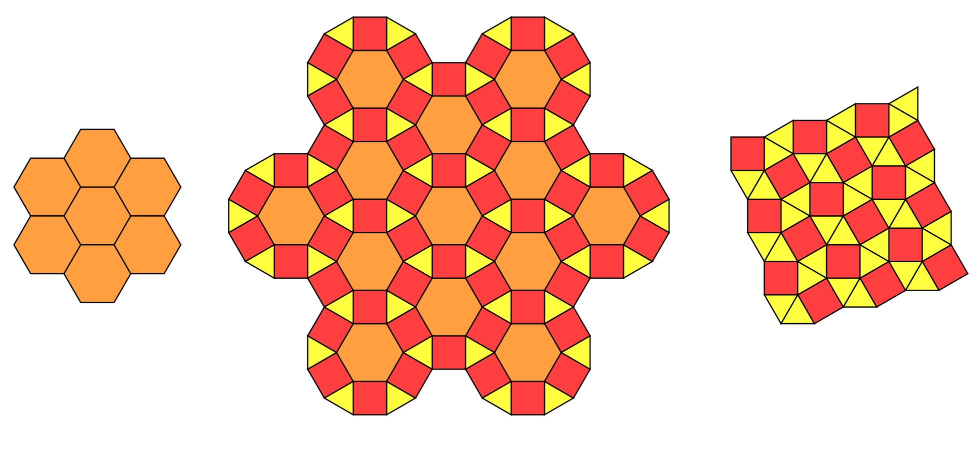

Figure 1: Periodic tilings of the plane.

A tiling of the plane is the division of the plane into regions or tiles. In this poster we will consider the special case where the tiles are polygons.

We consider a finite number of distinct polygons. These polygons are called *prototiles*. A *tiling* is made up of translations of these tiles so that:

- Every point in the plane is covered by a tile.
- No two tiles intersect beyond their boundary.

A patch of a tiling is a bounded subset of the tiling.

The simplest tilings that cover the plane are the periodic tilings. For example the tilings shown in Figure 1. These tilings have a patch that fits next to itself. One may therefore use copies of the patch to cover the entire plane, as shown in Figure 2.

Tilings which are not periodic are harder to construct. In this poster we consider two methods of constructing aperiodic tilings and the links between them.

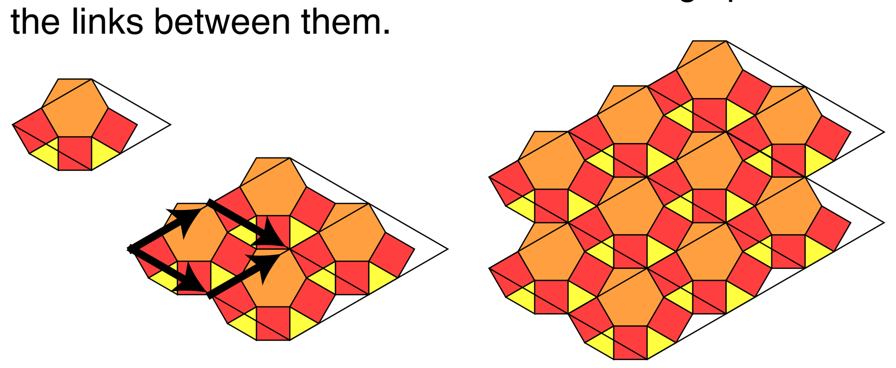

Figure 2: Constructing a periodic tiling

## Canonical Projection Tilings

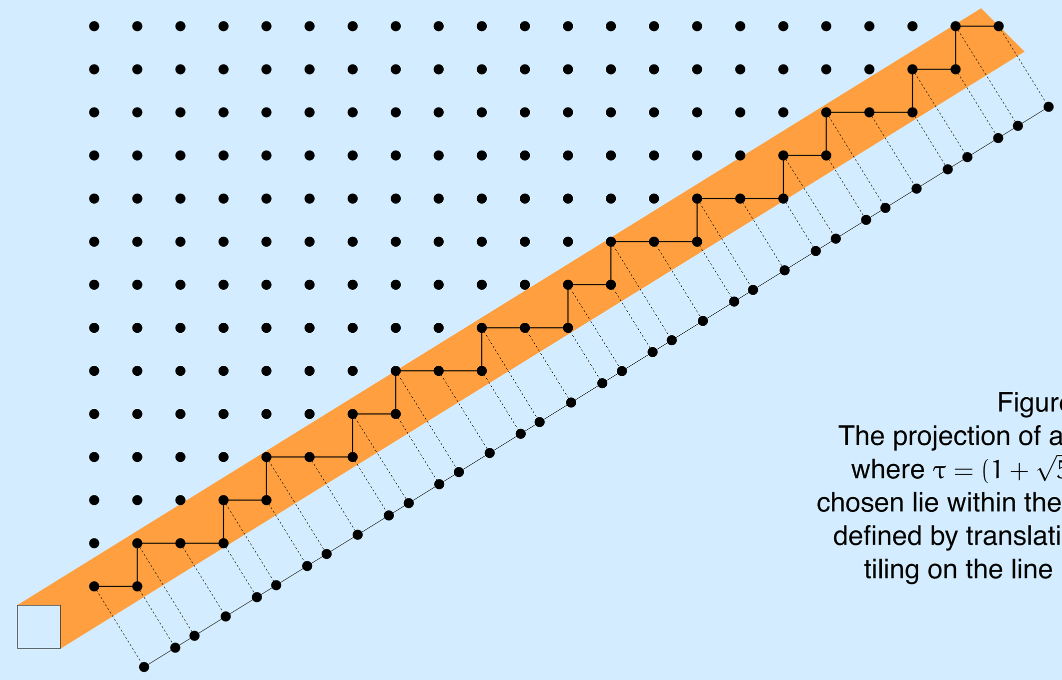

Figure 4: Fibonnacci Tiling. The projection of a slice of $\mathbb{Z}^2$ to a line of gradient $1/\tau$, where $\tau = (1+\sqrt{5})/2$ is the golden ratio. The points chosen lie within the region in orange, which is the region defined by translating a unit square along the line. The tiling on the line is known as the Fibonnacci tiling.

Like substitution rules, the canonical projection method is a way of generating non-periodic tilings. This considers the projection of a slice of a lattice in $\mathbb{R}^n$ to a subspace as follows.

Consider $\mathbb{Z}^n$ and $\mathbb{R}^n = R \oplus W \oplus V$, where $V \cap \mathbb{Z}^n = 0$ and $W \cap \mathbb{Z}^n = 0$. Also $R \cap \mathbb{Z}^n$ and $V + W \cap \mathbb{Z}^n$ are lattice of rank equal to the dimension of the space. A canonical projection point set is the set of points:

$$\Pi_V((V + \mathcal{H} + t) \cap \mathbb{Z}^n)$$

where $\Pi_V$ is projection down $W + R$ onto $V$ and $t \in \mathbb{R}^n$ is some translation vector. In $\mathbb{R}^n$ the points in $(V + \mathcal{H} + t) \cap \mathbb{Z}^n$ may be linked up by the $m$ dimensional facets of the lattice (where $m$ is the dimension of $V$). If the projections of these facets do not intersect, the the canonical projection point set may be linked together to form a canonical projection tiling.

The example of the Fibonnacci tiling is shown in Figure 4. The lattice is $\mathbb{Z}^2 \subset \mathbb{R}^2$. The unit square in the lattice is moved along a line of gradient $1/\tau$ giving the orange region. The points in the orange region are projected to the line. This produces a non-periodic set of points as $1/\tau$ is irrational. The intervals between the points are the projections of the standard lattice generators. In $\mathbb{R}^2$ these form a staircase linking the points in the orange region. Considering the projection of these line segments gives the tiling on the line.

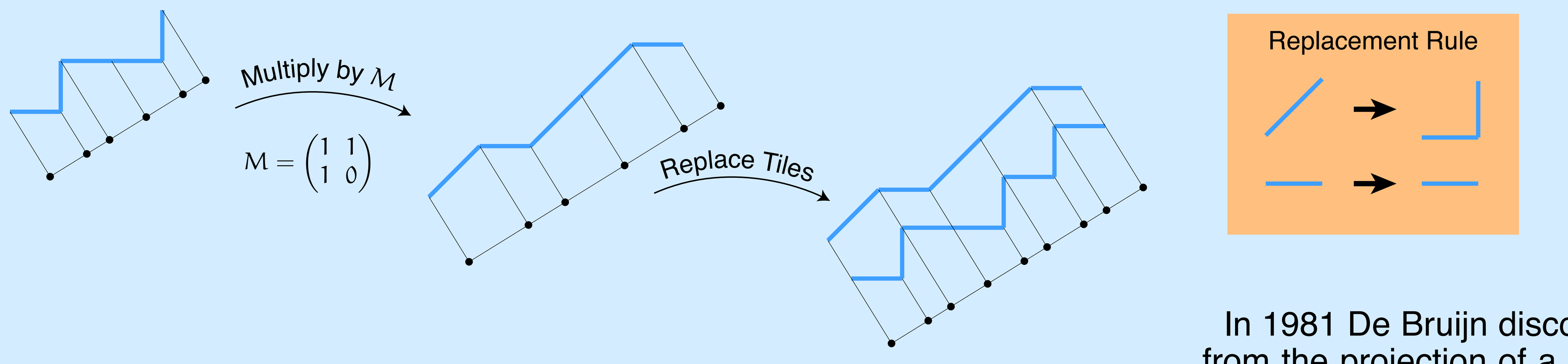


Figure 5: The Fibonnacci substitution rule.

The tiling shown in Figure 4 admits a substitution rule. In this figure we show how the action of this substitution rule is related to the matrix $M$. $M$ has two linear eigenspaces $V$ and $W$. The eigenvalues are $\tau = (1+\sqrt{5})/2$ on $V$ and $-\tau^{-1} = (1-\sqrt{5})/2$ on $W$.

Consider a section of the staircase for the Fibonnacci tiling on the left. Applying the matrix $M$ gives a new stretched staircase with vertices in $\mathbb{Z}^2$, as $M$ is an integer matrix. Applying the replacement rule shown on the right gives a longer section of Fibonnacci tiling staircase.

If we consider $V$ we start with the projection of the staircase, as section of tiling. Applying $M$ multiplies this by $\tau$. The replacement rule on the staircase then induces a replacement rule on $V$. We therefore have a substitution rule on $V$.

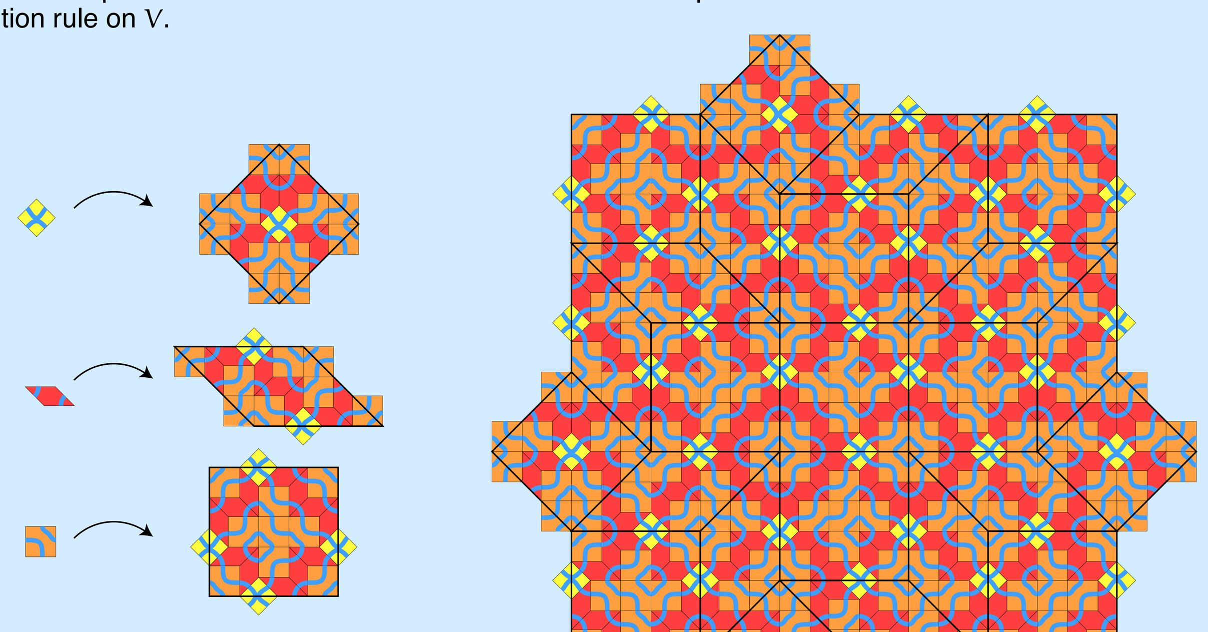

Figure 6: A substitution rule with matrix:

$$\begin{pmatrix} 3 & 2 & 0 & 2 \\ 1 & 1 & -1 & 0 \\ 0 & -2 & 3 & 2 \\ 1 & 0 & 1 & 1 \end{pmatrix}$$

In 1981 De Bruijn discovered [dB81] that the beautiful aperiodic substitution tilings of Penrose [Gar77] can be constructed from the projection of a slice of a five dimensional lattice. This led to the development of the canonical projection method, which has been studied in great detail [BM00].

We may pose the following general question:

**[Q] What canonical projection tilings admit substitution rules?**

Interestingly, despite the elementary nature of the question, and the amount of research in the area, [Q] has received relatively little attention. Recently, EOH and JSWL have given a characterisation of the substitution tilings which admit a substitution rule [Har03].

[Har03], shows that all canonical substitution rules are related to *quadratic expansion matrices*. These are integer matrices which act on three spaces $V$, $W$ and $R$. The space $R$ must be rational, that is $R \cap \mathbb{Z}^n$ is a lattice of rank equal to the dimension of $R$. On $V$ and $W$ the matrix must act as multiplication by $\lambda$ and $\pm\lambda^{-1}$, where $\lambda$ is a quadratic algebraic unit, that is $\lambda^2 + q\lambda \pm 1 = 0$, where $q$ is an integer. The following result is obtained:

THEOREM 1 *A canonical projection tiling is a canonical substitution tiling if and only if the spaces $V$, $W$ and $R$ of the canonical projection are the eigenspaces $V$, $W$ and $R$ of a quadratic expansion matrix.*

The role of the matrix in describing the substitution is illustrated in Figure 6. An example of a substitution tiling found using this theorem is shown in Figure 7.

### References

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## Example: The Ammann Tiling

$$N = \begin{pmatrix} 1 & 1 & 0 & -1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ -1 & 0 & 1 & 1 \end{pmatrix}$$

The Ammann substitution rule and tiling was discovered by Beenker following the work of De Bruijn [Bee82] and independantly by Ammann, considering substitution rules (later published in [AGS92]).

The matrix for this substitution rule is $N$. This has two eigenspaces, $V$ and $W$, which are orthogonal in $\mathbb{R}^4$. On $V$, $N$ acts as multiplication by $\lambda = 1 + \sqrt{2}$, and on $W$ it acts as multiplication by $-\lambda^{-1}$.

Figure 7 shows the projection of the unit hypercube $\mathcal{H}$ to $W$ and the projection of $N\mathcal{H}$. By considering $\mathbb{Z}^4 \cap (V + \mathcal{H})$ we may construct the canonical projection tiling $\mathcal{T}$. We may also construct a tiling using $\mathbb{Z}^4 \cap (V + N\mathcal{H})$, this turns out to be $\lambda\mathcal{T}$, as $N$ acts as multiplication by $\lambda$ on $V$. As $\Pi_W(N\mathcal{H}) \subset \mathcal{H}$ the vertices of $\lambda\mathcal{T}$ will be a subset of the vertices of $\mathcal{T}$. We may now apply a replacement rule to regain the tiling $\mathcal{T}$.

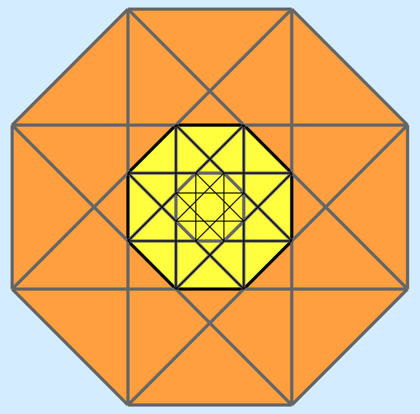

Figure 7: The plane $W$ for the Ammann Tiling The projection of the unit hypercube $\mathcal{H}$ to the plane $W$ for the Ammann tiling is shown in orange and the projection of $N\mathcal{H}$ is shown in yellow. The projections of the other edges are also shown.

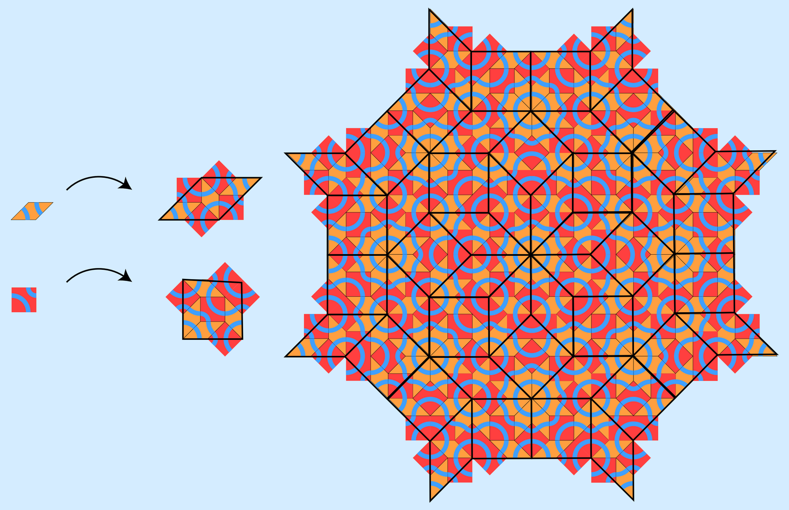

Figure 8: The Ammann Tiling and Substitution rule. On the right is a patch of $\mathcal{T}$ using coloured tiles. The edges of the tiling $\lambda\mathcal{T}$ are shown on top as black lines.