Hamiltonian Particle Mesh Methods for Non-Hydrostatic Weather Prediction Models

How Can Classical Mechanics Improve Weather Forecasting?

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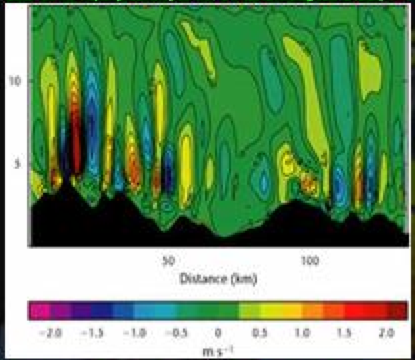
FIGURE 1: Vertical component of the wind velocity field over a slice of the Alps (Courtesy of the UK Meteorological Office)



INTRODUCTION

• The atmosphere is a highly complex chaotic system.

• To simplify the dynamics, weather prediction models typically neglect vertical acceleration of the wind velocity in the vertical momentum equation by assuming that the atmosphere is hydrostatic, that is the pressure gradient

$$\frac{\partial p}{\partial z} = -\rho g$$

• This is inadequate for high resolution weather prediction, especially where vertical dynamics are significant, such as mountainous regions as shown in the plot of the vertical wind velocity in Figure 1.

THE EULER VERTICAL SLICE MODEL

• A first step towards modelling non-hydrostatic dynamics is to consider just a vertical slice of the atmosphere.

• The hydrodynamic equations in u, the wind velocity vector, the potential temperature θ and modified density μ, defined to the right, are referred to as a Vertical Slice Euler Model.

$$\frac{d\mathbf{u}}{dt} = -c_p\theta\nabla_x\pi - g\mathbf{k},$$
$$\frac{d\theta}{dt} = 0,$$
$$\frac{d\mu}{dt} = -\mu\nabla_x\cdot\mathbf{u},$$

FIGURE 2: Momentum, entropy and mass conservation equations form the vertical slice Euler model.

($\pi = \mu^{R/c_v}$, c_p and c_v are the specific heat capacities of dry air at constant pressure and volume, and R is the gas constant for dry air)

• Conventional numerical methods typically smooth the solution field to improve the numerical stability but fail to preserve the flow structure, violating for example energy, mass or potential vorticity conservation laws.

• This can lead to numerical weather prediction models which do not accurately represent the real physical characteristics of the atmosphere because the discretisation does not preserve the flow's variational structure.

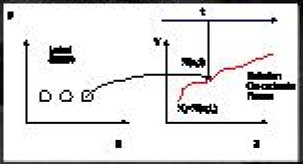
VARIATIONAL APPROACH

• In the eighteenth century, Euler and Lagrange, pictured top right, took a variational approach to classical mechanics by defining equations of motion that minimise an action principle.

• Hamilton extended this approach in defining Hamiltonian systems whose motion preserves a scalar called the *Hamiltonian*, which for the Euler model is energy, and whose solution has Hamiltonian structure.

KEY IDEA:

Atmospheric motion can be defined by particles in label space whose motion is Hamiltonian



• More precisely, a Hamiltonian fluid can be approximated by a finite system of particles each with position q_i and momentum p_i defined by Hamiltonian's equation of motion as shown in Figure 3.

How can particles defining atmospheric motion be modelled on a computer? …

HAMILTONIAN PARTICLE MESH (HPM) METHODS

• The HPM approximation of the Hamiltonian equations of motion for the Euler Vertical Slice Model is similar to a technique called *Smooth Particle Hydrodynamics* but adds a fixed grid to efficiently smooth the solution field.

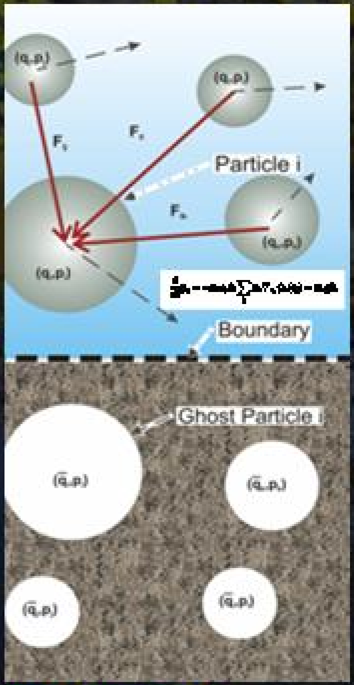
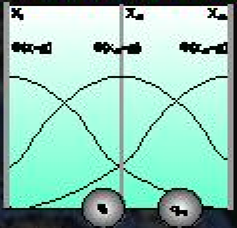


FIGURE 3: Particle mechanics and enforcement of boundary conditions

KEY IDEA:

THE HPM method uses a fixed grid to efficiently stabilise the particle motion and preserve Hamiltonian structure

FIGURE 4: Particle to grid interpolation of the solution field using basis functions

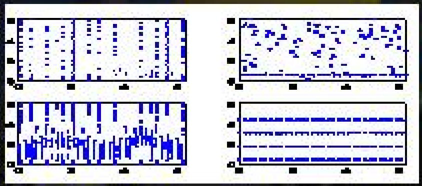


1) Each point of the mesh x_i is assigned a compact, basis function $\psi(x_i - .)$, such as a cubic spline, as shown in Figure 4 to the right.

2) Each particle's position q_i is initially distributed at uniform intervals between the grid points.

3) Each particle's momentum p_i and position are then updated by the differential of the Hamiltonian [illegible] with respect to q_i and p_i using a grid smoothed layer depth field.

• Ghost particles can be used to enforce boundary conditions with position and momentum defined by a reflection R(q, p) as shown in Figure 3.



FIGURES 5: (Left to right) velocity vector field and particle positions without (top) smoothing and smoothing across 8 grid cells (bottom)

ADVANTAGES	DISADVANTAGES
• Conserves important physical quantities	• More difficult to implement than traditional numerical approaches
• Scalable to the modelling of large regions	• Requires numerical smoothing to remain stable as shown in Figure 5 above.
• Well supported by the theory of classical mechanics	• Implementation of boundary conditions is still an open area of research