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Asymptotic behaviour of randomised fractional volatility models

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Introduction

- Large deviations are widely used in Physics as well as in Mathematics to model the exponential decay of probability measures of rare events.
- This set of techniques and results have recently been adopted to study small-time or tail behaviours of some variables of interest.
- These asymptotics have provided a deeper understanding of the behaviour of models, and, ultimately, allow for better calibration of real data.

Goal

- **We wish to prove small-time and tail behaviour of option prices, and implied volatilities, when the underlying stock price follows an extension of Stein-Stein stochastic volatility model.**
- The extensions considered:
  (i) Randomised model: the SDE driving the instantaneous volatility process is started from a random distribution;
  ex.: allows to understand the ST behaviour of the so-called forward volatility, simpler distributions also possible.
  (ii) Fractional model: the volatility is driven by a fractional Brownian motion (with Hurst exponent $H \in (0,1)$);
  (iii) Extended model: allow for a more general dependence of the stock price on the instantaneous volatility process.

Fractional Brownian Motion

- Fractional stochastic volatility models have recently been extended to the case $H < 1/2$ and have become the go-to types of models for estimation and calibration.
- A fractional Brownian motion (fBm) W^H is a continuous centered Gaussian process, starting from zero, with Hurst parameter $H \in (0,1)$ and covariance matrix

$$\left\langle W^H_t, W^H_s \right\rangle = \frac{1}{2}\left(|t|^{2H} + |s|^{2H} - |t-s|^{2H}\right),$$

for any $0 \le s, t$.
- Volterra representation of the fractional Brownian motion: for all $t \in [0,T]$,

$$W^H_t = \int_0^t K^H(t,s)\mathrm{d}B_s,$$

where B is a standard Brownian motion generating the same filtration as W^H, and K^H is the Volterra kernel.
- In this paper, we are interested in the case $H \in (0, \frac{1}{2})$.

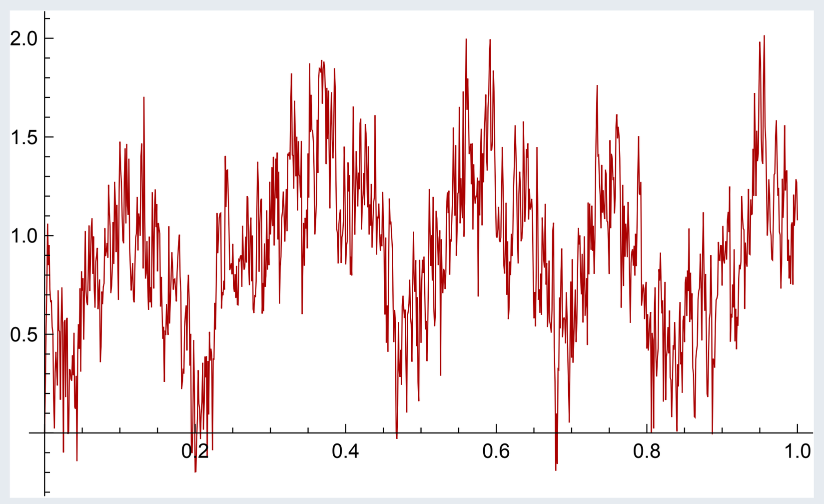

Figure: Simulation of a trajectory of a fractional Brownian motion with $H = 0.2$.

Model description

We study a generalised version of the fractional Stein-Stein model

$$\begin{cases} \mathrm{d}X_\tau = -\frac{1}{2}\sigma(Y_\tau)^2\mathrm{d}\tau + \sigma(Y_\tau)(\rho\mathrm{d}B_\tau + \bar\rho\mathrm{d}B^\perp_\tau), & X_0 = 0, \\ \mathrm{d}Y_\tau = (\lambda + \beta Y_\tau)\mathrm{d}\tau + \xi\mathrm{d}W^H_\tau, & Y_0 \sim \Theta, \end{cases} \tag{0.1}$$

where **Θ is a random variable**, W^H is a fractional Brownian motion, with Hurst parameter $H \in (0,1)$, $(B, B^\perp)$ is an independent two-dimensional standard Brownian motion, $\beta < 0$, $\lambda, \xi > 0$, $\bar\rho := \sqrt{1-\rho^2}$ with $\rho \in (-1,1)$ the correlation between W^H and B.

Assumptions:

- σ^2 is Lipschitz continuous, such that $|\sigma^2(x)| \le C(1+|x|)$ for $x \in \mathbb{R}$, derivable with its derivative locally Hölder continuous.
- **σ satisfies 'generalised homogeneity' properties:** $\exists\tilde\sigma : \mathbb{R} \to \mathbb{R} : \exists b > 0 : \forall x \in \mathbb{R}, \varepsilon^b\sigma(x/\varepsilon^b) = \tilde\sigma(x)$, for ε small enough.
- $\mathcal{A}^\Theta_b$: $\exists b > 0 : \limsup_{\varepsilon\downarrow 0} h_\varepsilon \log\mathbb{P}(\varepsilon^b\Theta > 1) = -\infty$.

Tails asymptotics

- Rescaling: for $b, \varepsilon > 0$, $X^\varepsilon := \varepsilon^{2b}X$, and $Y^\varepsilon := \varepsilon^b Y$.
- Model: for $b, \varepsilon > 0$, $\tau \in [0,T]$,

$$\begin{cases} \mathrm{d}X^\varepsilon_\tau = -\frac{1}{2}\tilde\sigma(Y^\varepsilon_\tau)^2\mathrm{d}\tau + \varepsilon^b\tilde\sigma(Y^\varepsilon_\tau)(\rho\mathrm{d}B_\tau + \bar\rho\mathrm{d}B^\perp_\tau), & X^\varepsilon_0 = 0, \\ \mathrm{d}Y^\varepsilon_\tau = \left(\varepsilon^b\lambda + \beta Y^\varepsilon_\tau\right)\mathrm{d}\tau + \varepsilon^b\xi\mathrm{d}W^H_\tau, & Y^\varepsilon_0 \sim \varepsilon^b\Theta. \end{cases} \tag{0.2}$$

Small-time asymptotics

- Rescaling: for $b > 0$ and $\tau \in [0,T]$, $X^\varepsilon_\tau := \varepsilon^{2H+2b-1}X_{\varepsilon^2\tau}$ and $Y^\varepsilon_\tau := \varepsilon^b Y_{\varepsilon^2\tau}$.
- Model: for $b, \varepsilon > 0$, $\tau \in [0,T]$,

$$\begin{cases} \mathrm{d}X^\varepsilon_\tau = -\frac{\varepsilon^{2H+1}}{2}\tilde\sigma(Y^\varepsilon_\tau)^2\mathrm{d}\tau + \varepsilon^{2H+b}\tilde\sigma(Y^\varepsilon_\tau)(\rho\mathrm{d}B_\tau + \bar\rho\mathrm{d}B^\perp_\tau), & X^\varepsilon_0 = 0, \\ \mathrm{d}Y^\varepsilon_\tau = \left(\varepsilon^{b+2}\lambda + \beta\varepsilon^2 Y^\varepsilon_\tau\right)\mathrm{d}\tau + \varepsilon^{2H+b}\xi\mathrm{d}W^H_\tau, & Y^\varepsilon_0 \sim \varepsilon^b\Theta. \end{cases} \tag{0.3}$$

Large deviations principle

- The sequence $(X^\varepsilon)_{\varepsilon>0}$ is said to satisfy a Large Deviations Principle on $\mathcal{C}([0,T],\mathbb{R}^n)$ as ε tends to 0, with rate function I and speed h_ε, if for any Borel subset $A \subset \mathcal{C}([0,T],\mathbb{R}^n)$, the following inequalities hold:

$$\begin{cases} -\inf_{A^o} I(\phi) \le \liminf_{\varepsilon\downarrow 0} h_\varepsilon \log\mathbb{P}(X^\varepsilon \in A), \\ \limsup_{\varepsilon\downarrow 0} h_\varepsilon \log\mathbb{P}(X^\varepsilon \in A) \le -\inf_{\bar A} I(\phi). \end{cases} \tag{0.4}$$

- We will denote $X^\varepsilon \sim \mathrm{LDP}(h_\varepsilon, I)$.
- In particular, if the rate function I is continuous on $\bar A$, then the lim inf and lim sup coincide and

$$\lim_{\varepsilon\downarrow 0} h_\varepsilon \log\mathbb{P}(X^\varepsilon \in A) = -\inf_{A} I(\phi).$$

Main results

- **Tail asymptotics**
  For any $H \in (0,1)$ and $b \ge \frac{1}{2}$, such that the Assumptions hold,

$$X^\varepsilon \sim \mathrm{LDP}(\varepsilon^{2b}, \widetilde\Lambda),$$

  with $\widetilde\Lambda$ defined by,

$$\widetilde\Lambda(\phi) := \inf\{I_6(\chi) \mid \phi = I(\varphi, \varphi\cdot\psi),\ \chi = (\varphi,\psi), \psi \in \mathrm{BV}\}.$$

  The good rate function $\widetilde\Lambda$ is known explicitly.

- **Small-time asymptotics**
  For any $H \in (0,1)$ and $b \ge \frac{1}{2} - 2H$ such that the Assumptions hold,

$$X^\varepsilon \sim \mathrm{LDP}(\varepsilon^{4H+2b}, \mathtt{I}),$$

  with $\mathtt{I}$ defined by

$$\mathtt{I}(\chi) := \inf\left\{\widetilde{I}_5(\varphi,\psi) : \varphi\cdot\psi = \chi,\ \psi \in \mathrm{BV}\right\}.$$

  The good rate function $\mathtt{I}$ is known explicitly.

Applications to Implied Volatility Asymptotics

(i) **Large-strike implied volatility asymptotics**
For any $H \in (0,1)$, any $b \ge 1/2$, and any $t \in \mathcal{T}$, we have

$$\lim_{k\uparrow\infty} \frac{\Sigma^2_t(k)t}{k} = \frac{1}{2}\left(\inf_{y\ge 1}\widetilde\Lambda(y)\right)^{-1}.$$

(ii) **Small-time Implied volatility asymptotics**
For any $H \in (0,1)$, any $b \ge 1/2 - 2H$, and any $k \ne 0$, we have

$$\lim_{t\downarrow 0} t^b\Sigma^2_t\left(t^{1/2-H-b}k\right) = \frac{k^2}{2}\left(\inf_{y\ge k}\mathtt{I}(y)\right)^{-1}.$$

$\Rightarrow$ The implied volatility explodes with rate t^{-b}.

Conclusion

We extend [1] and [4] by

- considering an fO-U process for the volatility;
- allowing for **random initial value in the volatility process**, which is, in manifold ways, natural to financial modelling setups: uncertain volatility, etc.

References

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