A Dual Backward Representation for the Quantile Hedging of Bermudean Options

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Problem Framework

On $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, $\mathbb{F} := \{\mathcal{F}_t, 0 \le t \le T\}$ we define W a d-dim. B.M., and $\forall (t,x,y) \in [0,T] \times (0,\infty)^d \times \mathbb{R}^+$, $T > 0$ and for $s \ge t$ the two processes

$$X_s^{t,x} := x + \int_t^s \mu(r, X_r^{t,x})\mathrm{d}r + \int_t^s \sigma(r, X_r^{t,x})\mathrm{d}W_r,$$

$$Y_s^{t,x,y,\nu} := y + \int_t^s \nu_r^\top \mathrm{d}X_r^{t,x}, \nu \in \mathcal{U}_{t,x,y},$$

where $\mathcal{U}_{t,x,y}$ is the set of predictable processes valued in $\mathbb{R}^d$ such that $\mathbb{E}[\int_t^T |\nu_s^\top \sigma(s, X_s^{t,x})|^2 \mathrm{d}s] < \infty$, $\mu(\cdot,x)$, $\sigma(\cdot,x)$ are Lipschitz continuous and s.t. $X^{t,x} \in (0,\infty)^d$ $\mathbb{P}$–a.s., σ is invertible and $\lambda(\cdot,x) := (\sigma^{-1}\mu)(\cdot,x)$ is bounded and Lipschitz continuous.

The process X and Y respectively stands for the underlying and portfolio process with ν the strategies.

Finally the market is complete and $\frac{\mathrm{d}\mathbb{P}}{\mathrm{d}\mathbb{Q}_{t,x}} = Q_{t,x}$ where for $s \ge t$: $\mathrm{d}Q_{t,x}(s) := \lambda(s, X_{t,x}(s))Q_{t,x}(s)dW_s^{\mathbb{Q}_{t,x}} \in (0,\infty)$, $Q_{t,x}(t) = 1$.

Problem Definition

Set $\mathbb{T}_t := \{t_0 = 0 \le \cdots \le t_i \le \cdots \le t_n = T\} \cap (t,T]$. $\forall (t,x,p) \in [0,T] \times (0,\infty)^d \times [0,1]$, we want to solve

$$v(t,x,p) := \inf \Gamma(t,x,p) \ ,$$

where $\Gamma(t,x,p) :=$

$$\Big\{ y \in \mathbb{R}^+ : \exists \nu \in \mathcal{U}_{t,x,y} \text{ s.t.} \mathbb{P}\left[\cap_{s \in \mathbb{T}_t} S_s^{t,x,y,\nu}\right] \ge p \Big\},$$

with $S_s^{t,x,y,\nu} := \Omega \mathbb{1}_{t \ge s} + \mathbb{1}_{t<s}\{Y_s^{t,x,y,\nu} \ge g(s, X_s^{t,x})\}$,

and $g : [0,T] \times (0,\infty)^d \to \mathbb{R}^+$ a Lipschitz continuous function. This is an extension to [2]-[4].

Preliminary Remarks

- $v(T,\cdot) = 0$, $v(t,x,p) = +\infty$ when $p > 1$, by convention, and $v(t,x,p) = 0$ for all $p \le p_{\min}(t,x)$ with $p_{\min}(t,x) := \mathbb{P}\left[g(s, X_s^{t,x}) = 0, \forall\, s \in \mathbb{T}_t\right]$.

- $v(t,x,1)$ coincides with the continuation value of the super-hedging price of the Bermudean option. In this complete market it should satisfy $v(t,x,1) = \mathbb{E}^{\mathbb{Q}_{t,x}}[(v \vee g)(t_{i+1}, X_{t_{i+1}}^{t,x}, 1)]$ where $g(t,x,p) := g(t,x)\mathbb{1}_{0<p\le 1} + \infty \mathbb{1}_{p>1}$, $p \in \mathbb{R}$.

- The Lipschitz continuity of g implies that we can restrict to strategies ν such that $0 \le Y^{t,x,y,\nu} \le C(1 + |X^{t,x}|)$ giving, in particular, $0 \le v(t,x,p) \le C(1+|x|)$.

Problem Reduction

In the spirit of [2] the following proposition holds.

Proposition 1. *Fix $(t,x,p) \in [0,T] \times (0,\infty)^d \times [0,1]$, then*

$$\Gamma(t,x,p) = \{ y \ge 0 : \exists\, (u,\alpha) \in \hat{\mathcal{U}}_{t,x,y,p},$$
$$\text{s.t. } Y^{t,x,y,\nu} \ge g(\cdot, X^{t,x}, P^{t,p,\alpha}) \text{ on } \mathbb{T}_t \},$$

with $\hat{\mathcal{U}}_{t,x,y,p} := \mathcal{U}_{t,x,y} \times \mathcal{A}_{t,p}$, with $\mathcal{A}_{t,p}$ the set of predictable square integrable processes valued in $\mathbb{R}^d$ and such that $P^{t,p,\alpha} \in [0,1]$ on $[0,T]$.

Probabilistic Representation

Theorem 1. *Fix $0 \le i \le n-1$ and $(t,x,p) \in [t_i, t_{i+1}) \times (0,\infty)^d \times [0,1]$,*

$$v(t,x,p) = \inf_{\alpha \in \mathcal{A}_{t,p}} \mathbb{E}^{\mathbb{Q}_{t,x}}\left[(v \vee g)(t_{i+1}, X_{t_{i+1}}^{t,x}, P_{t_{i+1}}^{t,p,\alpha})\right].$$

As a consequence, there exists $C > 0$ such that $|v(t,x,p) - v(t,x',p)| \le C(1 + |x| + |x'|)|x - x'|$.

Remark 1. $(v \vee g)$ can be replaced by its convex envelope with respect to p (see [1] and [2]).

Dual Representation of the Solution

From Theorem 1 standard arguments should lead to a characterization of v as a viscosity solution on each interval $[t_i, t_{i+1})$, $i < n$ of[a]

$$\sup_{a \in \mathbb{R}^d} \left\{ -\partial_t \varphi + a^\top \lambda \partial_p \varphi - \frac{1}{2}\left(\operatorname{Tr} \sigma\sigma^\top \partial_{xx}^2 \varphi + 2 \operatorname{Tr} a^\top \sigma^\top \partial_{xp}^2 \varphi + |a|^2 \partial_{pp}^2 \varphi\right) \right\} = 0,$$

with the boundary condition $v(t_{i+1}-, \cdot) = (v \vee g)(t_{i+1}, \cdot)$.

$\Rightarrow$ **BUT** the control $a \in \mathbb{R}^d$ is not bounded making the use of numerical schemes delicate in practice....

$\Rightarrow$ **IDEA**: consider the Fenchel transform of v, $v^\sharp := \sup_{p \in \mathbb{R}}(pq - v(t,x,p))$. Heuristically a formal change of variable argument suggests that $v^\sharp$ should be solution of the linear PDE (see [2] for the case $n = 1$)

$$-\partial_t \varphi - \frac{1}{2}\left(\operatorname{Tr}[\sigma\sigma^\top \partial_{xx}^2 \varphi] + 2q \operatorname{Tr} \lambda^\top \sigma^\top \partial_{xq}^2 \varphi + |\lambda|^2 q^2 \partial_{qq}^2 \varphi\right) = 0,$$

on the different time steps, with the boundary conditions $v^\sharp(t_{i+1}-, \cdot) = (v \vee g)^\sharp(t_{i+1}, \cdot)$.

By the Feynman-Kac representation, this corresponds to the following backward algorithm

$$\begin{cases} w(T,x,q) := q + \infty \mathbb{1}_{\{q<0\}}, \\ w(t,x,q) := \mathbb{E}^{\mathbb{Q}_{t,x}}\left[(w^\sharp \vee g)^\sharp(t_{i+1}, X_{t_{i+1}}^{t,x}, Q_{t_{i+1}}^{t,x,q})\right] \end{cases} \text{for } t \in [t_i, t_{i+1}),\ i < n,$$

$\Rightarrow$ **MAIN RESULT**:

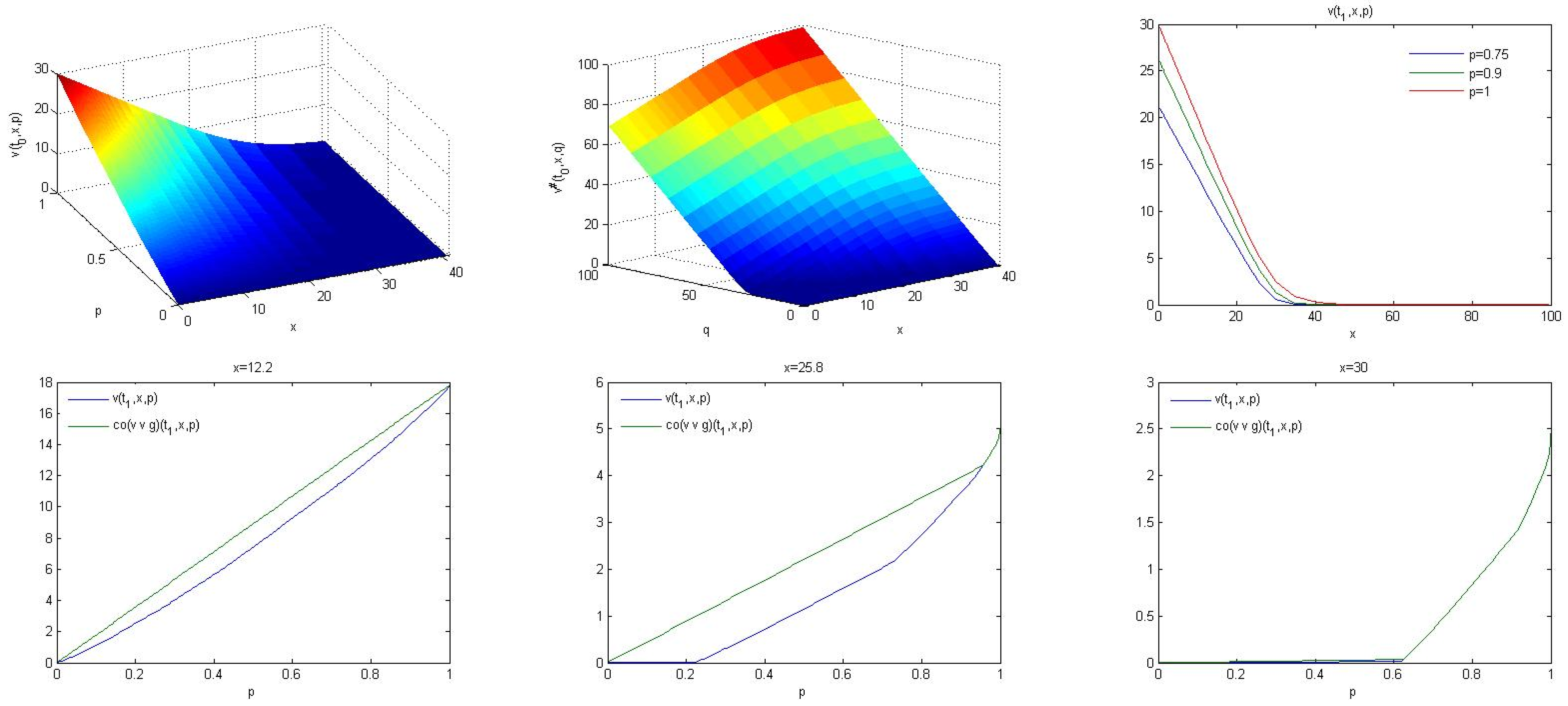
Theorem 2. $v = w^\sharp$ *on* $[0,T] \times (0,\infty)^d \times [0,1]$.

$\Rightarrow$ **MAIN DIFFICULTY**: control the propagation of the differentiability and growth properties of $w^\sharp$, backward in time.

[a] A precise statement would require a relaxation of the operator, see [2].

Numerical Application for a Put Option on X

Parameters: $r = 0$, $\sigma(t,x) = \sigma x = 0.25$, $\lambda(t,x) = \lambda = 0.2$, $T = 1$, $\mathbb{T}_t := \{t_0 = 0,\ t_1 = \frac{1}{3},\ t_2 = \frac{2}{3},\ t_3 = 1\} \cap (t, t_3]$, $t \in [0, t_3]$ and K (strike price) $= 30$. Notation: $\mathrm{co}(\psi)$ stands for the closed convex envelope of a given function ψ with respect to its last argument (see Remark 1).



Main References

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