Mathematical Problems in Climate Dynamics,
Nelder Fellow Lectures

3 March 2014

Lecture VI
Nonlinear & Stochastic Models or From DDS to RDS

Michael Ghil

Ecole Normale Supérieure, Paris, and University of California, Los Angeles

Please visit these sites for more info.

http://www.atmos.ucla.edu/tcd/

http://www.environnement.ens.fr/

Math of Climate Change, Hazards & Risks
IMU, IUGG & IUTAM

CIMAT, Guanajuato,
31 July 2013

Toward a Mathematical Theory of Climate Sensitivity

Michael Ghil
Ecole Normale Supérieure, Paris, and
University of California, Los Angeles
Joint work with M.D. Chekroun, D. Kondrashov, J. C. McWilliams
and J. D. Neelin (UCLA) + A. Bracco (Georgia Tech),
E. Simonnet (INLN, Nice), S. Wang (Indiana U.) and
I. Zaliapin (U. Nevada, Reno)

ENS

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Motivation

- The ***climate system*** is highly ***nonlinear and*** quite ***complex***.
- The system's ***major components*** — the atmosphere, oceans, ice sheets — ***evolve*** on many time and space scales.
- Its ***predictive understanding*** has to rely on the system's physical, chemical and biological modeling,

  but also on the thorough mathematical analysis of the models thus obtained: ***the forest vs. the trees***.
- The ***hierarchical modeling*** approach allows one to give proper weight to the understanding provided by the models *vs*. their realism: back-and-forth between ***"toy"*** (conceptual) and ***detailed*** ("realistic") ***models***, and between ***models*** and ***data***.
- How do we disentangle ***natural variability*** from ***the anthropogenic forcing***: ***can we & should we, or not?***

Climate and Its Sensitivity

Let's say CO_2 doubles:

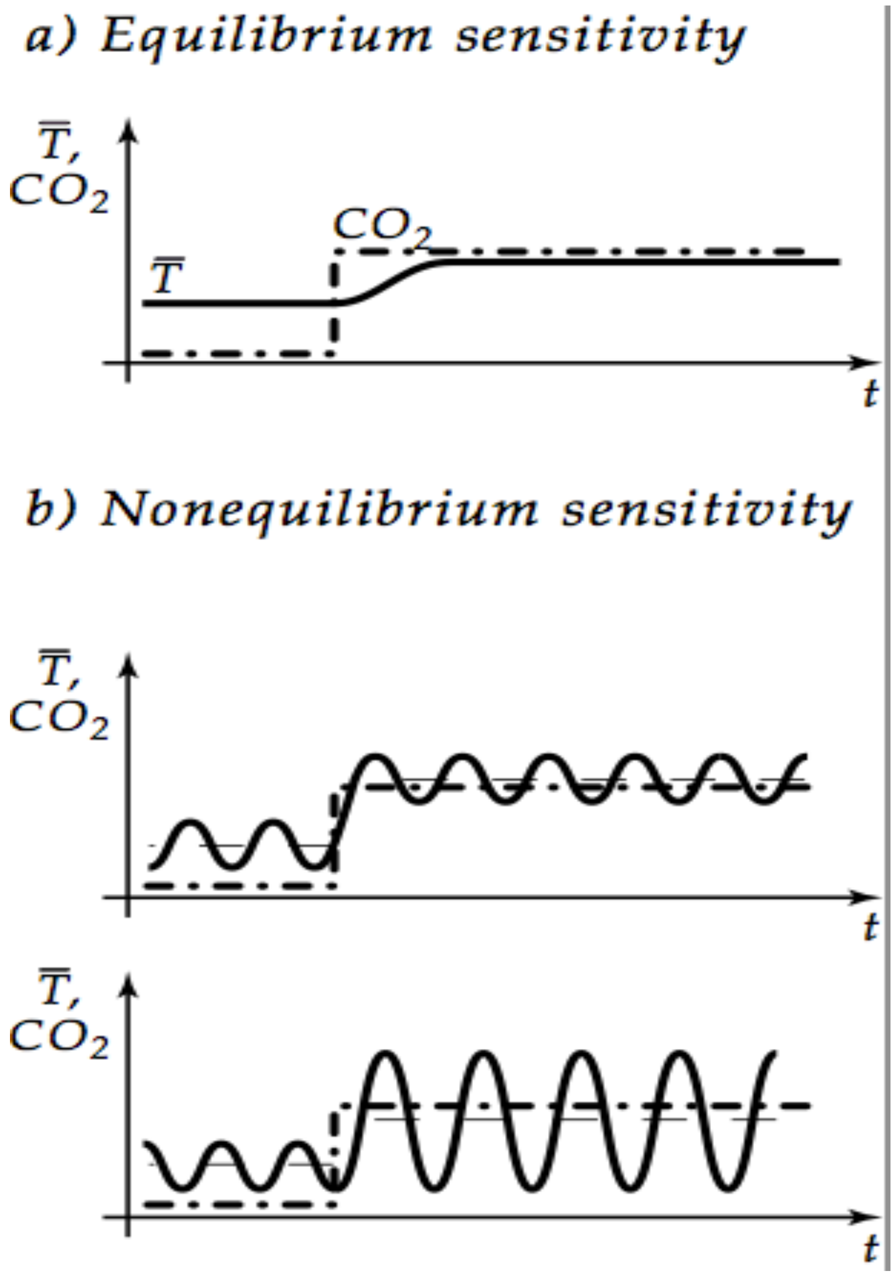
How will "climate" change?

1. Climate is in stable equilibrium (fixed point); if so, mean temperature will just shift gradually to its new equilibrium value.

2. Climate is purely periodic; if so, mean temperature will (maybe) shift gradually to its new equilibrium value. But how will the period, amplitude and phase of the limit cycle change?

3. And how about some "real stuff" now: chaotic + random?

Ghil (in *Encycl. Global Environmental Change*, 2002)



Outline

- The IPCC process: results and uncertainties
- Natural climate variability as a source of uncertainties
  - sensitivity to initial data ➔ error growth
  - sensitivity to model formulation ➔ see below!
- Uncertainties and how to fix them
  - structural stability and other kinds of robustness
  - non-autonomous and random dynamical systems (NDDS & RDS)
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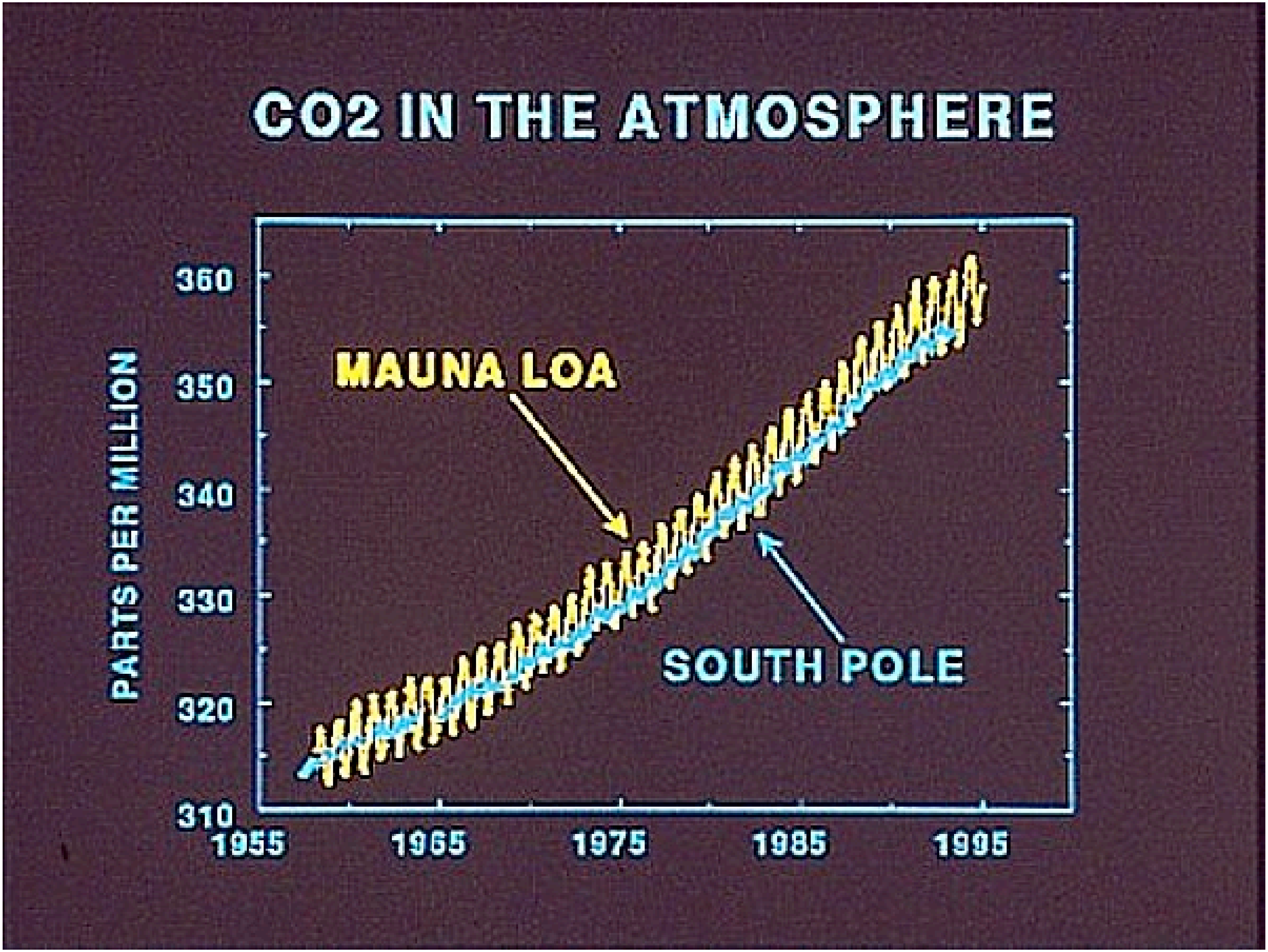
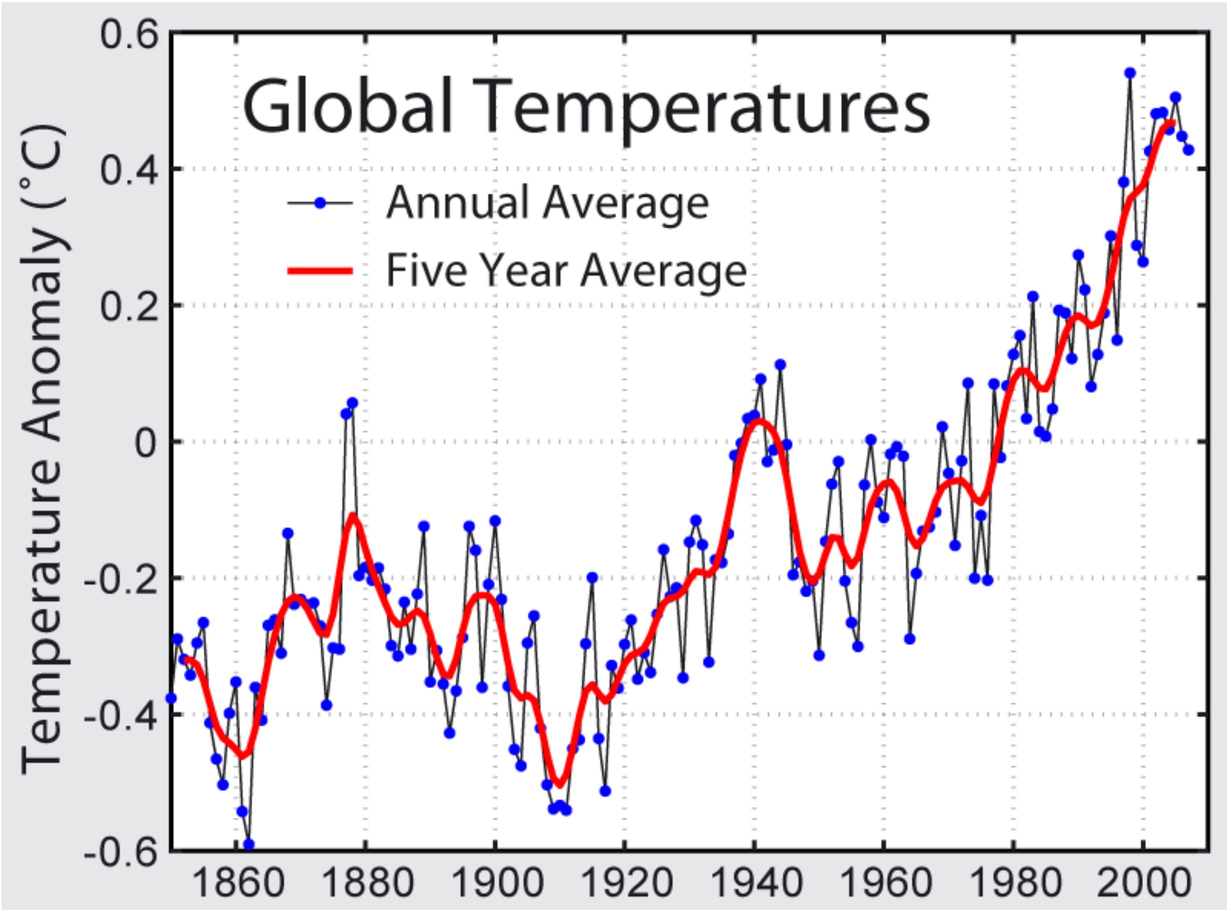

CO2 IN THE ATMOSPHERE
PARTS PER MILLION
360
350
340
330
320
310
1955
1965
1975
1985
1995
MAUNA LOA
SOUTH POLE

Temperatures and GHGs

Greenhouse gases (GHGs) go up,
temperatures go up:

It's gotta do with us, at least a bit,
doesn't it?



Wikicommons, from
Hansen *et al.* (*PNAS*, 2006);
see also http://data.giss.nasa.gov/gistemp/graphs/

Unfortunately, things aren't all that easy!

What to do?

Try to achieve better interpretation of, and agreement between, models …

Ghil, M., 2002: Natural climate variability, in *Encyclopedia of Global Environmental Change*, T. Munn (Ed.), Vol. 1, Wiley

Natural variability introduces additional complexity into the anthropogenic climate change problem

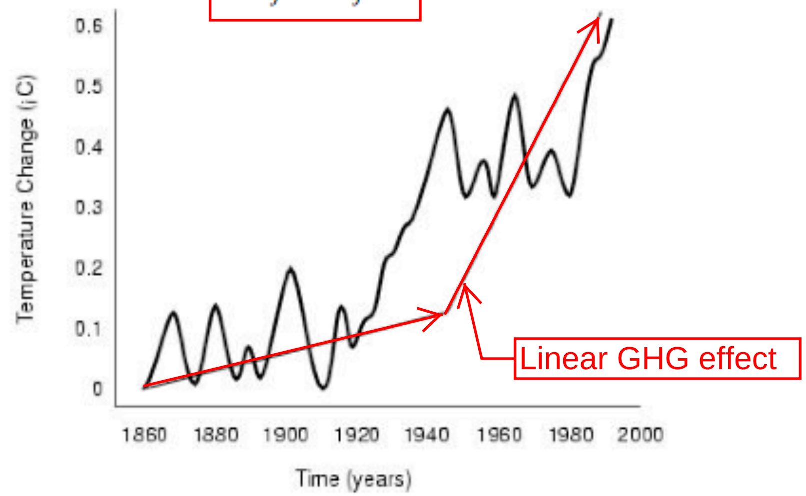
The most common interpretation of observations and GCM simulations of climate change is still in terms of a scalar, linear Ordinary Differential Equation (ODE)

$$c\frac{dT}{dt} = -kT + Q$$

$k = \sum k_i$ – feedbacks (+ve and -ve)

$Q = \sum Q_j$ – sources & sinks

$Q_j = Q_j(t)$



Linear response to CO_2 vs. observed change in T

Hence, we need to consider instead a system of nonlinear Partial Differential Equations (PDEs), with parameters and multiplicative, as well as additive forcing (deterministic + stochastic)

$$\frac{d\mathbf{X}}{dt} = \mathbf{N}(\mathbf{X}, t, \mu, \beta)$$

Global warming and its socio-economic impacts

Temperatures rise:

- What about impacts?
- How to adapt?

The answer, my friend, is blowing in the wind, *i.e.*, it depends on the accuracy and reliability of the forecast …

Source : IPCC (2007), *AR4, WGI, SPM*

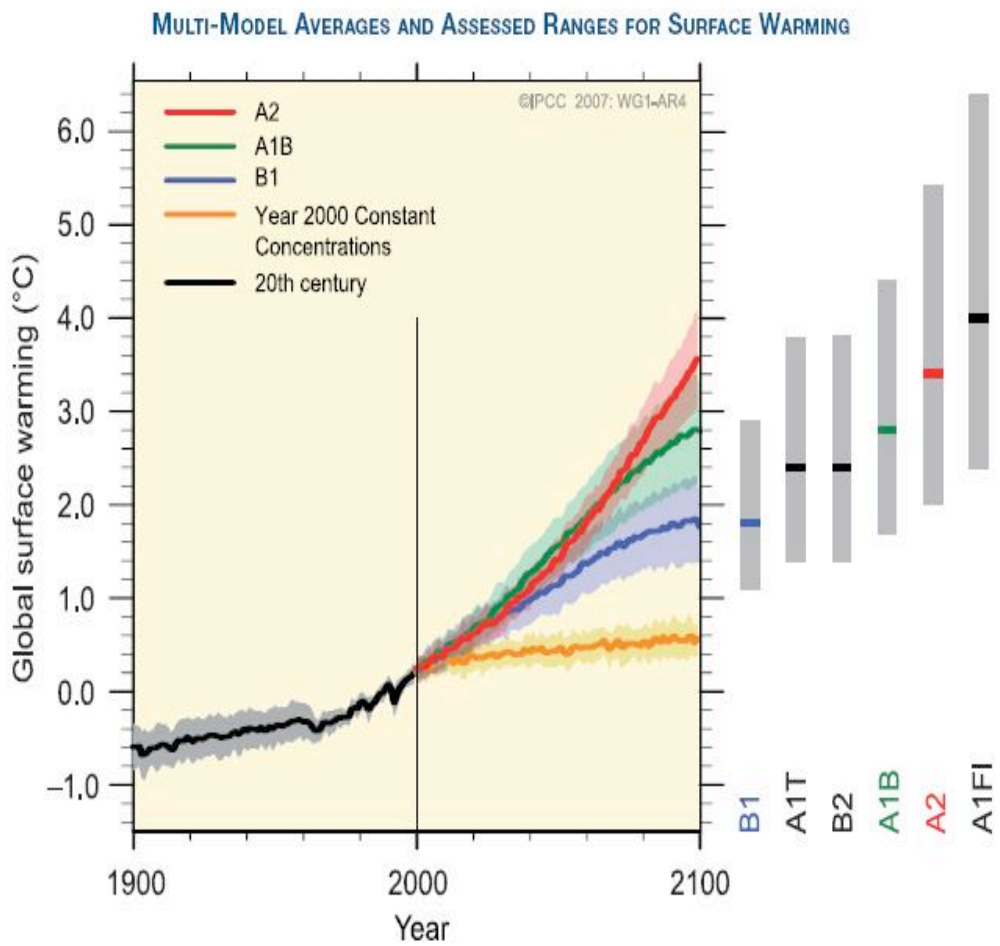


Figure SPM.5. *Solid lines are multi-model global averages of surface warming (relative to 1980–1999) for the scenarios A2, A1B and B1, shown as continuations of the 20th century simulations. Shading denotes the ±1 standard deviation range of individual model annual averages. The orange line is for the experiment where concentrations were held constant at year 2000 values. The grey bars at right indicate the best estimate (solid line within each bar) and the* ***likely*** *range assessed for the six SRES marker scenarios. The assessment of the best estimate and* ***likely*** *ranges in the grey bars includes the AOGCMs in the left part of the figure, as well as results from a hierarchy of independent models and observational constraints. {Figures 10.4 and 10.29}*

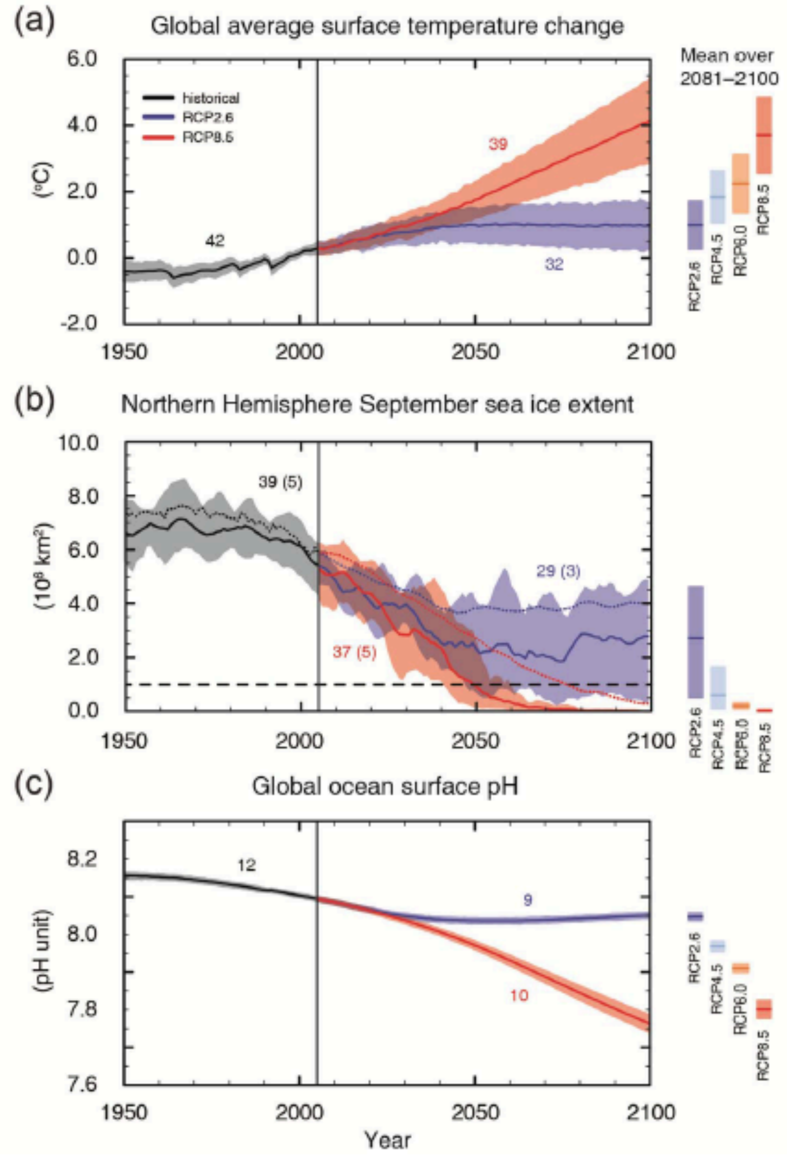
Global warming and its socio-economic impacts– II

Temperatures rise:

- What about impacts?
- How to adapt?

AR5 vs. AR4

A certain air of *déjà vu*:
GHG "scenarios" have been replaced by "representative concentration pathways" (RCPs), more dire predictions, but the uncertainties remain.



Source : IPCC (2013), AR5, WGI, SPM

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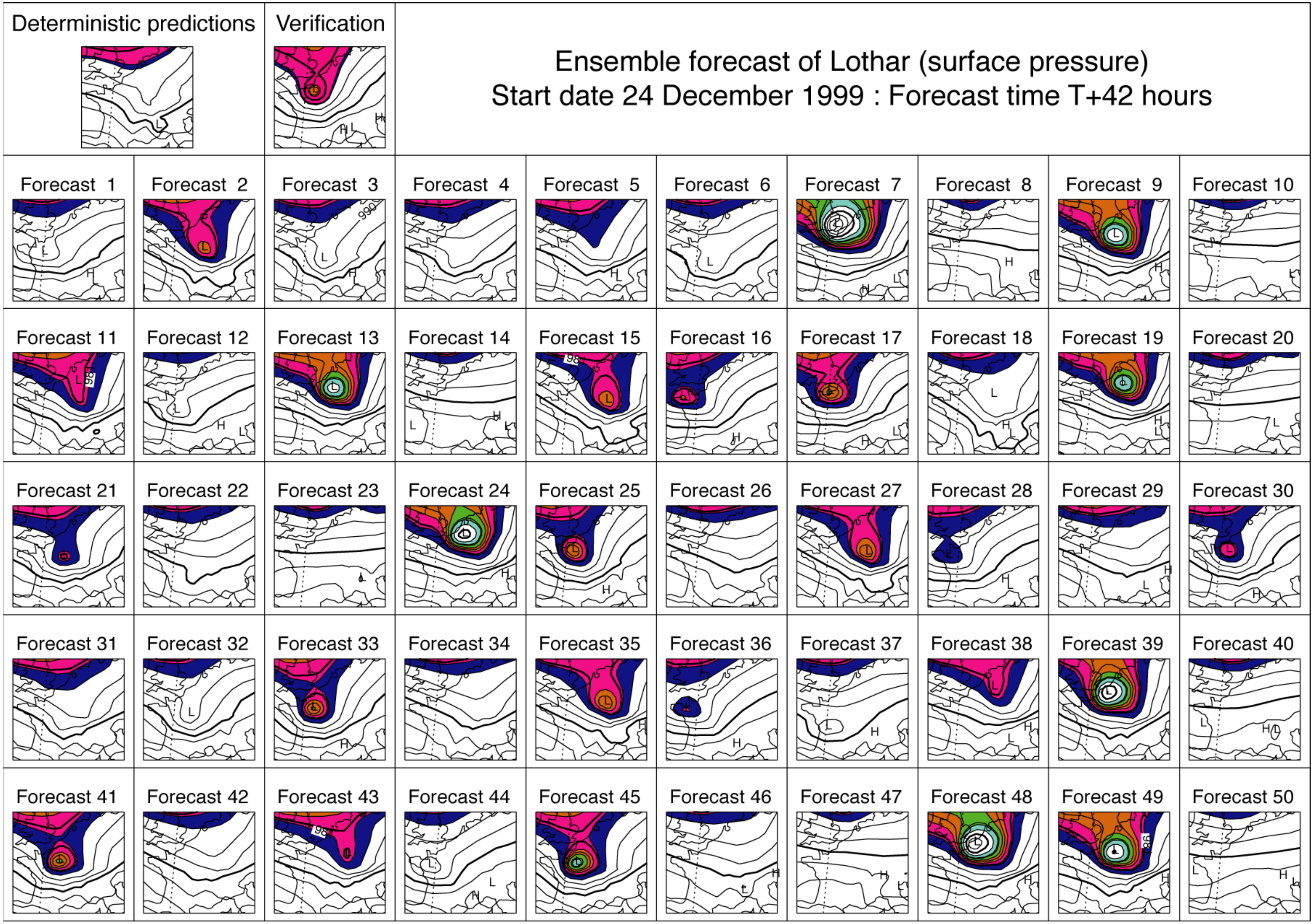


Courtesy Tim Palmer, 2009

Exponential divergence vs. "coarse graining"

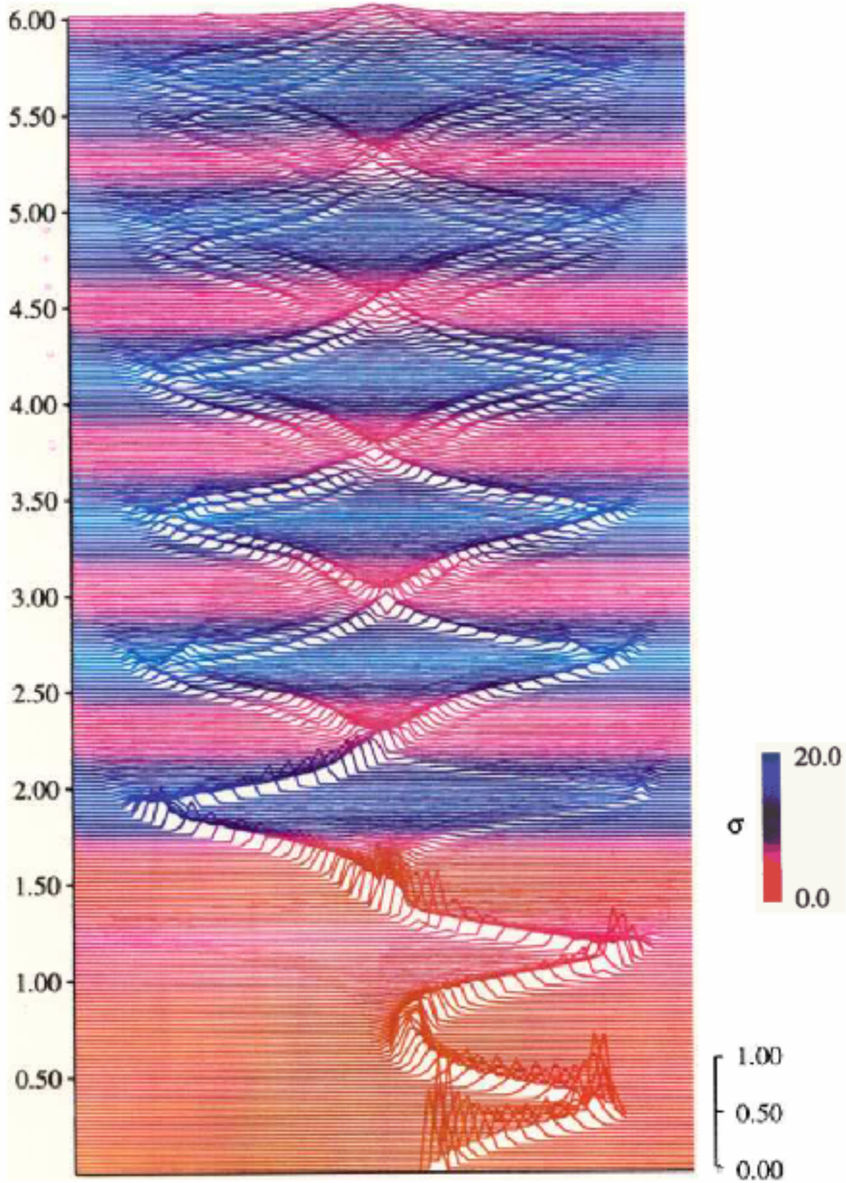
The classical view of dynamical systems theory is:

positive Lyapunov exponent ➔ trajectories diverge exponentially

But the presence of multiple regimes implies a much more structured behavior in phase space

Still, the probability distribution function (pdf), when calculated forward in time, is pretty smeared out

L. A. Smith (*Encycl. Atmos. Sci.*, 2003)



So what's it gonna be like, by 2100?

Table SPM.2. *Recent trends, assessment of human influence on the trend and projections for extreme weather events for which there is an observed late-20th century trend. {Tables 3.7, 3.8, 9.4; Sections 3.8, 5.5, 9.7, 11.2–11.9}*

Phenomenon[a] and direction of trend	Likelihood that trend occurred in late 20th century (typically post 1960)	Likelihood of a human contribution to observed trend[b]	Likelihood of future trends based on projections for 21st century using SRES scenarios
Warmer and fewer cold days and nights over most land areas	*Very likely*[c]	*Likely*[d]	*Virtually certain*[d]
Warmer and more frequent hot days and nights over most land areas	*Very likely*[e]	*Likely (nights)*[d]	*Virtually certain*[d]
Warm spells/heat waves. Frequency increases over most land areas	*Likely*	*More likely than not*[f]	*Very likely*
Heavy precipitation events. Frequency (or proportion of total rainfall from heavy falls) increases over most areas	*Likely*	*More likely than not*[f]	*Very likely*
Area affected by droughts increases	*Likely* in many regions since 1970s	*More likely than not*	*Likely*
Intense tropical cyclone activity increases	*Likely* in some regions since 1970	*More likely than not*[f]	*Likely*
Increased incidence of extreme high sea level (excludes tsunamis)[g]	*Likely*	*More likely than not*[f,h]	*Likely*[i]

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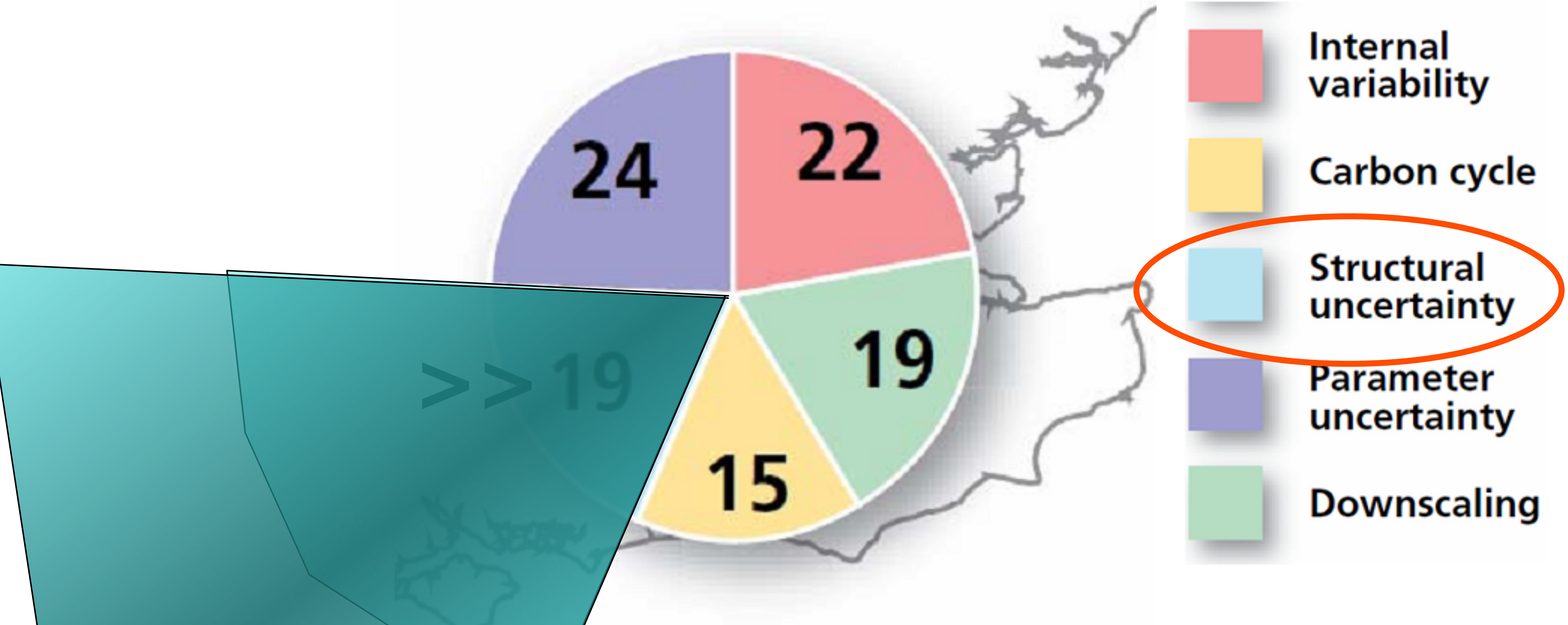
How important are different sources of uncertainty?

- Varies, but typically no single source dominates.



Uncertainties in winter precipitation changes for the 2080s relative to 1961-90, at a 25km box in SE England

Source: Met Office

Can we, nonlinear dynamicists, help?

The uncertainties
might be ***intrinsic***,
rather than mere
"tuning problems"

If so, maybe
stochastic structural stability could help!

Might fit in nicely with
recent taste for
"stochastic
parameterizations"

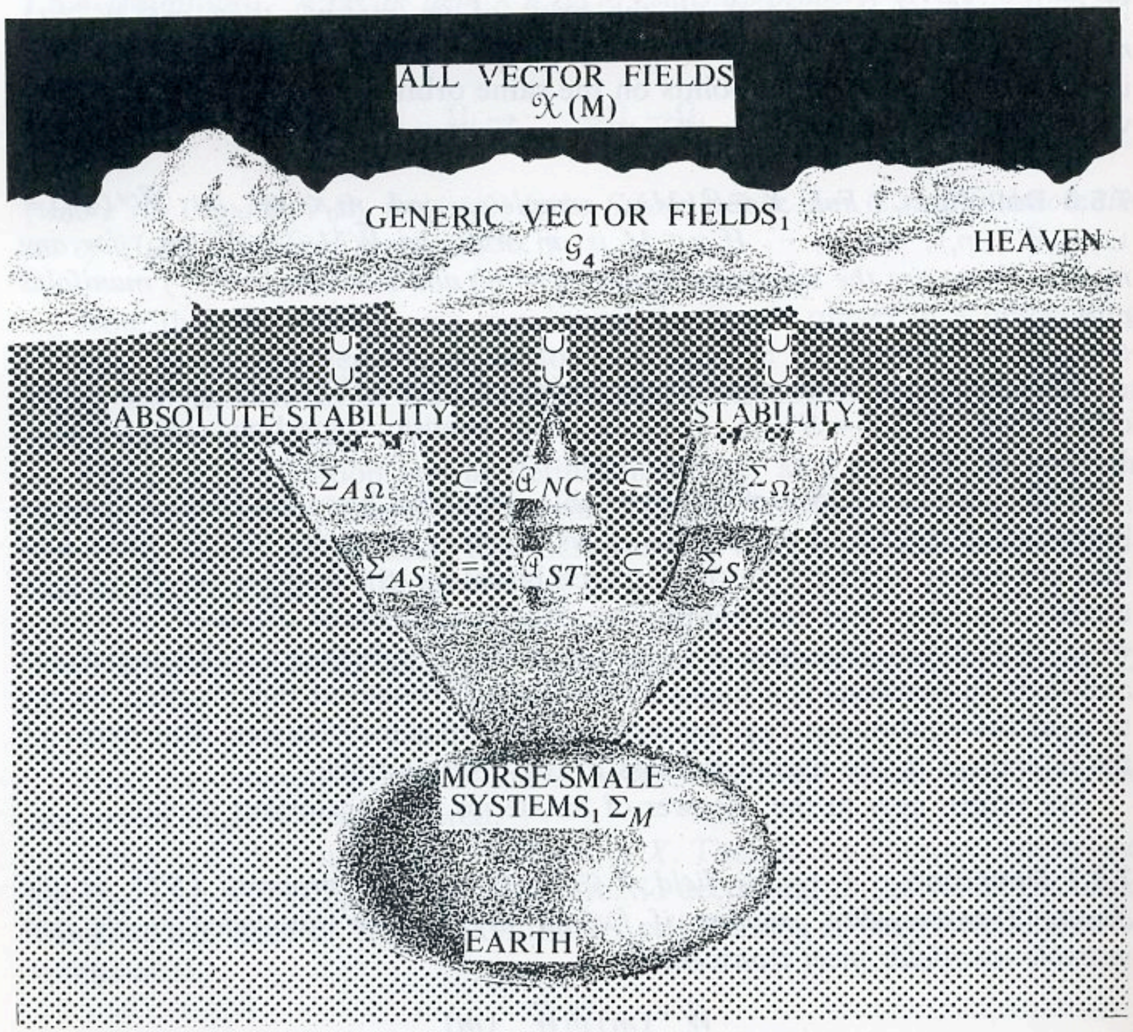


Figure 7.5-1. The three towers of differentiable dynamics.

The DDS dream of structural stability (from *Abraham & Marsden*, 1978)

Non-autonomous Dynamical Systems

A linear, dissipative, forced example: *forward vs. pullback attraction*

Consider the scalar, linear ordinary differential equation (ODE)

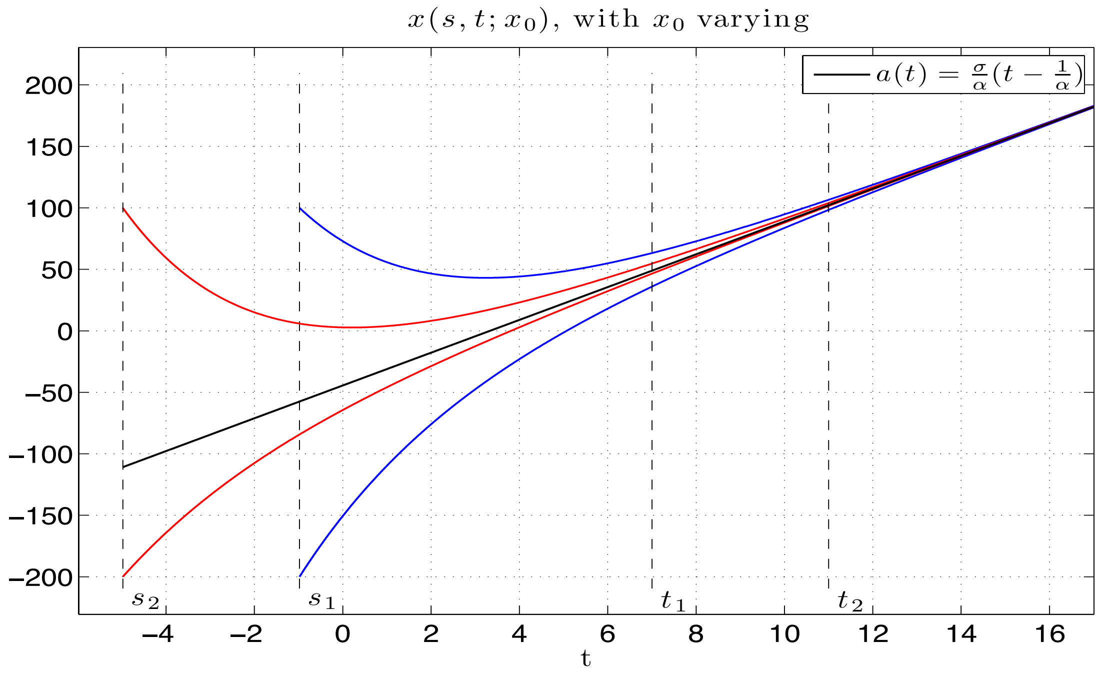
$$\dot{x} = -\alpha x + \sigma t\,, \quad \alpha > 0\,,\ \sigma > 0\,.$$

The autonomous part of this ODE, $\dot{x} = -\alpha x$, is dissipative and all solutions $x(t; x_0) = x(t; x(0) = x_0)$ converge to 0 as $t \to +\infty$.

What about the non-autonomous, forced ODE? As the energy being put into the system by the forcing is dissipated, we expect things to change in time. In fact, if we "pull back" far enough, replace $x(t; x_0)$ by $x(s, t; x_0) = x(s, t; x(s) = x_0)$,

and let $s \to -\infty$, we get th pullback attractor $a = a(t)$ in the figure,

$$a(t) = \frac{\sigma}{\alpha}(t - \frac{1}{\alpha})\,.$$



Non-autonomous Dynamical Systems - II

Remarks

- We've just shown that:

$$|x(t, s; x_0) - a(t)| \underset{s \to -\infty}{\longrightarrow} 0 \text{ ; for every } t \text{ fixed,}$$

 and for all initial data x_0, with $a(t) = \frac{\sigma}{\alpha}(t - 1/\alpha)$.

- We've just encountered the concept of pullback attraction; here $\{a(t)\}$ is the pullback attractor of the system (1).
- What does it mean physically?
 The pullback attractor provides a way to assess an asymptotic regime at time t — the time at which we observe the system — for a system starting to evolve from the remote past s, $s << t$.
- This asymptotic regime evolves with time: it is a dynamical object.
- Dissipation now leads to a dynamic object rather than to a static one, like the strange attractor of an autonomous system.

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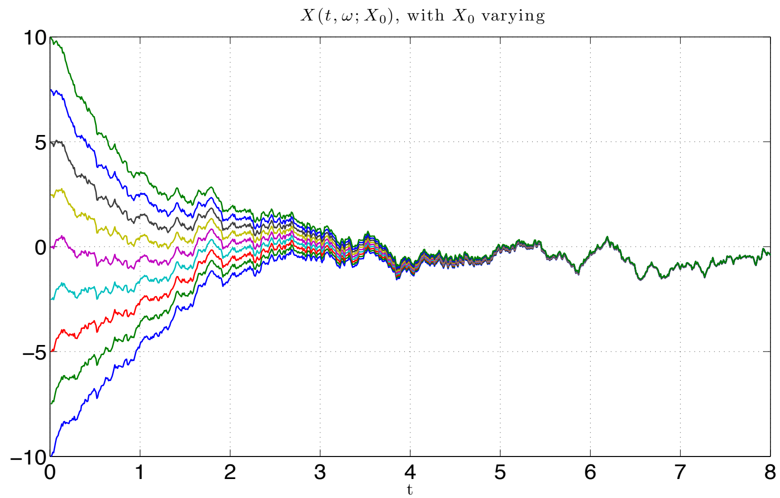
A little history of climate & stochasticity

- A. Einstein's (1905) Brownian motion paper.
- K. Itō (prof. at Kyoto U., RIMS director) formulates Itō calculus in 1942, enables solution of stochastic differential equations (SDEs); Itō's lemma is the stochastic counterpart of Leibniz's chain rule for differentiation.
- K. Hasselmann (*Tellus*, 1976) describes climate as Brownian motion, with weather the stochastic driver.
- In this view, the deterministic part of the model is stable, and random perturbations decay to the mean.

Kiyoshi Itō



Auto-regressive (AR) decay

Random Dynamical Systems (RDS), I - RDS theory

This theory is the counterpart for randomly forced dynamical systems (RDS) of the *geometric theory* of ordinary differential equations (ODEs). It allows one to treat stochastic differential equations (SDEs) — and more general systems driven by noise — as flows in (phase space)$\times$(probability space).

SDE$\sim$ODE, RDS$\sim$DDS, L. Arnold (1998)$\sim$V.I. Arnol'd (1983).

Setting:

(i) A phase space X. **Example**: $\mathbb{R}^n$.

(ii) A probability space $(\Omega, \mathcal{F}, \mathbb{P})$. **Example**: The Wiener space $\Omega = C_0(\mathbb{R}; \mathbb{R}^n)$ with Wiener measure $\mathbb{P}$.

(iii) A model of the noise $\theta(t) : \Omega \to \Omega$ that preserves the measure $\mathbb{P}$, i.e. $\theta(t)\mathbb{P} = \mathbb{P}$; θ is called the driving system.
Example: $W(t, \theta(s)\omega) = W(t + s, \omega) - W(s, \omega)$;
it starts the noise at s instead of $t = 0$.

(iv) A mapping $\varphi : \mathbb{R} \times \Omega \times X \to X$ with the cocycle property.
Example: The solution operator of an SDE.

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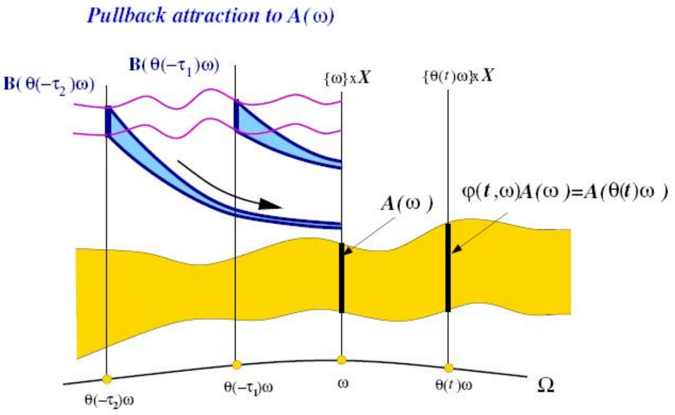
Example: The solution operator of an SDE.

A random attractor $\mathcal{A}(\omega)$ is both *invariant* and "pullback" *attracting*:

(a) **Invariant**: $\varphi(t,\omega)\mathcal{A}(\omega) = \mathcal{A}(\theta(t)\omega)$.

(b) **Attracting**: $\forall B \subset X$, $\lim_{t\to\infty} \mathrm{dist}(\varphi(t,\theta(-t)\omega)B, \mathcal{A}(\omega)) = 0$ a.s.



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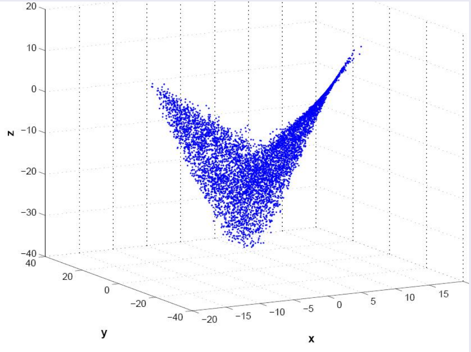
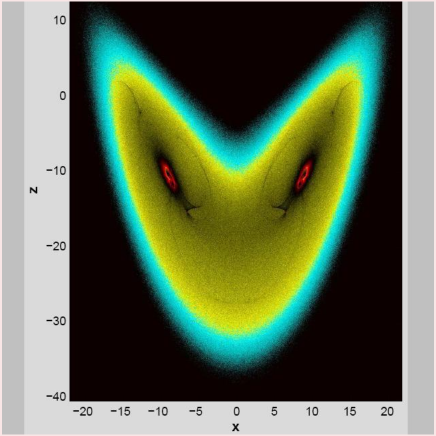
Random attractor of the stochastic Lorenz system

Snapshot of the random attractor (RA)



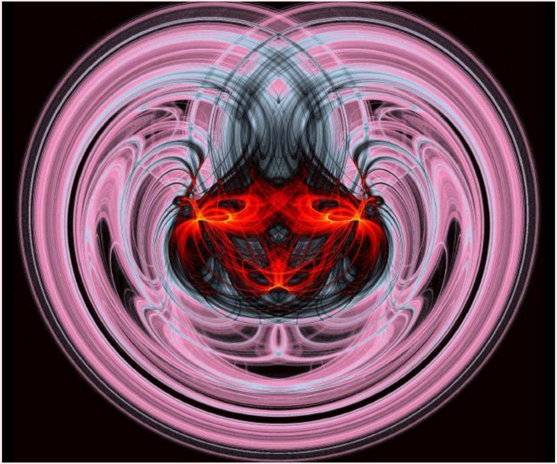
- A snapshot of the RA, $\mathcal{A}(\omega)$, computed at a fixed time t and for the same realization ω; it is made up of points transported by the stochastic flow, from the remote past $t - T$, $T >> 1$.
- We use multiplicative noise in the deterministic Lorenz model, with the classical parameter values $b = 8/3$, $\sigma = 10$, and $r = 28$.
- Even computed pathwise, this object supports meaningful statistics.

Sample measures supported by the R.A.



- We compute the probability measure on the R.A. at some fixed time t, and for a fixed realization ω. We show a "projection", $\int \mu_\omega(x, y, z)\mathrm{d}y$, with multiplicative noise: $\mathrm{d}x_i$=Lorenz$(x_1, x_2, x_3)\mathrm{d}t + \alpha\ x_i \mathrm{d}W_t; i \in \{1, 2, 3\}$.
- 10 million of initial points have been used for this picture!

Sample measure supported by the R.A.



- Still 1 Billion I.D., and $\alpha = 0.5$. Another one?

Sample measure supported by the R.A.

Sample measures evolve with time.

- Recall that these sample measures are the frozen statistics at a time t for a realization ω.
- How do these frozen statistics evolve with time?
- Action!

A day in the life of the Lorenz (1963) model's random attractor, or LORA for short; see SI in Chekroun, Simonnet & Ghil (2011, *Physica D*)

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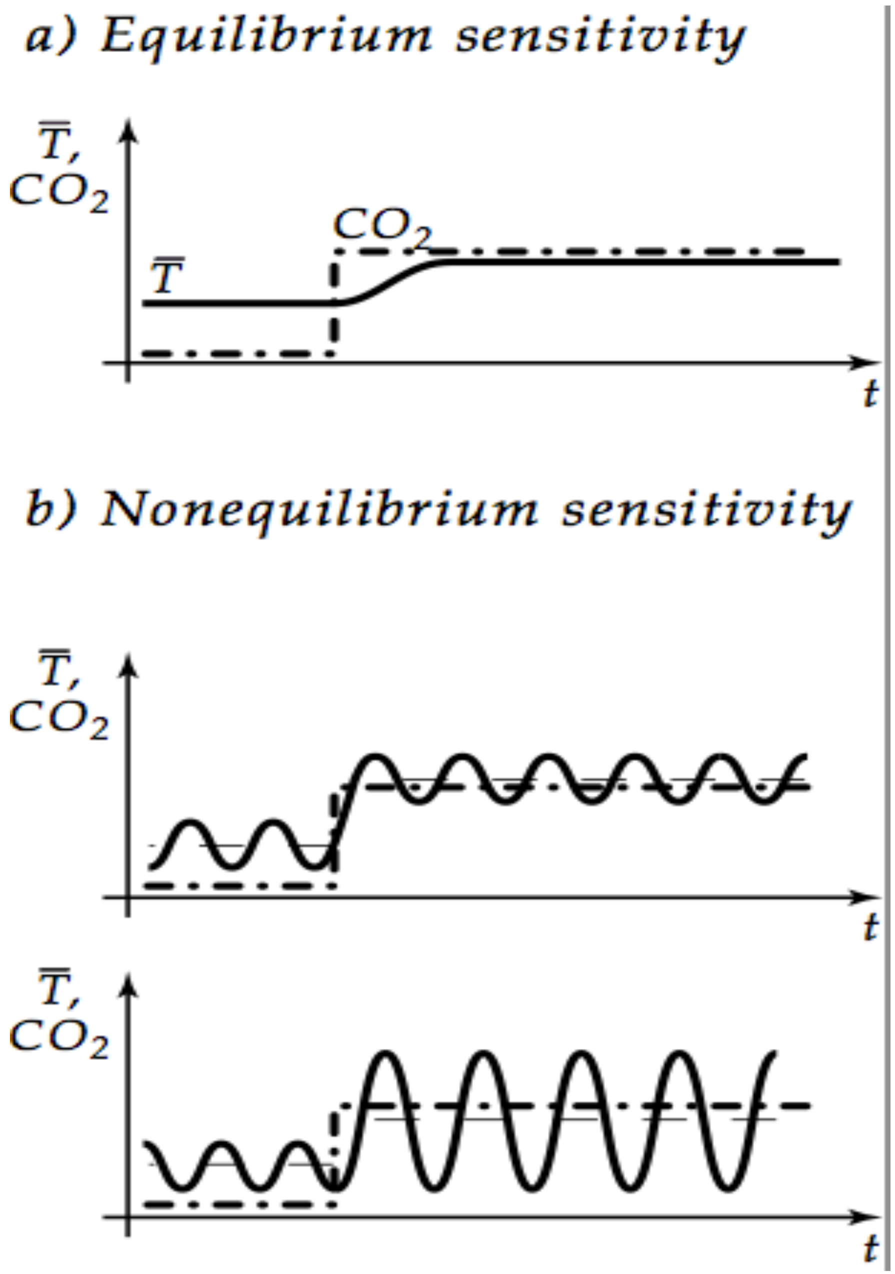
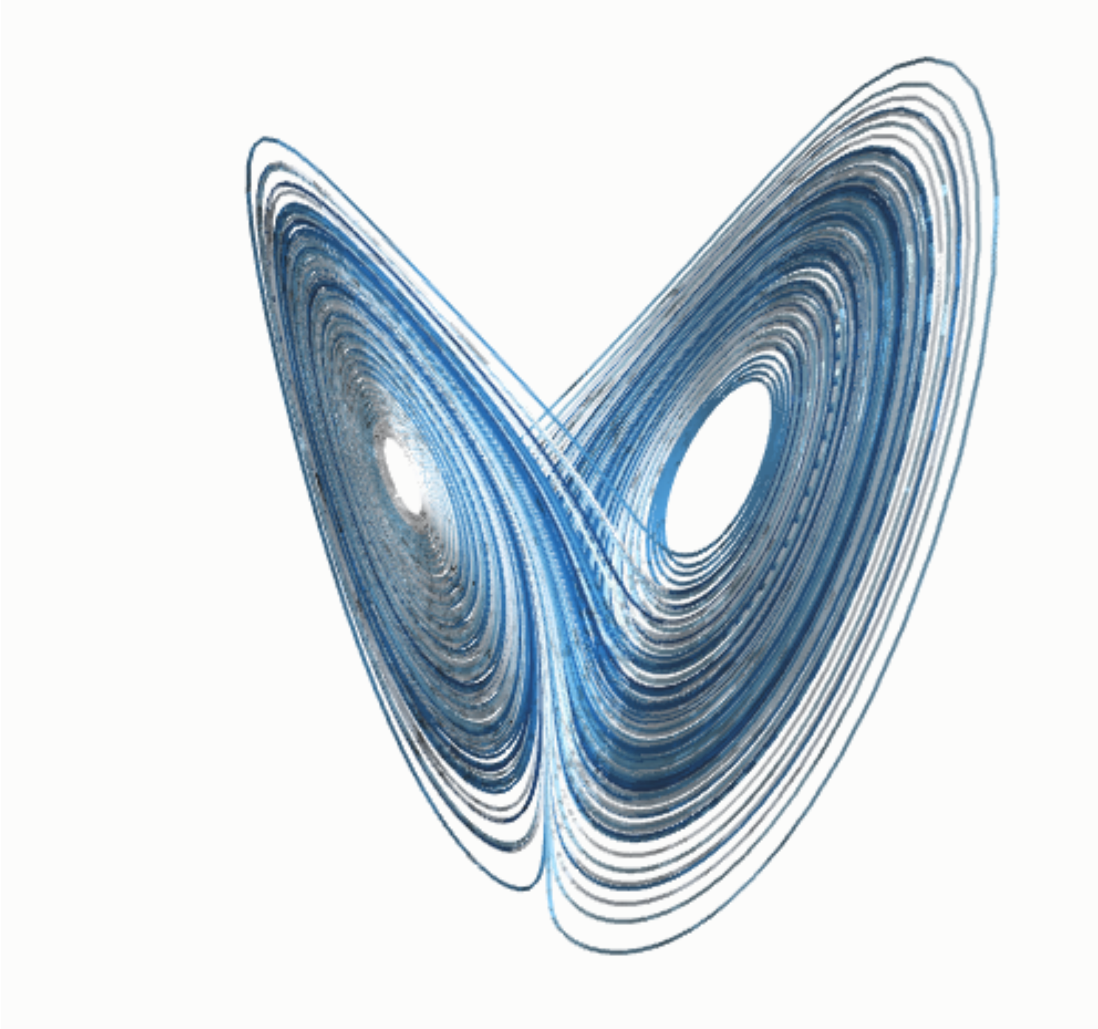


Classical Strange Attractor

Physically closed system, modeled mathematically as autonomous system: neither deterministic (anthropogenic) nor random (natural) forcing.

The attractor is strange, but still fixed in time ~ “irrational” number.

Climate sensitivity ~ change in the average value (first moment) of the coordinates (x, y, z) as a parameter λ changes.



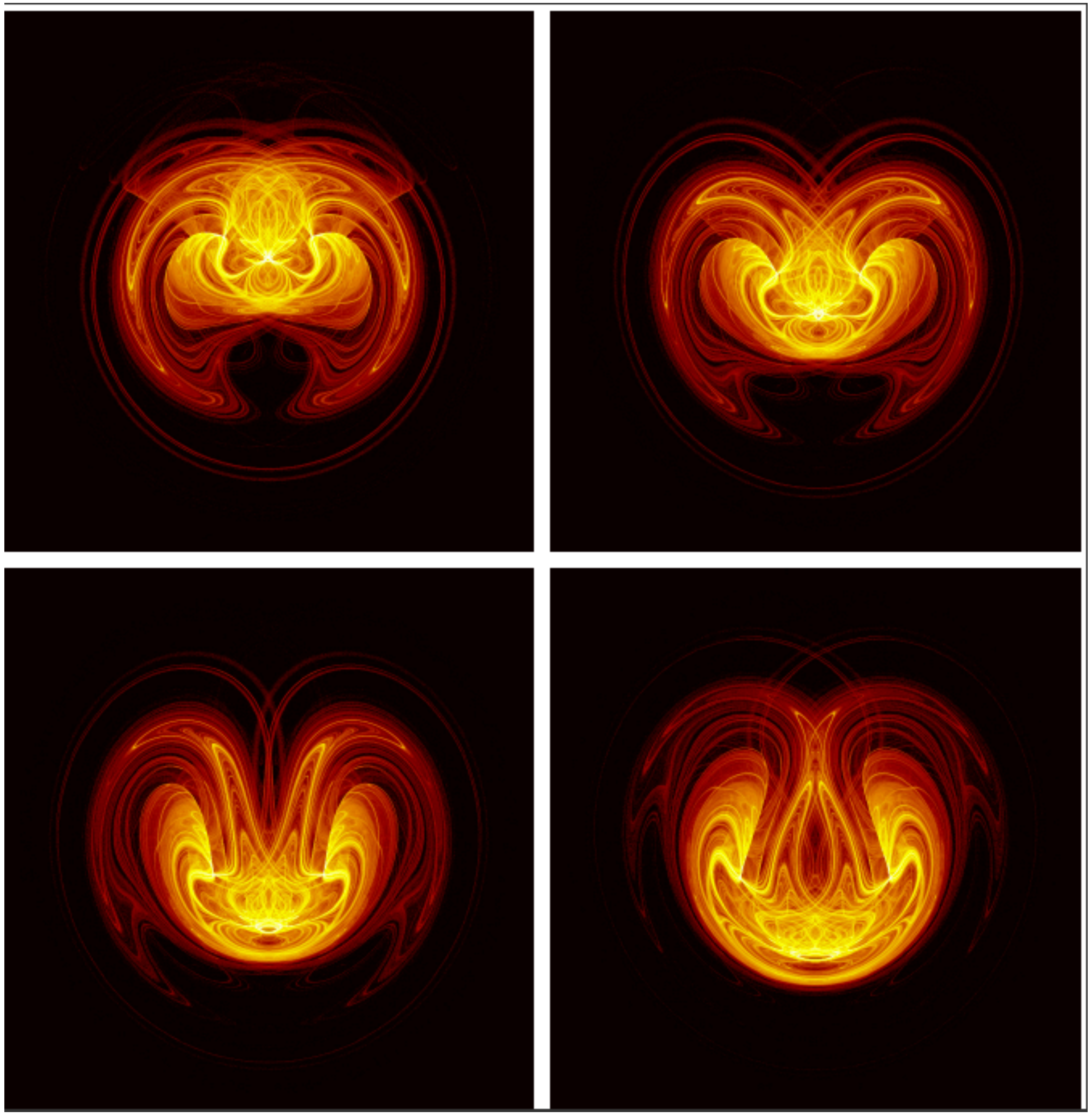
Random Attractor

Physically open system, modeled mathematically as non-autonomous system: allows for deterministic (anthropogenic) as well as random (natural) forcing.

The attractor is "pullback" and evolves in time ~ "imaginary" or "complex" number.

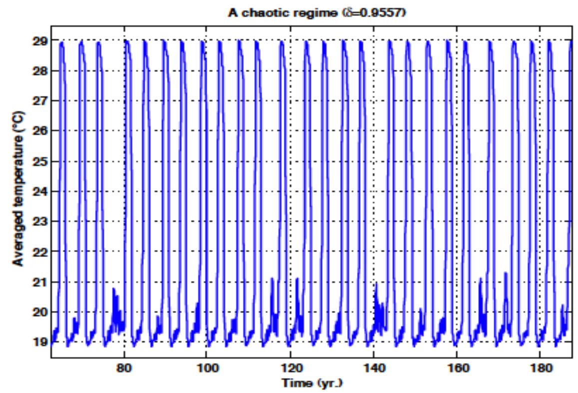
Climate sensitivity ~ change in the statistical properties (first and higher-order moments) of the attractor as one or more parameters (λ, μ, …) change.

Ghil (*Encyclopedia of Atmospheric Sciences*, 2nd ed., 2012)



Parameter dependence – I

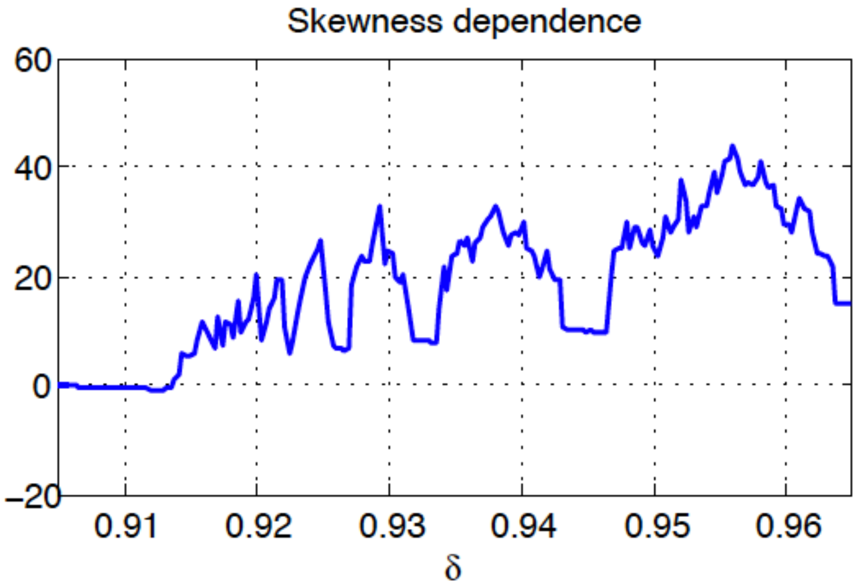
$\delta = 0.9557$

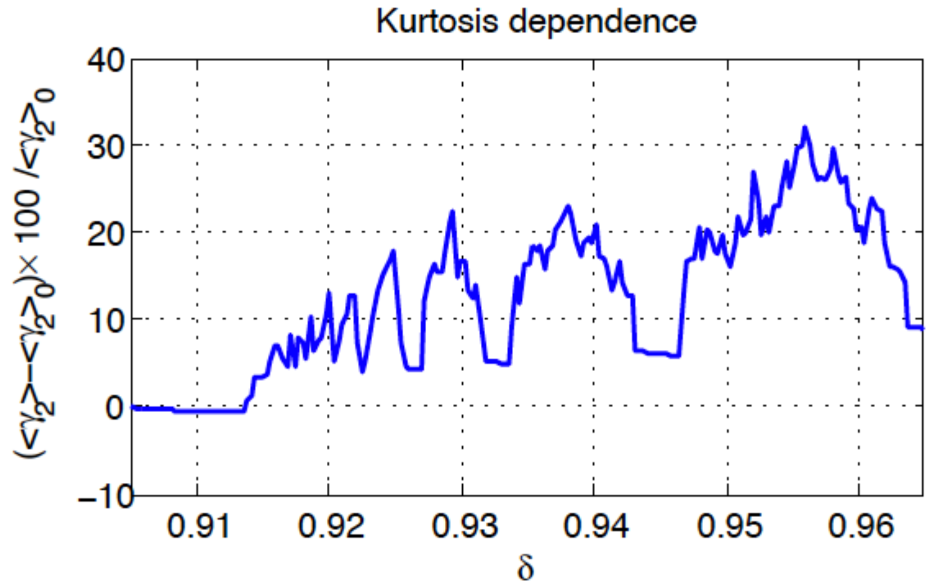


It can be smooth or it can be rough:
Niño-3 SSTs from intermediate coupled model for ENSO (Jin, Neelin & Ghil, 1994, 1996)

Skewness & kurtosis of the SSTs:
time series of 4000 years,

$$\Delta\delta = 3 \cdot 10^{-4}$$





M. Chekroun & D. Kondrashov (work in progress)

Sample measures for an NDDE model of ENSO

The Galanti-Tziperman (GT) model (*JAS*, 1999)

$$\frac{dT}{dt} = -\epsilon_T T(t) - M_0(T(t) - T_{sub}(h(t))),$$

$$h(t) = M_1 e^{-\epsilon_m(\tau_1+\tau_2)} h(t-\tau_1-\tau_2) - M_2\tau_1 e^{-\epsilon_m(\frac{\tau_1}{2}+\tau_2)}\mu(t-\tau_2-\frac{\tau_1}{2})T(t-\tau_2-\frac{\tau_1}{2}) + M_3\tau_2 e^{-\epsilon_m\frac{\tau_2}{2}}\mu(t-\frac{\tau_2}{2})T(t-\frac{\tau_2}{2}).$$

Neutral delay-differential equation (NDDE), derived from Cane-Zebiak and Jin-Neelin models for ENSO: *T* is East-basin SST and *h* is thermocline depth.

Seasonal forcing given by $\mu(t) = 1 + \epsilon\cos(\omega t + \phi)$. The pullback attractor and its invariant measures were computed.

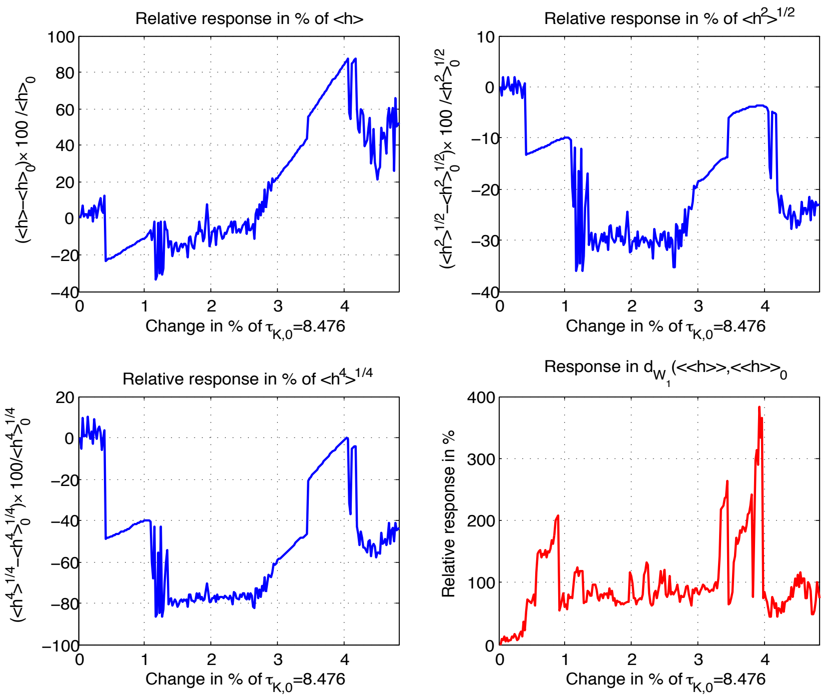


Figure shows the changes in the mean, 2nd & 4th moment of $h(t)$, along with the Wasserstein distance d_W, for changes of 0–5% in the delay parameter $\tau_{\kappa,0}$.

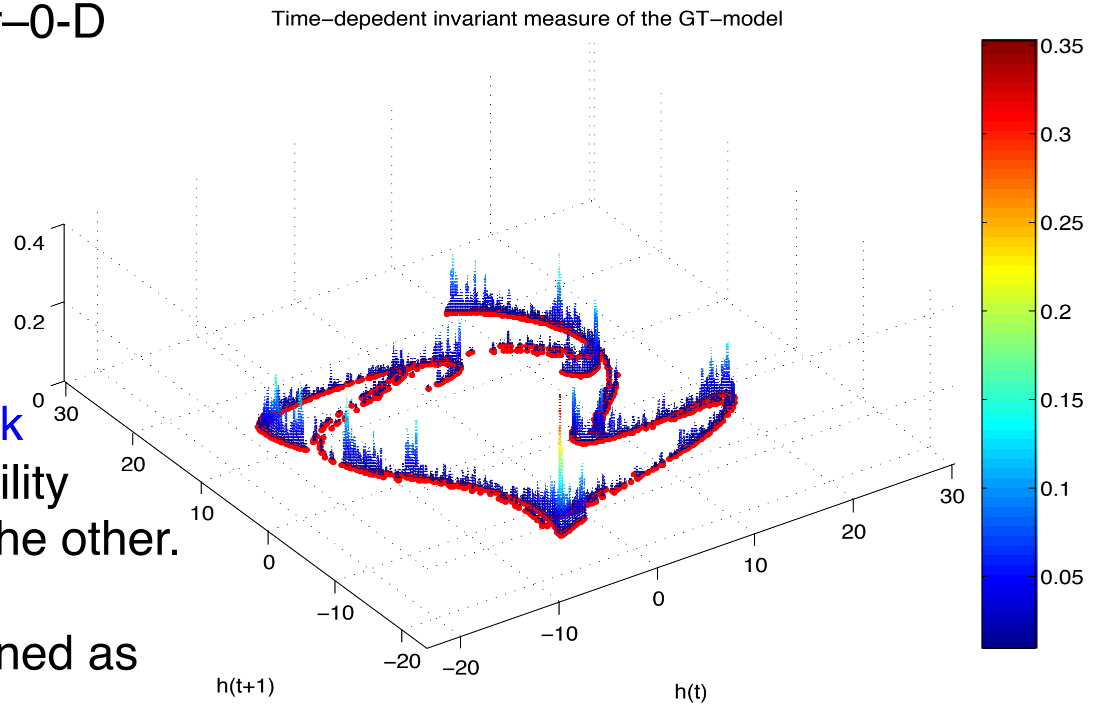
Note intervals of both smooth & rough dependence!

Pullback attractor and invariant measure of the GT model

The time-dependent pullback attractor of the GT model supports an invariant measure $\nu = \nu(t)$, whose density is plotted in 3-D perspective.

The plot is in delay coordinates *h*(*t*+1) *vs.* *h*(*t*) and the density is highly concentrated along 1-D filaments and, furthermore, exhibits sharp, near–0-D peaks on these filaments.

The Wasserstein distance d_W between one such configuration, at given parameter values, and another one, at a different set of values, is proportional to the work needed to move the total probability mass from one configuration to the other.

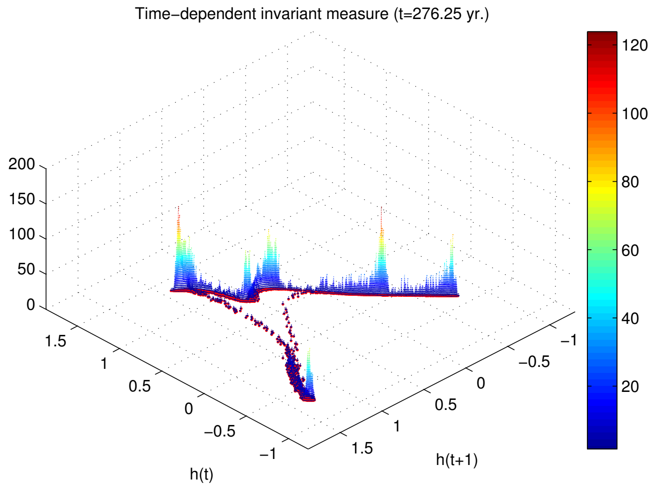


Climate sensitivity γ can be defined as

$$\gamma = \partial d_{\mathrm{W}} / \partial \tau$$

How to define climate sensitivity or, What happens when there's natural variability?

This definition allows us to watch how "the earth moves," as it is affected by both natural and anthropogenic forcing, in the presence of natural variability, which includes both chaotic & random behavior:



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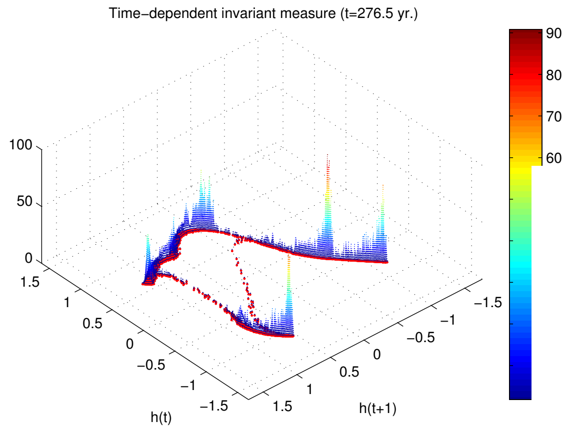
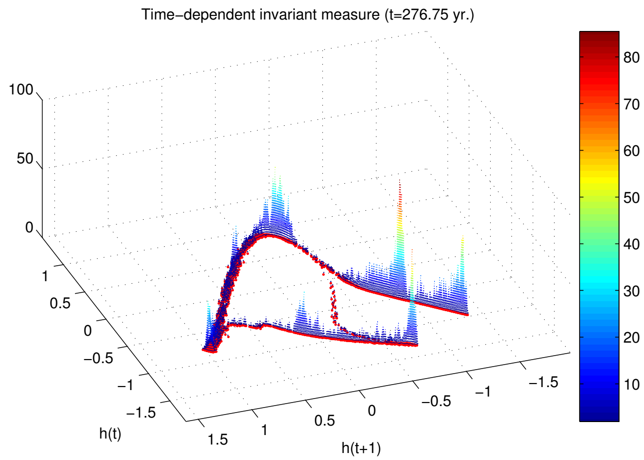
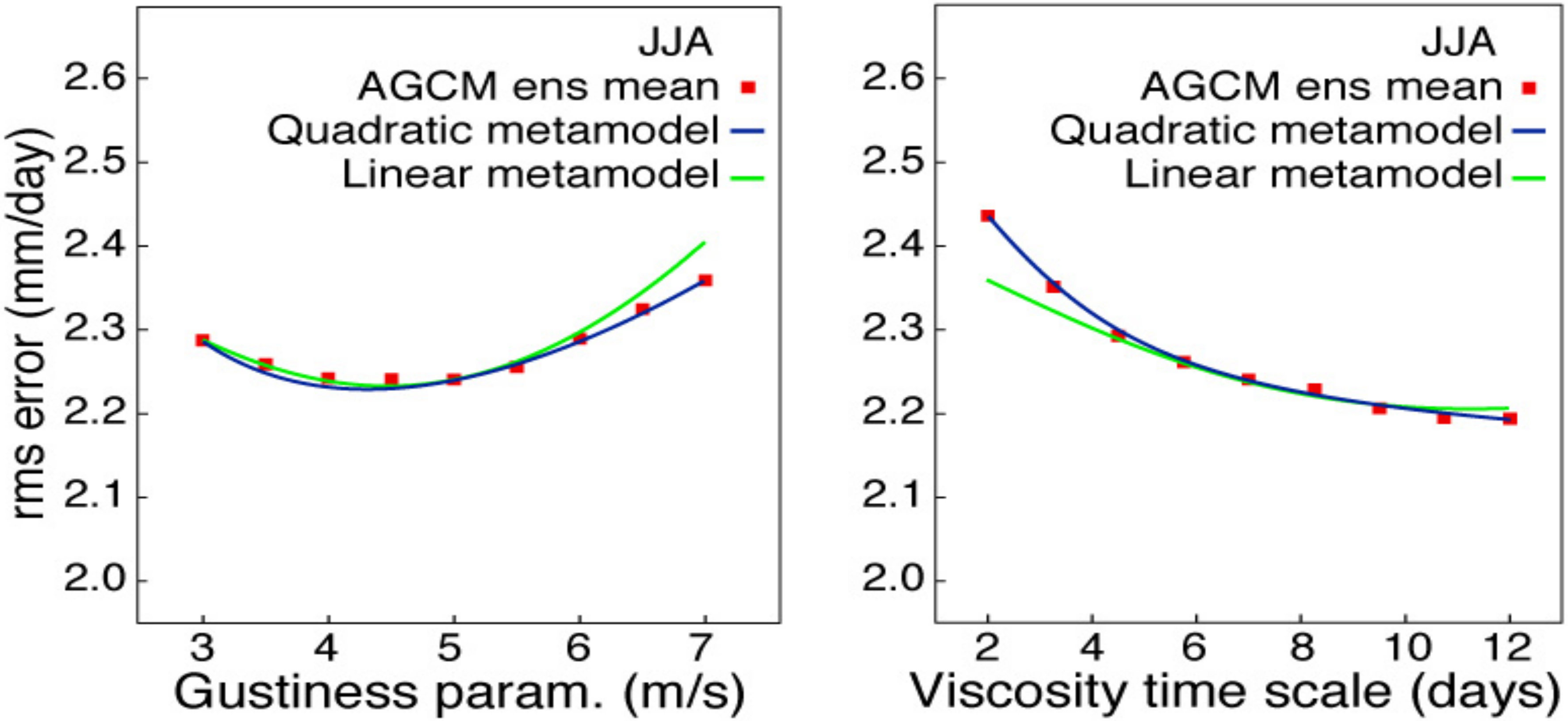




Parameter dependence – II

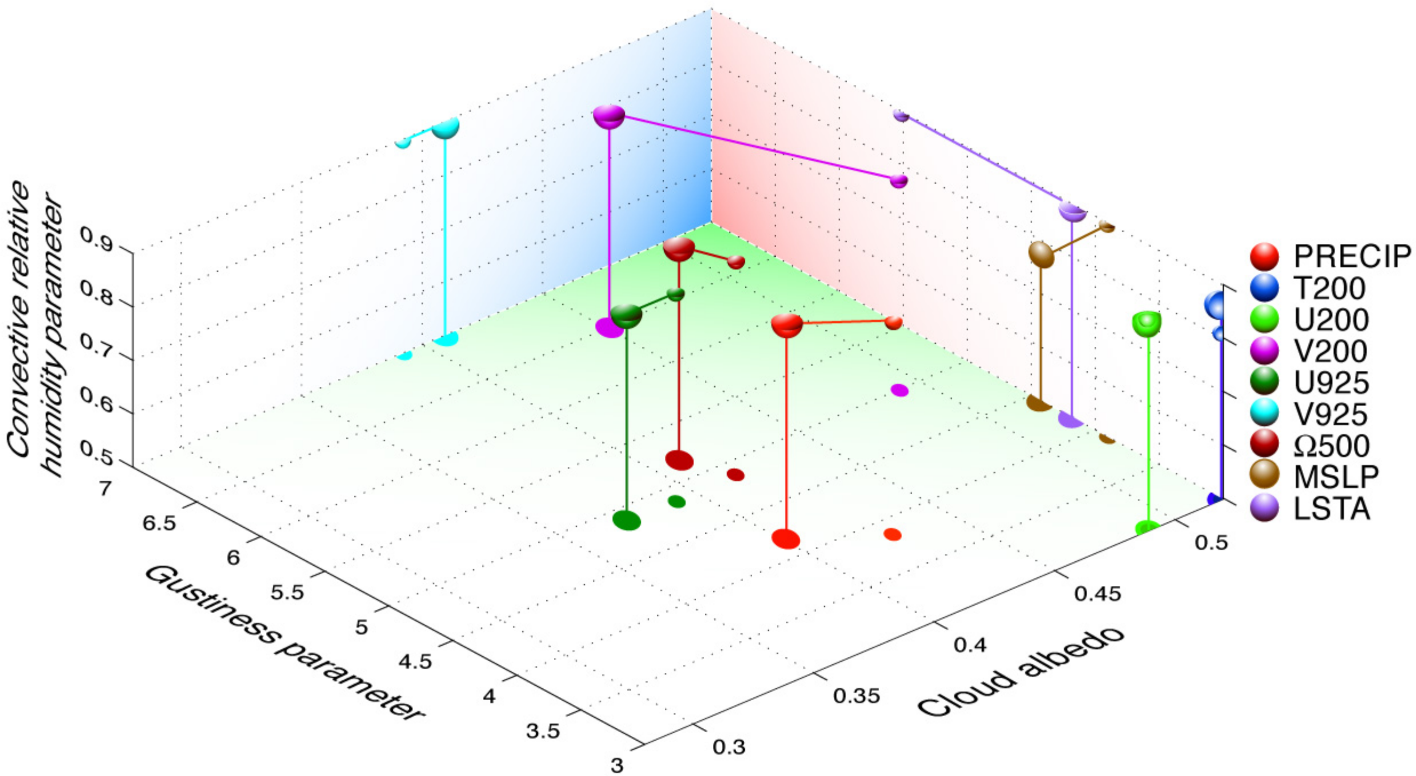
When it is smooth, one can optimize a GCM's single-parameter dependence



ICTP AGCM (Neelin, Bracco, Luo, McWilliams & Meyerson, *PNAS*, 2010)

Parameter dependence – III

Multi-objective algorithms avoid arbitrary weighting of criteria
in a unique cost function:



Optimization algorithms that are $\mathcal{O}(N)$ and $\mathcal{O}(N^2)$, rather than $\mathcal{O}(S^N)$,
where N is the number of parameters and S is the sampling density.

ICTP AGCM (Neelin, Bracco, Luo, McWilliams & Meyerson, *PNAS*, 2010)

Outline

- The IPCC process: results and uncertainties
- Natural climate variability as a source of uncertainties
  - sensitivity to initial data ➔ error growth
  - sensitivity to model formulation ➔ see below!
- Uncertainties and how to fix them
  - structural stability and other kinds of robustness
  - non-autonomous and random dynamical systems (NDDS & RDS)
- Two illustrative examples
  - the Lorenz convection model
  - an El Niño–Southern Oscillation (ENSO) model
- Linear response theory and climate sensitivity
- Conclusions and references
  - natural variability and anthropogenic forcing: the "grand unification"
  - selected bibliography

Concluding remarks, I – RDS and RAs

Summary

- A change of paradigm from **closed, autonomous systems** to **open, non-autonomous ones.**
- Random attractors are (i) spectacular, (ii) useful, and (iii) just starting to be explored for climate applications.

Work in progress

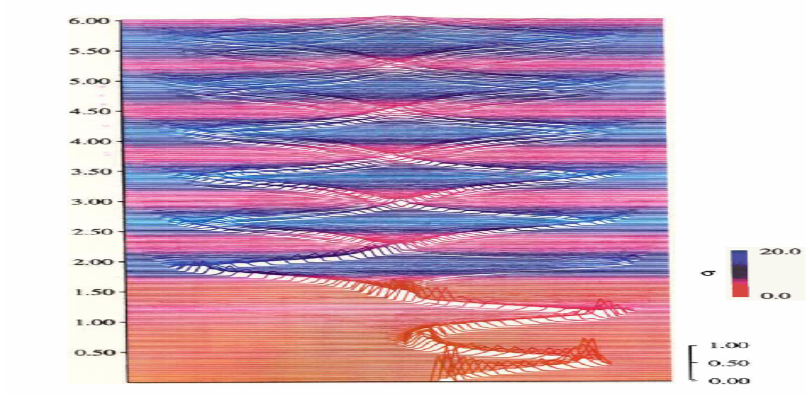
- Study the effect of specific stochastic parametrizations on model robustness.
- Applications to intermediate models and GCMs.
- Implications for climate sensitivity.
- Implications for predictability?

Yet another (grand?) unification

Lorenz (*JAS*, 1963)

Climate is deterministic and autonomous,
    but highly nonlinear.

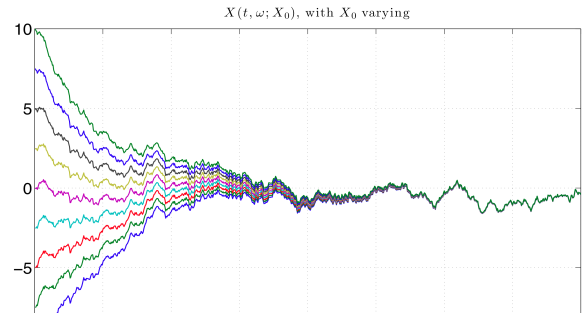
Trajectories diverge exponentially,
    forward asymptotic PDF is multimodal.



Hasselmann (*Tellus*, 1976)

Climate is stochastic and noise-driven,
    but quite linear.

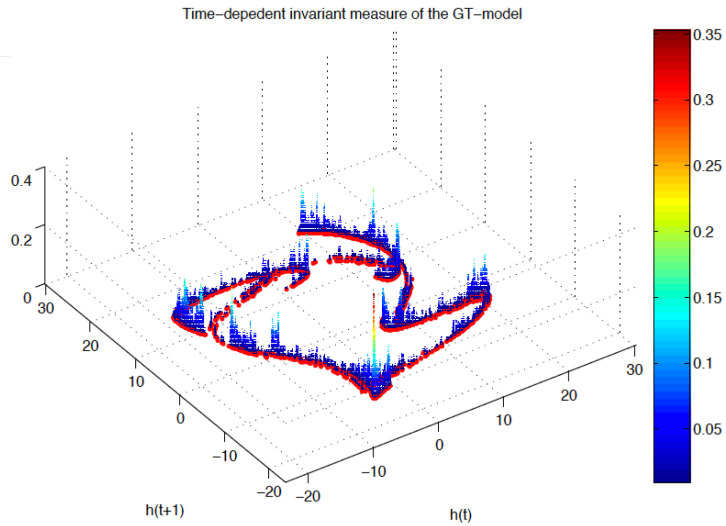
Trajectories decay back to the mean,
    forward asymptotic PDF is unimodal.



Grand unification (?)

**Climate is deterministic + stochastic,
    as well as highly nonlinear.**

**Internal variability and forcing interact
    strongly, change and sensitivity
    refer to both mean and higher moments.**



The Lorenz (1963a) convection model

Problem 2: Find the appropriate software to compute the
Lorenz “butterfly” and use it to do so.

Problem 4: Find the appropriate software to compute the statistics
of the Lorenz “butterfly” – e.g., pdf, EOFs – and use it to do so.

Glossary
pdf = probability density function
EOF = empirical orthogonal function

Problem 8: Add some noise to the Lorenz convection model and compute:

a) some sample solutions;
b) the invariant measure (more precisely, an approximate pdf);
c) the random attractor; and
d) its sensitivity to parameter changes.

Concluding remarks, II –
Climate change & climate sensitivity

What do we know?

- It's getting warmer.
- We do contribute to it.
- So we should act as best we know and can!

What do we know less well?

- By how much?
  - Is it getting warmer …
  - Do we contribute to it …
- How does the climate system (atmosphere, ocean, ice, etc.) really work?
- How does natural variability interact with anthropogenic forcing?

What to do?

- Better understand the system and its forcings.
- Explore the models', and the system's, robustness and sensitivity
  - stochastic structural and statistical stability!
  - linear response = response function + susceptibility function!!

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Reserve slides

Galileo on math, science & opinions, 1

La filosofia è scritta in questo grandissimo libro che continuamente ci sta aperto innanzi a gli occhi (io dico l'universo), ma non si può intendere se prima non s'impara a intender la lingua, e conoscer i caratteri, ne' quali è scritto. Egli è scritto in lingua matematica, e i caratteri son triangoli, cerchi ed altre figure geometriche, senza i quali mezzi è impossibile a intenderne umanamente parola; senza questi è un aggirarsi vanamente per un oscuro laberinto.

G. Galilei, *Il Saggiatore*, VI, 232)

Philosophy is written in this grand book — I mean the Universe—which stands continually open to our gaze, but it cannot be understood unless one first learns to comprehend the language and interpret the characters in which it is written. It is written in the language of mathematics, and its characters are triangles, circles and other geometrical figures, without which it is humanly impossible to understand a single word of it.

nature 28 March 1991

letters to nature

Nature **350**, 324 - 327 (1991); doi:10.1038/350324a0

Interdecadal oscillations and the warming trend in global temperature time series

M. Ghil & R. Vautard

THE ability to distinguish a warming trend from natural variability is critical for an understanding of the climatic response to increasing greenhouse-gas concentrations. Here we use singular spectrum analysis[1] to analyse the time series of global surface air tem-peratures for the past 135 years[2], allowing a secular warming trend and a small number of oscillatory modes to be separated from the noise. The trend is flat until 1910, with an increase of 0.4 °C since then. The oscillations exhibit interdecadal periods of 21 and 16 years, and interannual periods of 6 and 5 years. The interannual oscillations are probably related to global aspects of the El Niño-Southern Oscillation (ENSO) phenomenon[3]. The interdecadal oscillations could be associated with changes in the extratropical ocean circulation[4]. The oscillatory components have combined (peak-to-peak) amplitudes of 0.2 °C, and therefore limit our ability to predict whether the inferred secular warming trend of 0.005 °Cyr^{-1} will continue. This could postpone incontrovertible detection of the greenhouse warming signal for one or two decades.

Galileo on math, science & opinions, 2

Sì perché l'autorità dell'opinione di mille nelle scienze non val per una scintilla di ragione di un solo.

In questions of science, the authority of a thousand is not worth the humble reasoning of a single individual.

Galileo Galilei, *Venere, Luna e Pianeti Medicei,*
e nuove apparenze di Saturno, p. 8/20

Applications to a nonlinear stochastic El Niño model

Chekroun, Simonnet and Ghil, 2011

Timmerman & Jin (*Geophys. Res. Lett.*, 2002) have derived the following low-order, tropical-atmosphere–ocean model. The model has three variables: thermocline depth anomaly h, and
SSTs T_1 and T_2 in the western and eastern basin.

$$\begin{aligned}
\dot{T}_1 &= -\alpha(T_1 - T_r) - \frac{2\varepsilon u}{L}(T_2 - T_1), \\
\dot{T}_2 &= -\alpha(T_2 - T_r) - \frac{w}{H_m}(T_2 - T_{sub}), \\
\dot{h} &= r(-h - bL\tau/2).
\end{aligned}$$

The related diagnostic equations are:

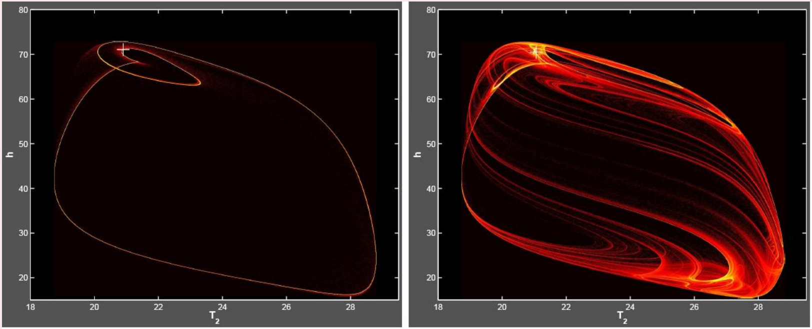
$$\begin{aligned}
T_{sub} &= T_r - \frac{Tr - T_{r0}}{2}[1 - \tanh(H + h_2 - z_0)/h^*] \\
\tau &= \frac{a}{\beta}(T_1 - T_2)[\xi_t - 1].
\end{aligned}$$

- τ: the wind stress anomalies, $w = -\beta\tau/H_m$: the equatorial upwelling.
- $u = \beta L\tau/2$: the zonal advection, T_{sub}: the subsurface temperature.

Wind stress bursts are modeled as white noise ξ_t of variance σ,
and ε measures the strength of the zonal advection.

Random attractors: the frozen statistics

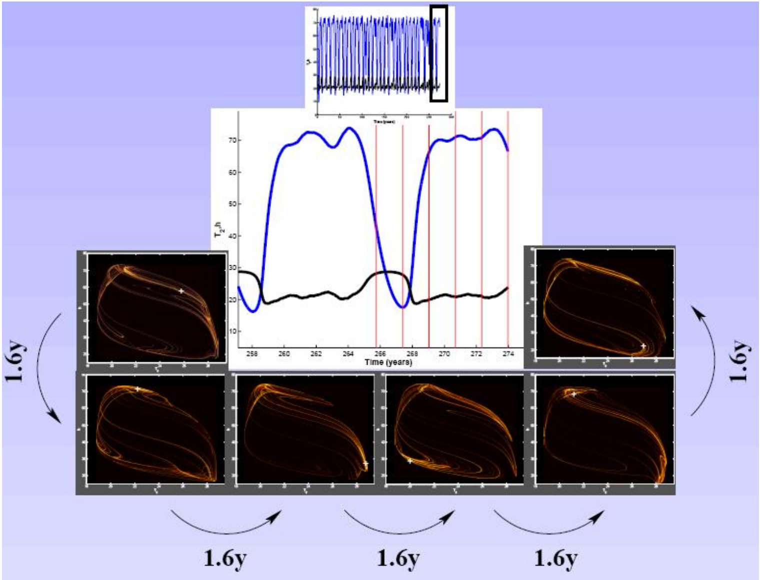
Random Shil'nikov horseshoes

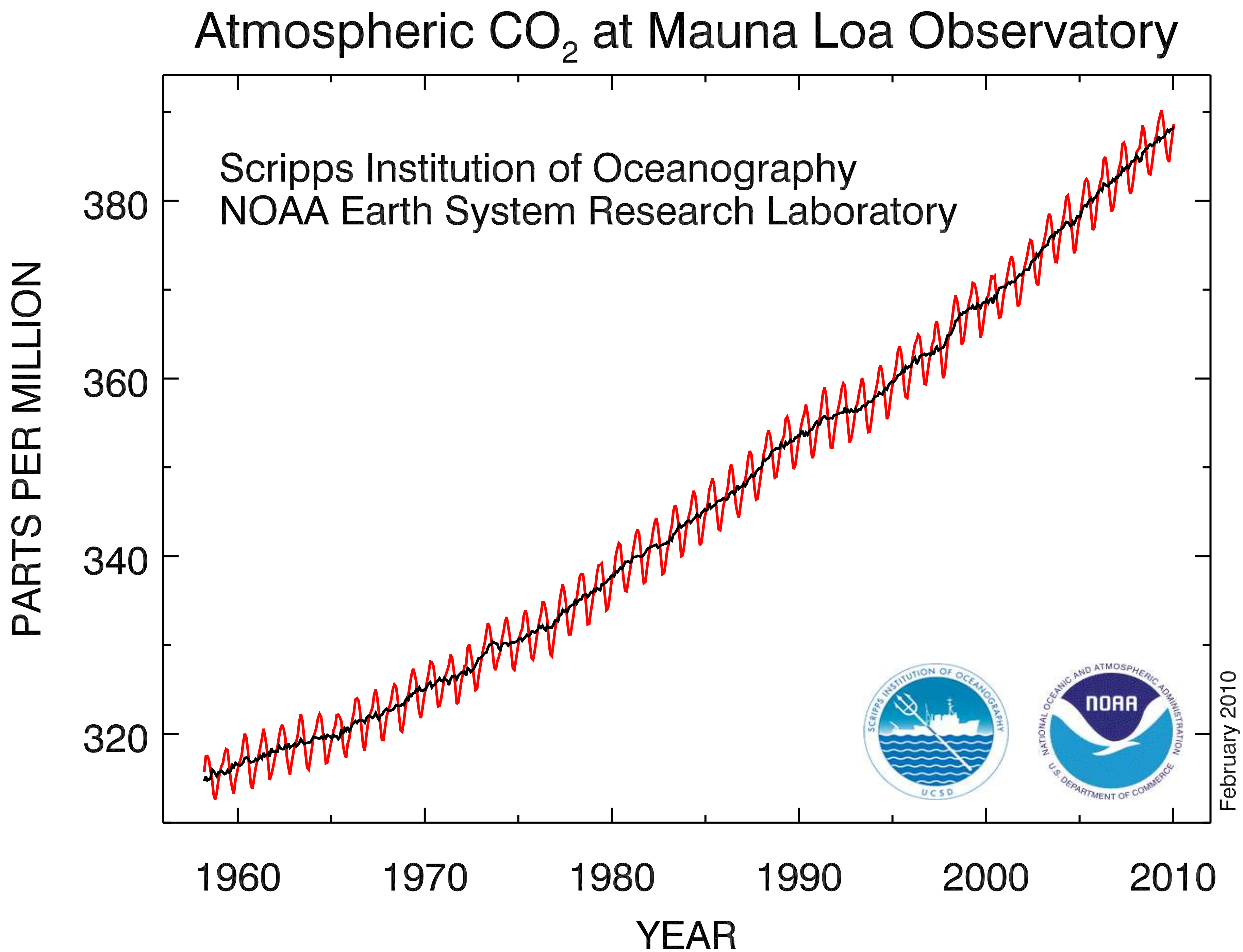


σ=0.005 σ=0.05

- Horseshoes can be noise-excited,
  left: a weakly-perturbed limit cycle, right: the same with larger noise.
- Golden: most frequently-visited areas; white 'plus' sign: most visited.

An episode in the random's attractor life



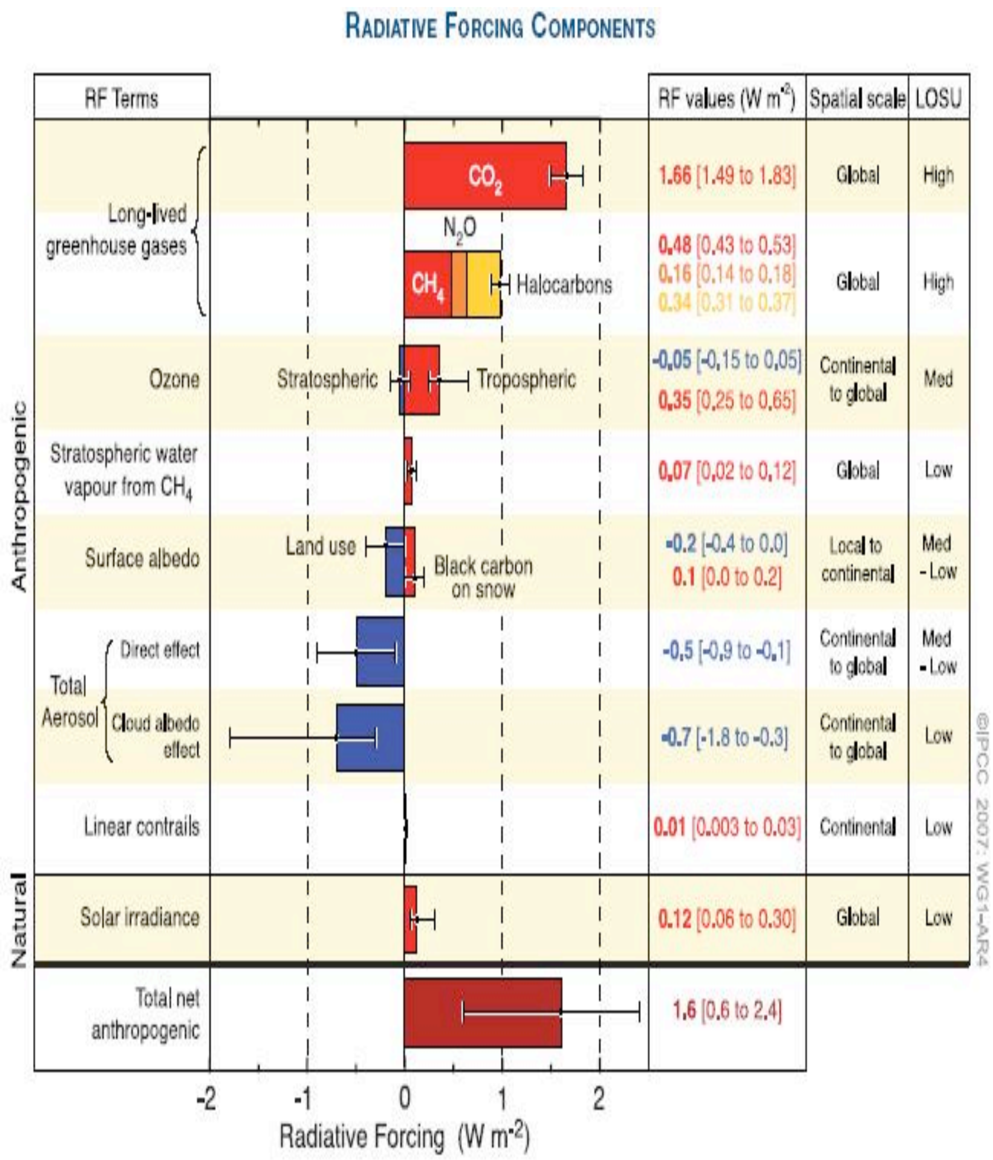


Atmospheric CO2 at Mauna Loa Observatory
Scripps Institution of Oceanography
NOAA Earth System Research Laboratory
PARTS PER MILLION
320
340
360
380
1960
1970
1980
1990
2000
2010
YEAR
February 2010

GHGs rise!

It's gotta do with us, at least a bit, ain't it?

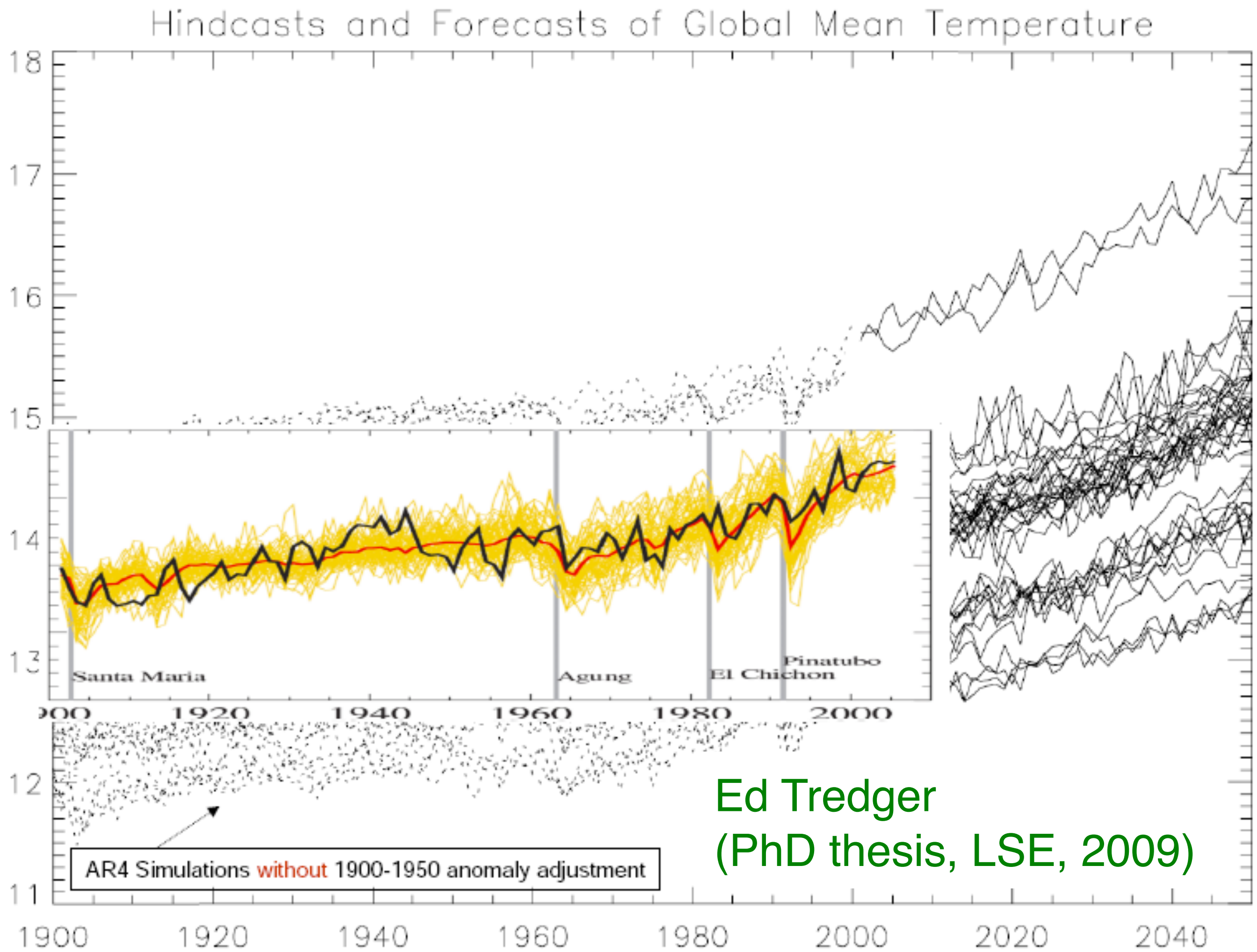
But just how much?



IPCC (2007)

www.lseca

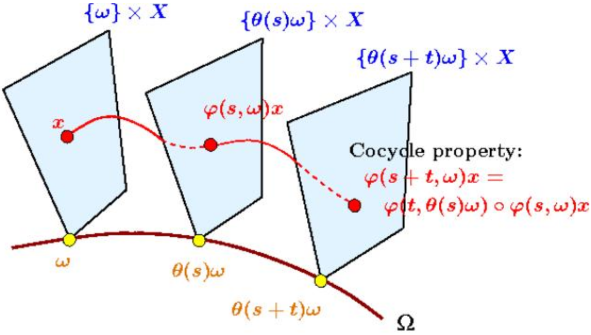
AR4 adjustment of 20th century simulation



Ed Tredger
(PhD thesis, LSE, 2009)

Grantham Research Institute on
Climate Change and
the Environment

L.A. ("Lenny") Smith (2009)
personal communication



- φ **is a random dynamical system (RDS)**
- $\Theta(t)(x,\omega) = (\theta(t)\omega, \varphi(t,\omega)x)$ **is a flow on the bundle**

A linear example as a paradigm

Let us first start with a very difficult problem:

$$\text{Study the “dynamics" of} \quad \dot{x} = -\alpha x + \sigma t, \quad \alpha, \sigma > 0. \tag{1}$$

First remarks:

- The system $\dot{x} = -\alpha x$, i.e. the autonomous part of (1), is dissipative. All the solutions of $\dot{x} = -\alpha x$ converge to 0 as $t \to +\infty$.
- Is it the case for (1)? Certainly not! The autonomous part is forced; we even introduce an infinite energy over an infinite time interval: $\int_0^{+\infty} t \, dt = +\infty$! Forward attraction seems to be ill adapted to time-dependent forcing.

Goal:

Find a concept of attraction that is:

(i) compatible with the forward concept, when there is no forcing; and

(ii) provides a way to assess the effect of dissipation in some sense.

For that let's do some computations...

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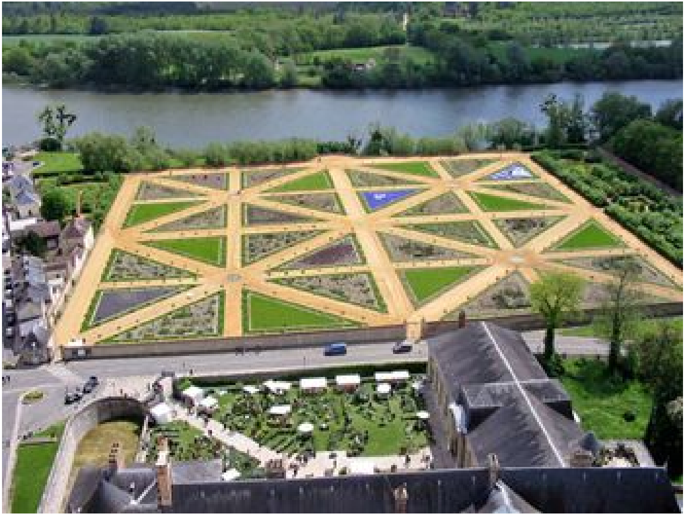
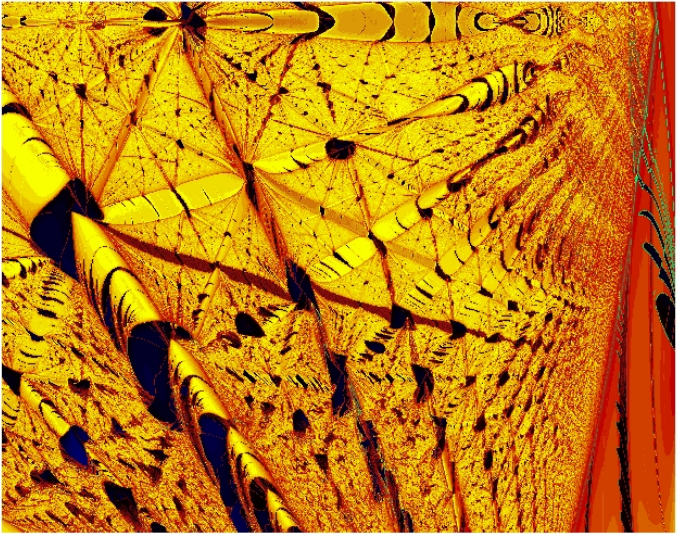
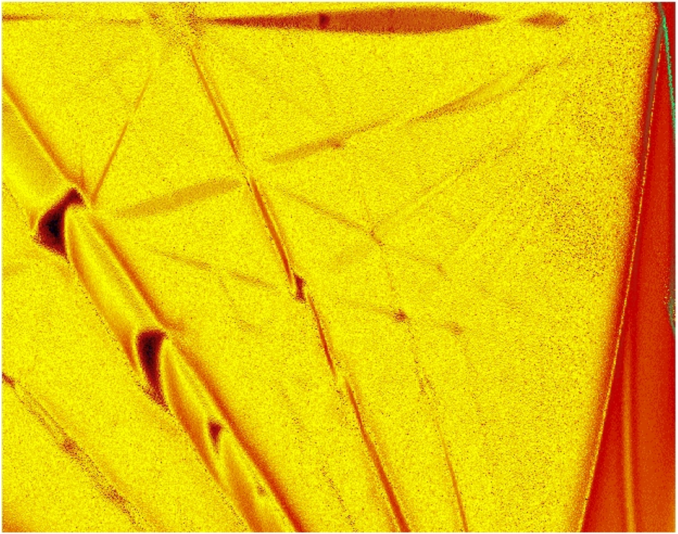
A French garden near the castle of La Roche-Guyon



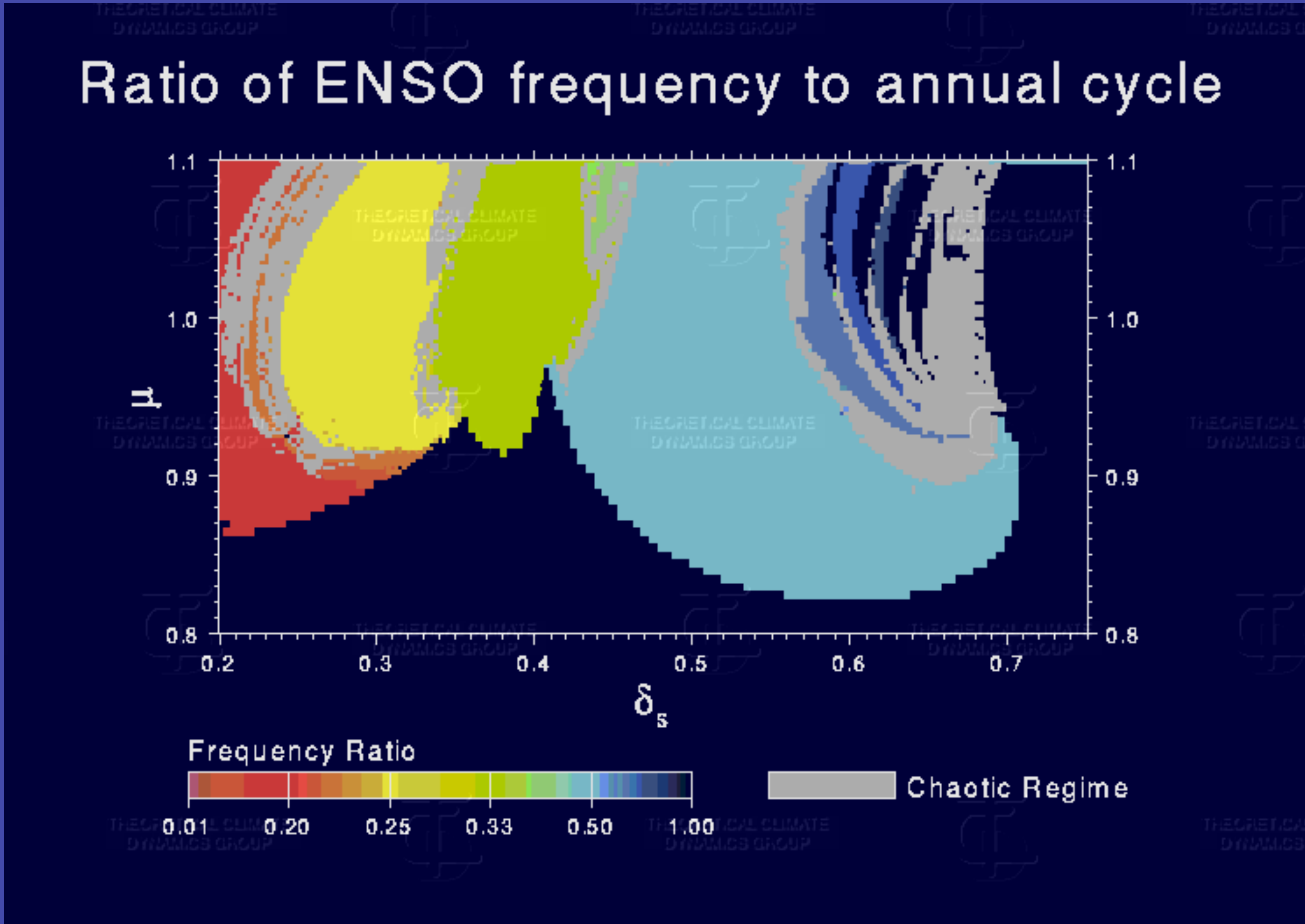
Devil's quarry for a coupling parameter $\varepsilon = 0.15$: a web of resonances



Effect of the noise on Devil's quarry

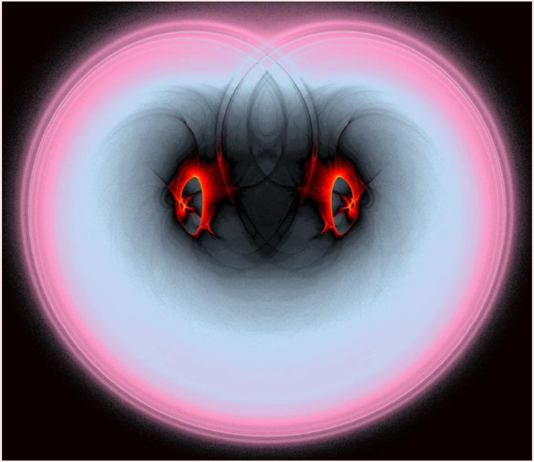


Devil's Bleachers in a 1-D ENSO Model



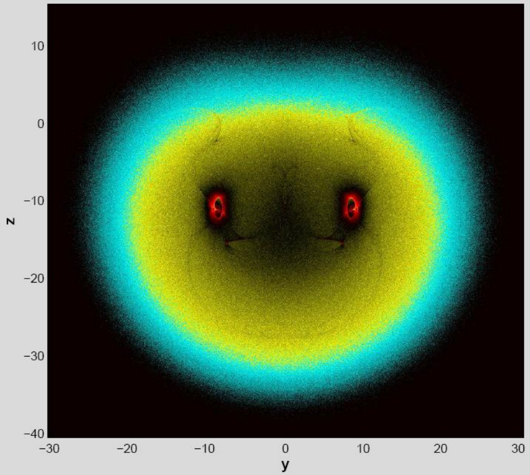
F.-F. Jin, J.D. Neelin & M. Ghil, *Physica D*, **98**, 442-465, 1996

Sample measure supported by the R.A.



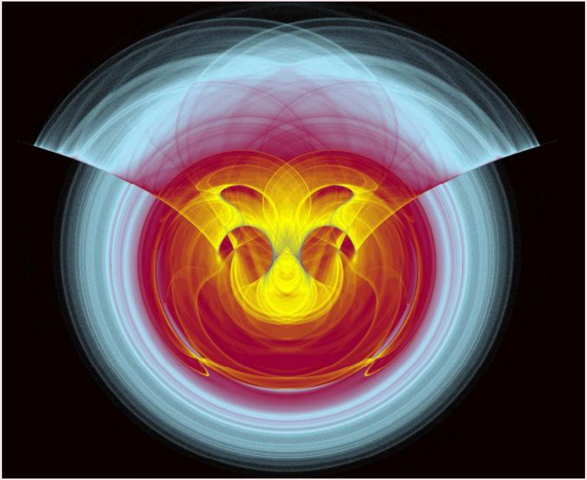
- 1 Billion I.D., and a different color palette!
- Intensity is $\alpha = 0.2$.
- Do you want different noise intensities?

Another proj. of the sample measure, “friendlier"



- The next slides are similar, with different noise level α and more I.D....

Sample measure supported by the R.A.



- Here $\alpha = 0.4$. The sample measure is approximated for another realization ω of the noise, starting from 8 billion I.D.
- Now more serious stuff is coming...

Sample measure supported by the R.A.

Sample measures evolve with time.

- Recall that these sample measures are the frozen statistics at a time t for a realization ω.

- How do these frozen statistics evolve with time?

- Action!

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Property of μ_ω for chaotic stochastic systems-I

The Sinai-Ruelle-Bowen (SRB) property

- RDS theory offers a rigorous way to define random versions of stable and unstable manifolds, via the Lyapunov spectrum, the Oseledec multiplicative theorem, and a random version of the Hartman-Grobman theorem.
- When the sample measures μ_ω of an RDS have absolutely continuous conditional measures on the random unstable manifolds, then μ_ω is called a *random SRB measure*.
- If the sample measure of an RDS φ is SRB, then its a “physical" measure in the sense that:

$$\lim_{s \to -\infty} \frac{1}{t-s} \int_s^t G \circ \varphi(s, \theta_{-s}\omega)x \, \mathrm{d}s = \int_{\mathcal{A}(\theta_t \omega)} G(x) \mu_{\theta_t \omega}(\mathrm{d}x), \qquad (3)$$

  for almost every $x \in X$ (in the Lebesgue sense), and for every continuous *observable* $G : X \to \mathbb{R}$.
- The measure μ_ω is also the image of the Lebesgue measure under the stochastic flow φ: for each region of $\mathcal{A}(\omega)$, it gives the probability to end up on that region, when starting from a volume.

Property of μ_ω for chaotic stochastic systems-II

A remarkable theorem of Ledrappier and Young (1988)

- Ledrappier and Young have proved that, that if the stationary solution, ρ, of the Fokker-Planck equation associated to an SDE presenting a Lyapunov exponent > 0, has a density w.r.t. the Lebesgue measure, then:

  μ_ω is a random SRB measure.

- This theorem applies to a large class of dissipative stochastic systems, namely the hypoelliptic ones that exhibit a Lyapunov exponent > 0: they all support a random SRB measure.
- Furthermore, we have the important relation:

$$\mathbb{E}(\mu_\bullet) = \rho, \tag{4}$$

where ρ is the stationary solution of the Fokker-Planck equation, when the latter is unique.

Mathematics of climate sensitivity-I

The Ruelle response formula

- Physically, the challenge is to find the trade-off between the physics present in the model and the stochastic parameterizations of the missing physics.
  From a mathematical point of view, climate sensitivity can be related to sensitivity of SRB measures.
- The thermodynamic formalism *à la* Ruelle, in the RDS context, helps to understand the response of systems out-of-equilibrium, to changes in the parameterizations (Gundlach, Kifer, Liu).
- The Ruelle response formula: Given an SRB measure μ of an autonomous chaotic system $\dot{x} = f(x)$, an *observable* $G : X \to \mathbb{R}$, and a smooth time-dependent perturbation X_t, the time-dependent variations $\delta_t \mu$ of μ are given by:

$$\delta_t \mu(G) = \int_{-\infty}^{t} d\tau \int \mu(dx) X_\tau(x) \cdot \nabla_x (G \circ \varphi_{t-\tau}(x)),$$

  where φ_t is the flow of the unperturbed system $\dot{x} = f(x)$.

Mathematics of climate sensitivity-II

The susceptibility function

- In the case $X_t(x) = \phi(t)X(x)$, the Ruelle response formula can be written:

$$\delta_t \mu(G) = \int dt' \kappa(t - t')\phi(t'),$$

  where κ is called the response function. The Fourier transform $\hat{\kappa}$ of the response function is called the susceptibility function.
- In this case $\widehat{\delta_t \mu(G)}(\xi) = \hat{\kappa}(\xi)\hat{\phi}(\xi)$ and since the r.h.s. is a product, there are no frequencies in the linear response that are not present in the signal.
- In general, the situation can be more complicated and the theory gives the following criterion of high sensitivity:

  $\mathfrak{C}$: **Poles of the susceptibility function** $\hat{\kappa}(\xi)$ **in the upper-half plane** $\Rightarrow$ **High sensitivity of the system's response function** $\kappa(t)$.
- RDS theory offers a path for extending this criterion when random perturbations are considered.

Climatic uncertainties & moral dilemmas

♥ … *keep today's climate for tomorrow?*

♥ *Feed the world today or…*

Davos, Feb. 2008, photos by *TIME Magazine*, 11 Feb. '08;

see also Hillerbrand & Ghil, *Physica D*, 2008, **237**, 2132–2138, doi:10.1016/j.physd.2008.02.015 .

The Biofuel Myth

- Fine illustration of the moral dilemmas (*).
- Conclusion: "**festina lentae**," as the Romans (**) used to say..

(**) ~ Han dynasty

(*) Hillerbrand & Ghil, *Physica D*, 2008, doi:10.1016/j.physd.2008.02.015, available on line.

Climate Change 1816–2008

M. Gillot, 2008,
Le Monde

T. Géricault, 1819,
Le Louvre