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Control under Communication Constraints and Invariance Entropy

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Goal

Determine **fundamental limitations** in control

Here: Describe the "information" needed to make a subset invariant for a control system

A recent survey on various definitions and application areas of **entropy** is Amigó et al. DCDS B (2015).

Classically, entropy is used in dynamical systems theory in order to describe the information generated by the systems and to classify them.

Control systems:

Delchamps (1990) (ergodic theory for quantized feedback)

Topological versions have been analyzed, in particular, by

Nair, Evans, Mareels and Moran (2004)

Kawan, Springer LNM Vol. 2089 (2013)

Control systems

We consider control system in discrete time given by

$$x_{n+1} = f(x_n, u_n), n \in \mathbb{N} = \{0, 1, ...\},$$

where $f : M \times \Omega \to M$ is continuous and M and Ω are metric spaces. The solution with $x_0 = x$ and $u = (u_n) \in \mathcal{U} := \Omega^N$ is denoted by $\varphi(n, x_0, u)$, $n \in \mathbb{N}$.

We assume that for every $x_0 \in Q \subset M$ there is $u(x) \in \Omega$ with $f(x, u(x)) \in Q$.
What is the "**information**" necessary to keep the system in Q?

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Motivation: Suppose that the present state x_n of the system is measured. If the controller has complete information about the present state, it can adjust a feedback control $u(x)$ appropriately. However, if the measurement is sent to the controller via a (noiseless) digital channel with bounded data rate it is of interest to determine the minimal data rate needed to make Q invariant. More abstractly: What is the minimal average information needed to make Q invariant?

Overview

This talk consists of three parts:

- Some motivation from classical entropy of dynamical systems

- Topological and measure-theoretic invariance entropy for control systems

- Relations to controllability properties

Topological entropy for dynamical systems

Let $T : X \to X$ be a continuous map on a compact metric space.
Suppose $\mathcal{B}$ is a finite open cover of X, i.e., the sets in $\mathcal{B}$ are open, their union is X.
For an **itinerary** $\alpha = (B_0, B_1, ..., B_{n-1}) \in \mathcal{B}^n$ let

$$\mathcal{B}_n(\alpha) = \{x \in X \,\middle|\, T^j(x) \in B_j \text{ for } j = 0, .., n-1\} = B_0 \cap .. \cap T^{-(n-1)} B_{n-1}.$$

They again form an open cover of X,

$$\mathfrak{B}^{(n)} = \{\mathcal{B}_n(\alpha) \,|\alpha \in \mathcal{B}^n\}.$$

Denote the minimal number of elements of a subcover by $N(\mathfrak{B}^{(n)})$.

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$$\mathfrak{B}^{(n)} = \{\mathcal{B}_n(\alpha) \,|\alpha \in \mathcal{B}^n\}.$$

Denote the minimal number of elements of a subcover by $N(\mathfrak{B}^{(n)})$.
Then the entropy of $\mathcal{B}$ is given by

$$h(\mathcal{B}, T) = \lim_{n\to\infty} \frac{1}{n} \log N(\mathfrak{B}^{(n)})$$

and the **topological entropy** of T is

$$h_{top}(T) = \sup_{\mathcal{B}} h(\mathcal{B}, T).$$

Adler, Konheim, McAndrew (1965)

A classical example

Consider the **logistic map** on the interval $X = [0, 1]$ given by

$$F_4(x) = 4x(1 - x), x \in [0, 1].$$

The topological entropy of F_4 is

$$h_{top}(F_4) = \log_2 2 = 1 > 0.$$

Hence this is a **chaotic map**.

Metric entropy for dynamical systems

For a probability measure μ and a partition $\mathcal{P}$ of X the **Shannon entropy** is

$$H_\mu(\mathcal{P}) = -\sum_{P\in\mathcal{P}} \mu(P)\log\mu(P).$$

Let μ be invariant for a map S on X, i.e., $\mu(S^{-1}B) = \mu(B)$ for all $B \subset X$.

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For an **itinerary** $\alpha = (P_0, P_1, ..., P_{n-1}) \in \mathcal{P}^n$ let

$$P_n(\alpha) = \{x \in X \,|\, S^j(x) \in P_j \text{ for all } j\} = P_0 \cap S^{-1}P_1 \cap \cdots \cap S^{-(n-1)}P_{n-1}.$$

They yield a partition $\mathcal{P}^{(n)} = \{P_n(\alpha) \,|\alpha \in \mathcal{P}^n\}$ and

$$h_\mu(\mathcal{P}, S) := \lim_{n\to\infty} \frac{1}{n} H_\mu\left(\mathcal{P}^{(n)}\right).$$

The **Kolmogorov-Sinai entropy** of S is

$$h_\mu(S) = \sup_{\mathcal{P}} h_\mu(\mathcal{P}, S).$$

The logic map again

Recall

$$F_4(x) = 4x(1-x) \text{ on } [0,1].$$

A (trivial) invariant measure is $\mu = \delta_0$ with entropy $h_{\delta_0}(F_4) = 0$.
A nontrivial invariant measure is given by its density (with respect to Lebesgue measure)

$$\frac{1}{\pi\sqrt{x(1-x)}}, x \in [0,1].$$

The corresponding **metric entropy** is

$$h_\mu(F_4) = \log_2 2 = 1$$

(hence equal to the topological entropy).

Comments

The **Variational Principle** states that

$$\sup_{\mu} h_{\mu}(T) = h_{top}(T)$$

and invariant measures μ with maximal entropy, i.e., $h_{\mu}(T) = h_{top}(T)$, are of special relevance.

Often, entropy can be characterized by (the positive) **Lyapunov exponents**.

Describe the **minimal information** to make a compact $Q \subset M$ invariant for

$$x_{n+1} = f(x_n, u_n), \;\; u_n \in \Omega,$$

with solutions $\varphi(n, x_0, u), n \in \mathbb{N}$, in M.

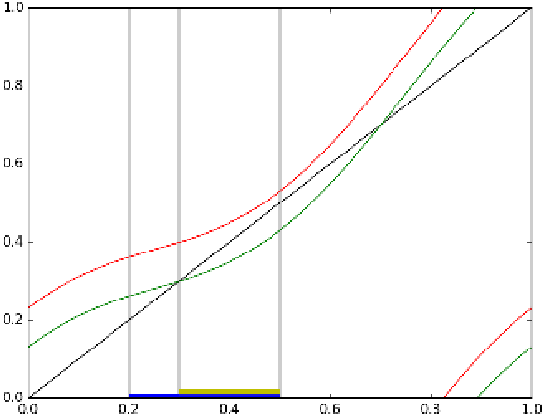
This can be done in a topological or in a measure-theoretic framework. Topological invariance entropy is well developed. It is based on **itineraries in Q** corresponding to **invariant open covers** of Q. They are constructed by feedbacks keeping the system in Q and replace the open covers.

Observe: This is not directly related to the entropy of the uncontrolled system which may behave very wildly in Q, while Q itself may be invariant. Hence the entropy of the dynamical system may be positive while the invariance problem is trivial.

Example

$$f_\alpha(x,\omega) = x + \sigma\cos(2\pi x) + A\omega + \alpha \mod 1,\ \omega \in \Omega = [-1,1].$$

With $A = 0.05, \sigma = 0.1, \alpha = 0.08$ consider the set $Q = [0.2, 0.5]$.



Topological invariance entropy for control systems

An **invariant open cover** $\mathcal{C}_\tau = (\mathcal{B}, F)$ is given by $\tau \in \mathbb{N}$, an open cover $\mathcal{B}$ of Q and $F : \mathcal{B} \to \Omega^\tau$ with

$$\varphi(j, B, F(B)) \subset \mathrm{int}Q \text{ for } j = 1, ..., \tau \text{ and } B \in \mathcal{B}.$$

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For a **$\mathcal{C}_\tau$-itinerary** $\alpha = (B_0, ..., B_{n-1}) \in \mathcal{B}^n$ define $u_\alpha = (F(B_0), F(B_1), ...)$ and

$$B_n(\alpha) = \{x \in Q \,|\, \varphi(i\tau, x, u_\alpha) \in B_i \text{ for } i = 0, ..., n-1\}.$$

These sets again form an open cover of Q,

$$\mathfrak{B}^{(n)} = \{B_n(\alpha) \,|\, \alpha \in \mathcal{B}^n\}.$$

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These sets again form an open cover of Q,

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The invariance entropy of $\mathcal{C}_\tau$ is

$$h(\mathcal{C}_\tau, Q) := \lim_{n\to\infty} \frac{1}{n} \log N(\mathfrak{B}^{(n)} \,|\, Q)$$

and the **topological invariance entropy** of Q is

$$h_{top}^{inv}(Q) := \inf_{\mathcal{C}_\tau(\mathcal{B},F)} h(\mathcal{C}_\tau, Q).$$

Relations to data rates and coder-controllers

A **coder-controller** has the form $\mathcal{H} = (S, \gamma, \delta, \tau)$ where
- $S = (S_k)_{k \in \mathbb{N}}$ denotes finite coding alphabets
- the coder mapping $\gamma_k : M^{k+1} \to S_k$ associates to the present and past states the symbol $s_k \in S_k$
- at time $k\tau$ the controller mapping is $\delta_k : S_0 \times \cdots \times S_k \to \Omega^\tau$.

The **transmission data rate** is

$$R(\mathcal{H}) = \liminf_{k \to \infty} \frac{1}{k\tau} \sum_{j=0}^{k-1} \log \# S_j .$$

$\mathcal{H}$ renders Q invariant if for every $x_0 \in Q$ the sequence

$$x_{k+1} := \varphi(\tau, x_k, u_k), k \in \mathbb{N}, \qquad (1)$$

with

$$u_k = \delta_k(\gamma_0(x_0), \gamma_1(x_0, x_1), \ldots, \gamma_k(x_0, x_1, \ldots, x_k)) \in \Omega^\tau \qquad (2)$$

satisfies

$$\varphi(i, x_k, u_k) \in Q \text{ for all } i \in \{1, \ldots, \tau\} \text{ and all } k \in \mathbb{N}. \qquad (3)$$

Theorem. For a compact and controlled invariant set Q it holds that

$$h_{inv}^{top}(Q) = \inf R(\mathcal{H}),$$

where the infimum is taken over all coder-controllers $\mathcal{H}$ that render Q invariant.

Comments and some further results

- Let $K \subset Q$ be compact. Then one can define (using spanning sets of controls) the invariance entropy $h_{inv}(K, Q)$ of K with respect to Q.

- For **linear control systems** in $\mathbb{R}^d$

$$x_{n+1} = Ax_n + Bu_n, u_n \in \Omega \subset \mathbb{R}^m,$$

with $\mathrm{int}K \neq \emptyset$ and (A, B) controllable, A hyperbolic and Ω a compact nbhd of 0, one has for K contained in the unique control set D

$$h_{top}^{inv}(K, D) = \sum_{\lambda \in \sigma(A)} \max(0, \log |\lambda|).$$

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- hyperbolicity of the control flow on $\mathcal{U} \times Q$ gives a formula in terms of Lyapunov exponents for periodic solutions

Kawan (2014),

- also for linear control systems on Lie groups

da Silva (2014)

Comments and some further results

Da Silva and Kawan, Disc.Cont.Dynam.Syst. (2016):

Theorem. Consider a uniformly hyperbolic chain control set E with nonempty interior of a control-affine system. Additionally assume that
(i) the Lie Algebra Rank Condition holds on $\mathrm{int} D$ and
(ii) for each $u \in \mathcal{U}$ there exists a unique $x \in E$ with $(u, x) \in E$, i.e., E is a graph over $\mathcal{U}$.
Then E is the closure of a control set D and for every compact set $K \subset D$ with positive volume,

$$h_{inv}(K, D) = \inf_{(u,x)\in\mathcal{E}} \limsup_{t\to\infty} \log J^{+}\varphi_{t,u}(x)$$

where $J^{+}\varphi_{t,u}(x)$ is the unstable determinant of $d\varphi_{t,u}(x)$.

Introduce a potential $f \in C(\Omega, \mathbb{R})$ for the control values.
Let $K \subset Q$ be compact s.t. $\forall x \in K\ \exists u \in \mathcal{U} : \varphi(\mathbb{R}_+, , u) \subset Q$.
A set $\mathcal{S} \subset \mathcal{U}$ is a (τ, K, Q)-spanning set if

$$\forall x \in K\ \exists u \in \mathcal{S} : \varphi([0, \tau], x, , u) \subset Q.$$

With $(S_\tau f)(u) := \int_0^\tau f(u(t))dt$ let

$$a_\tau(f, K, Q) := \inf \left\{ \sum_{u \in \mathcal{S}} e^{(S_\tau f)(u)};\ \mathcal{S} \text{ is } (\tau, K, Q)\text{-spanning} \right\}.$$

The **invariance pressure** is

$$P_{inv}(f, K, Q) = \limsup_{\tau \to \infty} \frac{1}{\tau} \log a_\tau(f, K, Q).$$

Consider a **linear control systems** in $\mathbb{R}^d$

$$\dot{x} = Ax + Bu, u(t) \in \Omega \subset \mathbb{R}^m,$$

with a compact neighborhood Ω of 0 and assume (A, B) controllable, A hyperbolic.
For $K \subset D$, the unique control set with $\mathrm{int} D \neq \varnothing$, one has:

$$P_{inv}(f, K, D) \leq \sum\nolimits_{\lambda \in \sigma(A)} \max(0, \mathrm{Re}\,\lambda) + \inf{}_{T,u(\cdot)} \frac{1}{T} \int_0^T f(u(s))ds,$$

where the infimum is taken over all $T > 0$ and all T-periodic controls $u(\cdot)$ with values in a compact subset of $\mathrm{int}\Omega$ and a T-periodic $x(\cdot) \subset \mathrm{int} D$.

Describe the **minimal information** to make a compact $Q \subset M$ invariant for

$$x_{n+1} = f(x_n, u_n), \;\; u_n \in \Omega,$$

with solutions $\varphi(n, x_0, u)$, $n \in \mathbb{N}$, in M.

We will need **itineraries in Q** corresponding to **invariant partitions** of Q. They will be constructed by feedbacks keeping the system in Q and replace the partitions.

Measure-theoretic invariance entropy: the ingredients

We need

- partitions and itineraries in Q for a map S

- a probability measure (quasi-stationary)

- a notion of entropy

Construction of metric invariance entropy: partitions

Let η be a probability measure on Q.
An **invariant partition** $\mathcal{C}_\tau = (\mathcal{P}, F)$ is given by $\tau \in \mathbb{N}$, a partition $\mathcal{P}$ of Q and $F : \mathcal{P} \to \Omega^\tau$ such that for $P \in \mathcal{P}$

$$\varphi(j, x, F(P)) \in Q \text{ for } j = 1, ..., \tau \text{ and } \eta\text{-a.a. } x \in P.$$

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$$\varphi(j, x, F(P)) \in Q \text{ for } j = 1, ..., \tau \text{ and } \eta\text{-a.a. } x \in P.$$

Define

$$A(P) = \{u \in \mathcal{U} \,|\, \varphi(j, x, u) \in Q,\ j = 1, ..., \tau, \eta\text{-a.a. } x \in P\} \times P \subset \mathcal{U} \times Q$$

and

$$\mathfrak{A} = \mathfrak{A}(\mathcal{C}_\tau) = \{A(P) \,|\, P \in \mathcal{P}\}.$$

Then $\mathfrak{A}$ consists of pairwise disjoint subsets in $\mathcal{U} \times Q$.

Construction of metric invariance entropy: control flow

Let the shift θ on $\mathcal{U} = \Omega^{\mathbb{N}_0}$ be $(\theta u)_n := u_{n+1}, n \in \mathbb{N}$. The control system

$$x_{n+1} = f(x_n, u_n),\ \ u_n \in \Omega,$$

is described by the **control flow** given by S on $\mathcal{U} \times M$ and its iterations,

$$S(u, x) := (\theta u, f(x, u_0)) \text{ for } u = (u_n) \in \mathcal{U} \text{ and } x \in M.$$

Then

$$S^n(u, x) = (\theta^n u, \varphi(n, x, u)), n \in \mathbb{N}.$$

We are interested in the restriction

$$S_Q : \mathcal{U} \times Q \to \mathcal{U} \times M.$$

Construction of metric invariance entropy: itineraries

A sequence $\alpha = (A(P_0), \dots, A(P_{n-1}))$ is called an **itinerary** if for $u_\alpha := (F(P_0), F(P_1), \dots, F(P_{n-1}))$

$$\eta\{x \in Q \,|\, \varphi(j\tau, x, u_\alpha) \in P_j, j = 0, 1, \dots, n-1\} > 0.$$

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$$\eta\{x \in Q \,|\, \varphi(j\tau, x, u_\alpha) \in P_j, j = 0, 1, \ldots, n-1\} > 0.$$

Let

$$A(\alpha) = A(P_0) \cap S^{-\tau} A(P_1) \cap \cdots \cap S^{-(n-1)\tau} A(P_{n-1}) \subset S_Q^{-(n-1)\tau}(\mathcal{U} \times Q)$$

be the set of all (u, x) following this itinerary and

$$\mathfrak{A}^{(n)} = \{A(\alpha) \,|\, \alpha \text{ an itinerary of length } n\}.$$

Then $\mathfrak{A}^{(n)}$ consists of pairwise disjoint sets in $S_Q^{-(n-1)\tau}(\mathcal{U} \times Q)$.

Choice of the probability measure

For

$$x_{n+1} = f(x_n, u_n), u_n \in \Omega, \ n \in \mathbb{N} = \{0, 1, ...\},$$

let ν be a probability measure on Ω and define Markov transition probabilities by

$$p(x, B) := \nu\{\omega \in \Omega \,|\, f(x, \omega) \in B\}, B \subset M.$$

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$$p(x, B) := \nu\{\omega \in \Omega \,|\, f(x, \omega) \in B\}, B \subset M.$$

Let η be a **quasi-stationary measure**, i.e. a probability measure on $Q \subset M$ with

$$\rho \cdot \eta(B) = \int_Q p(x, B)\eta(dx) \text{ for } B \subset Q,$$

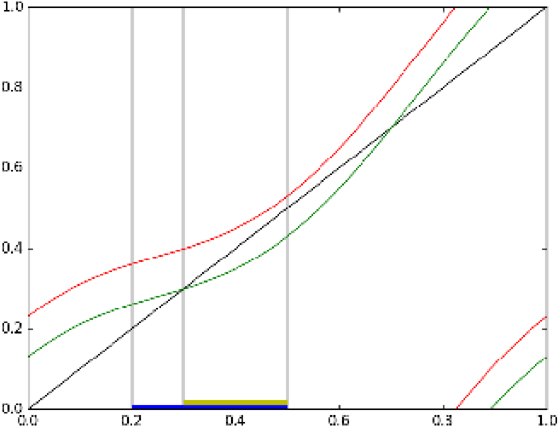
with $\rho := \int_Q p(x, Q)\eta(dx) \in (0, 1)$. η is stationary iff $\rho = 1$.

The measure $\mu := \nu^{\mathbb{N}} \times \eta$ on $\mathcal{U} \times Q$ is a **conditionally invariant measure** for the control flow S.

Collett, Martinez, San Martin (2013), Méléard, Villemonais (2012)

Pianigiani and Yorke (1979), Demers and Young (2006)

$$f(x,\omega) = x + \sigma\cos(2\pi x) + A\omega + \alpha \bmod 1, \omega \in \Omega = [-1,1], Q = [0.2, 0.5].$$



For the uniform distribution ν on $\Omega = [-1,1]$ one can prove that there is a quasi-stationary measure η for Q.

The entropy notion

Recall that $\mathfrak{A}^{(n)}$ is the collection of all sets

$$A_n(\alpha) = A(P_0) \cap S^{-\tau} A(P_1) \cap \cdots \cap S^{(n-1)\tau} A(P_{n-1}) \subset S_Q^{-(n-1)\tau}(\mathcal{U} \times Q)$$

consisting of the pairs (u, x) following an **itinerary** $\alpha = (P_0, ..., P_{n-1}) \in \mathcal{P}^n$.

It *does not work* to use the Shannon entropy of $\mathfrak{A}^{(n)}$ w.r.t. μ

$$H_\mu(\mathfrak{A}^{(n)}) = -\sum_\alpha \mu(A_n(\alpha)) \log \mu(A_n(\alpha)),$$

since η is only quasi-stationary with constant $\rho \in (0,1)$ and $\mu = \nu^{\mathbb{N}} \times \eta$.

Construction of metric invariance entropy

Since $\rho^{-1} \cdot \mu$ is a probability measure on $S_Q^{-1}(\mathcal{U} \times Q)$ consider

$$H_{\rho^{-(n-1)\tau}\mu}(\mathfrak{A}^{(n)}(\mathcal{C}_\tau))$$

for the partition $\mathfrak{A}^{(n)}(\mathcal{C}_\tau)$ in $S_Q^{-(n-1)\tau}(\mathcal{U} \times Q)$ and then take the average of the required information as time tends to ∞ to get

$$h(\mathcal{C}_\tau, Q) = \limsup_{n\to\infty} \frac{1}{n\tau} H_{\rho^{-(n-1)\tau}\mu}(\mathfrak{A}^{(n)}(\mathcal{C}_\tau)).$$

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$$h(\mathcal{C}_\tau, Q) = \limsup_{n\to\infty} \frac{1}{n\tau} H_{\rho^{-(n-1)\tau}\mu}(\mathfrak{A}^{(n)}(\mathcal{C}_\tau)).$$

Define the **metric invariance entropy** for the control system as

$$\boxed{h_\eta^{inv}(Q) := \limsup_{\tau\to\infty} \inf_{\mathcal{C}_\tau} h(\mathcal{C}_\tau, Q),}$$

where the infimum is taken over all invariant partitions $\mathcal{C}_\tau(\mathcal{P}, F)$.

Relation to topological invariance entropy

Theorem. For every quasi-stationary measure η on Q the η-invariance entropy is bounded by the topological invariance entropy,

$$\boxed{h_{\eta}^{inv}(Q) \leq h_{top}^{inv}(Q).}$$

Note that metric entropy is **invariant** under appropriate **conjugacies**.

Conjugacies

The metric entropy is **invariant** under appropriate **conjugacies** preserving the measure: Consider

$$x_{n+1} = f_1(x_n, u_n) \text{ and } y_{n+1} = f_2(y_n, u_n) \text{ with } (u_n) \in \mathcal{U}.$$

Let μ_1 and μ_2 be conditionally invariant measures for Q_1 and Q_2, resp. A bimeasurable bijection $\pi : Q_1 \to Q_2$ is a **conjugacy**, if

$$\pi\varphi_1(n, x_0, u) = \varphi_2(n, \pi x_0, u) \text{ for all } n \geq 0$$

and $\mathrm{id}_{\mathcal{U}} \times \pi : \mathcal{U} \times Q_1 \to \mathcal{U} \times Q_2$ maps μ_1 onto μ_2, i.e.,

$$\mu_1\left((\mathrm{id}_{\mathcal{U}} \times \pi)^{-1}(B) \right) = \mu_2(B) \text{ for all } B \in \mathcal{B}(\mathcal{U} \times Q_2).$$

Then

$$\boxed{h^{inv}_{\mu_1}(Q_1, S_1) = h^{inv}_{\mu_2}(Q_2, S_2).}$$

Invariance Entropy and Controllability Properties

For **dynamical systems** it is well known that the metric and the topological entropy are already determined on the recurrent set.

What about invariance entropy?

For **control systems** recurrence properties are replaced by controllability properties.

Here subsets of complete approximate controllability (in Q) are of relevance, called **control sets**. They are analogous to communicating classes.

W-control sets

For an **open** subset W of the state space et $\varphi_W(n,x,u)$ be the trajectories within W and define the **reachable and controllable set within W** by

$$\begin{aligned}\mathcal{R}_W(x) &= \{\varphi_W(n,x,u) \text{ for some } n \in \mathbb{N} \text{ and } u \in \mathcal{U}\}\\ \mathcal{C}_W(x) &= \{y \in W \,|\, \varphi_W(n,y,u) = x \text{ for some } n \in \mathbb{N} \text{ and } u \in \mathcal{U}\}.\end{aligned}$$

W-control sets

For an **open** subset W of the state space et $\varphi_W(n,x,u)$ be the trajectories within W and define the **reachable and controllable set within W** by

$$\begin{aligned} \mathcal{R}_W(x) &= \{\varphi_W(n,x,u) \text{ for some } n \in \mathbb{N} \text{ and } u \in \mathcal{U}\} \\ \mathcal{C}_W(x) &= \{y \in W \,|\, \varphi_W(n,y,u) = x \text{ for some } n \in \mathbb{N} \text{ and } u \in \mathcal{U}\}. \end{aligned}$$

Definition. A set D is called an **invariant W-control set** if
(i)

$$\overline{D}^W = \overline{\mathcal{R}_W(x)}^W \text{ for all } x \in D,$$

where the closure is taken with respect to W and
(ii) there is $x \in D$ with $x \in \mathrm{int}\mathcal{C}_W(x)$.

Remark. Condition (ii) is crucial for discrete-time systems.

Existence of invariant W-control sets

Theorem. Assume

- the state space M is a connected analytic Riemannian manifold
- $W \subset M$ is connected open and relatively compact
- the control range $\Omega \subset \overline{\mathrm{int}\Omega} \subset \mathbb{R}^m$ and $f : M \times \Omega \to M$ is analytic
- $\Omega_{sub} := \{\omega \in \Omega \,|\, f(\cdot, \omega) \text{ is submersive}\}$ is the complement of a proper analytic subset.

Then the following are equivalent:

(i) There are at least one and at most finitely many **invariant W-control sets** D and for every $x \in W$ there is D with

$$\mathcal{R}_W(x) \cap D \neq \emptyset.$$

(ii) There is a compact set $F \subset W$ with

$$F \cap \overline{\mathcal{R}_W(x)} \neq \emptyset \text{ for all } x \in W.$$

Albertini and Sontag (1993), Wirth (1998), Patrão and San Martin (2007)

Invariance entropy and W-control sets

Theorem. Under the assumptions of (i) in the previous theorem let $Q := \overline{W} \subset M$. Assume

(i) for the finitely many invariant W-control sets D_i

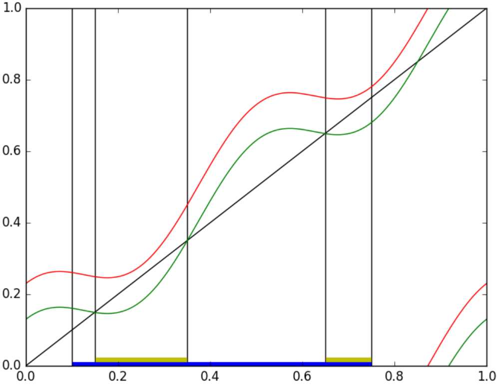
$$f(\bigcup_i \overline{D_i}, \Omega) \cap (\partial Q \setminus \bigcup_i \overline{D_i}) = \emptyset.$$

(ii) the maps $f(\cdot, \omega) : M \to M, \omega \in \Omega$, are nonsingular for a quasi-stationary measure η (i.e. preimages of null sets are null sets). Then

$$\boxed{h_\eta^{inv}(Q) = h_\eta^{inv}(\bigcup_i \overline{D_i}).}$$

Remark. In the continuous-time case a similar result for the topological invariance entropy has been shown in FC/Lettau (2016).

$$f_\alpha(x,\omega) = x + \sigma\cos(4\pi x) + A\omega + \alpha \mod 1.$$



Two W-control sets D_1 and D_2 (to the right) in $W = (0.1, 0.7)$. The invariance entropies for η on $Q = [0.1, 0.7]$ and on $\overline{D_2}$ coincide.

Final remarks

Classical entropy of dynamical systems describes the **total information** generated by the system topologically or with respect to an **invariant measure**.

In contrast, entropy for control systems describes the **minimal information** for invariance either topologically or with respect to a **quasi-stationary measure**.

The data rate theorem relates the topological invariance entropy to the minimal bit rate needed for invariance.

Some references

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