Uncertainty Quantification for multiscale kinetic equations with random inputs III

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III. Regularity and local sensitivity analysis in random space

- Regularity is important to understand the solution, as well as numerical accuracy
- Local sensitivity analysis helps to identify sensitive or insensitivy input parameters
- Our analysis is based on hypocoervicity theory, originally developed by deterministic kinetic equations (Herou-Nier, Villani, Guo, Desvillettes, Mouhot-Newmann, Briant, Doubeault-Mouhot-Schmieser)
- It allows us to establish regularity, local sensitivity, long-time behavior of the solution in random space; it also allows to prove (uniform) spectral accuracy and long time exponential decay of gPC-SG errors

Energy estimate (linear)

- Model problem $\partial_t f + v \cdot \nabla_x f = \sigma(z) L(f)$

- Energy estimate

$$\frac{1}{2}\partial_t \|f\|_{L^2}^2 = \sigma(z) \int\int L(f) f \, \mathrm{d}v \, \mathrm{d}x \ \leq -c\sigma_0 \|f^\perp\|_{L^2}^2$$

- Spectral gap of collision operator

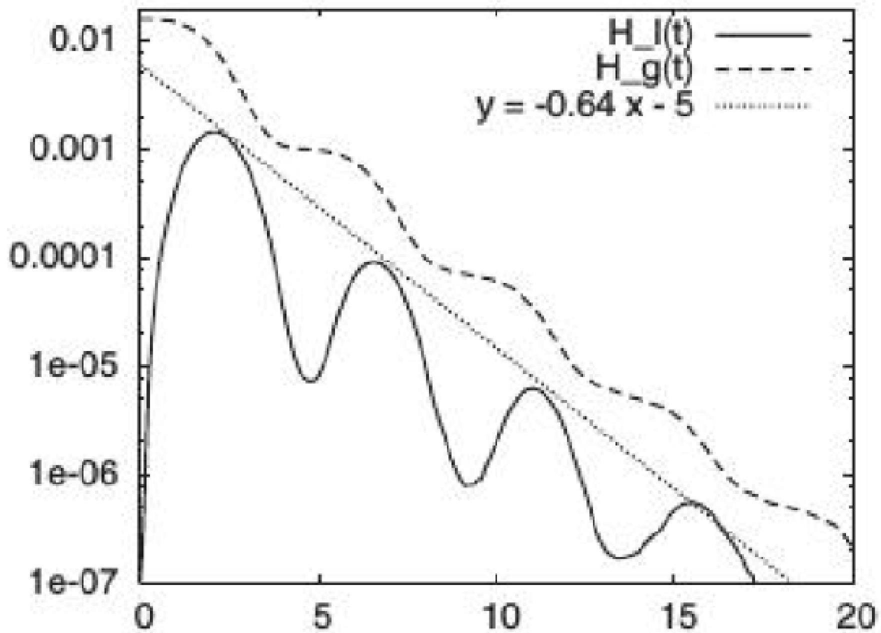
$$\int L(f) f \, \mathrm{d}v \leq -c \|f^\perp\|_{L^2}^2,$$

$f^\perp$:projection onto the orthogonal complement of N(L)

- To obtain exponential decay in time, one needs the dissipation in macroscopic quantities (inside N(L))

entropy decay in inhomogeneous Boltzmann equation

- simulation of 1+2 D Boltzmann equation: wavy entropy decay
- $- - -$ $H_g(t)$: relative entropy w.r.t. the global Maxwellian



Ref: [Filbet-Mouhot-Pareschi] 2006

Hypocoercivity

$$\partial_t f + v\partial_x f = \partial_{vv} f \qquad \partial_t \frac{1}{2}\|f\|^2 = -\|\partial_v f\|^2$$

$$\partial_t \partial_x f + v\partial_x \partial_x f = \partial_{vv} \partial_x f \qquad \partial_t \frac{1}{2}\|\partial_x f\|^2 = -\|\partial_{xv} f\|^2$$

$$\partial_t \partial_v f + v\partial_x \partial_v f + \partial_x f = \partial_{vv} \partial_v f \qquad \partial_t \frac{1}{2}\|\partial_v f\|^2 = -\|\partial_{vv} f\|^2 + \langle \partial_x f, \partial_v f\rangle$$

$$\partial_t \langle \partial_x f, \partial_v f\rangle = -\|\partial_x f\|^2 + \cdots$$

$$V(f) = \|f\|^2 + \|\partial_x f\|^2 + \alpha\|\partial_v f\|^2 + \beta\langle \partial_x f, \partial_v f\rangle$$

$$\partial_t V(f) \le -cV(f)$$

Lyapunov functional

Herau-Nier (2004), Mouhot-Neumann (2006), …

z-derivatives

$$\partial_t f + v \cdot \nabla_x f = \sigma(z) L(f)$$

$$\partial_t \partial_z^k f + v \cdot \nabla_x \partial_z^k f = \sigma(z) L(\partial_z^k f) + \mathcal{S}_k, \qquad \mathcal{S}_k = \sum_{j=1}^{k} \binom{k}{j} \partial_z^j \sigma L(\partial_z^{k-j} f),$$

Main part is the same equation

source terms only involve LOWER order z-derivatives

- Use good terms from lower order z-derivative estimate
- Energy functional: $E = \|\partial_z^k f\|^2 + \sum_{j=0}^{k-1} c_j \|\partial_z^j f\|^2$
- Hypocoercivity can be applied similarly

Jin-Liu-Ma (2017), Jin-Liu (2017), Li-Wang (2017), Liu (2017)

Linear transport equation with uncertainty:
Jin-J.-G. Liu-Ma RMS '17

- Define the following norms

$$\langle f, g\rangle_\omega = \int_{\mathbb{R}^d} f(z) g(z)\, \omega(z)\, \mathrm{d}z, \quad \|f\|_\omega^2 = \langle f, f\rangle_\omega$$

$$\|f(t,x,v,\cdot)\|_{H^k}^2 := \sum_{\alpha \le k} \|D^\alpha f(t,x,v,\cdot)\|_\omega^2$$

$$\|f(t,\cdot,\cdot,\cdot)\|_{\Gamma(t)}^2 := \int_Q \|f(t,x,v,\cdot)\|_\omega^2 \, \mathrm{d}x\, \mathrm{d}v.$$

Uniform regularity

- The regularity in the random space is preserved in time, uniformly in ε

$$D^k f(t,x,v,z) := \partial_z^k f(t,x,v,z)$$

Theorem 4.1 (Uniform regularity). *Assume*

$$\sigma(z) \geq \sigma_{\min} > 0\,.$$

If for some integer $m \geq 0$,

$$\|D^k \sigma(z)\|_{L^\infty} \leq C_\sigma, \qquad \|D^k f_0\|_{\Gamma(0)} \leq C_0, \qquad k = 0, \ldots, m,$$

then

$$\|D^k f\|_{\Gamma(t)} \leq C, \qquad k = 0, \cdots, m, \qquad \forall t > 0,$$

where C_σ, C_0 *and* C *are constants independent of* ε.

- A good problem to use the gPC-SG for UQ

Key estimates

Energy estimate: We will establish the following energy estimate by using Mathematical Induction with respect to k: for any $k \geq 0$, there exist k constants $c_{kj} > 0$, $j = 0, \ldots, k-1$ such that

$$\varepsilon^2 \partial_t \Big(\|D^k f\|^2_{\Gamma(t)} + \sum_{j=0}^{k-1} c_{kj} \|D^j f\|^2_{\Gamma(t)} \Big) \leq \begin{cases} -2\sigma_{\min} \big\| [f] - f \big\|^2_{\Gamma(t)}, & k = 0, \\ -\sigma_{\min} \big\| D^k([f] - f) \big\|^2_{\Gamma(t)}, & k \geq 1. \end{cases}$$

Theorem 4.2 **(Estimate on $[\boldsymbol{f}] - \boldsymbol{f}$).** *With all the assumptions in Theorem 4.1 and Lemma 4.2, for a given time $T > 0$, the following regularity result of $[f] - f$ holds:*

$$\begin{aligned} &\|D^k([f] - f)\|^2_{\Gamma(t)} \\ &\leq \mathrm{e}^{-\sigma_{\min} t / 2\varepsilon^2} \|D^k([f_0] - f_0)\|^2_{\Gamma(0)} + C'(T)\varepsilon^2 \\ &\leq C(T)\varepsilon^2, \end{aligned} \tag{54}$$

for any $t \in (0, T]$ and $0 \leq k \leq m_{,}$, where $C'(T)$ and $C(T)$ are constants depending on T.

uniform spectral convergence (sAP)

Theorem 4.3 (Uniformly convergence in ε). *Assume*

$$\sigma(z) \geq \sigma_{\min} > 0\,.$$

If for some integer $m \geq 0$,

$$\|\sigma(z)\|_{H^k} \leq C_\sigma, \quad \|D^k f_0\|_{\Gamma(0)} \leq C_0, \quad \|D^k(\partial_x f_0)\|_\omega \leq C_x, \quad k = 0, \ldots, m, \tag{82}$$

Then the error of the whole gPC-SG method is

$$\|f - f_N\|_{\Gamma(t)} \leq \frac{C(T)}{N^k}, \tag{83}$$

where $C(T)$ is a constant independent of ε.

Uniform stability

- For a fully discrete scheme based on the deterministic micro-macro decomposition ($f=M+g$) based approach (Klar-Schmeiser, Lemou-Mieusseun) approach, we can also prove the following uniform stability:

$$\Delta t \le \frac{\sigma_{\min}}{3}\Delta x^2 + \frac{2\varepsilon}{3}\Delta x.$$

General collisional kinetic equations (Jin-L. Liu MMS '18)

$$\begin{cases} \partial_t f + \frac{1}{\epsilon^\alpha} v \cdot \nabla_x f = \frac{1}{\epsilon^{1+\alpha}} \mathcal{Q}(f), \\ f(0,x,v,z) = f_{in}(x,v,z), \qquad x \in \Omega \subset \mathbb{T}^d,\ v \in \mathbb{R}^d,\ z \in I_z \subset \mathbb{R}. \end{cases}$$

perturbative setting $\quad f = \mathcal{M} + \epsilon M h$

(avoid compressible Euler limit, thus shocks):

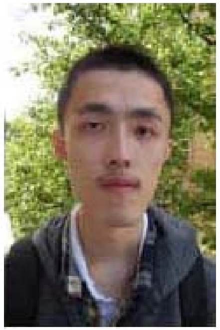
Global Maxwellian $\quad \mathcal{M} = \frac{1}{(2\pi)^{\frac{d}{2}}} e^{-\frac{|v|^2}{2}} \qquad M = \sqrt{\mathcal{M}}$

Euler (acoustic scaling)

$$\partial_t h + v \cdot \nabla_x h = \frac{1}{\epsilon}\mathcal{L}(h) + \mathcal{F}(h,h).$$

(incompressible) Navier-Stokes scaling

$$\partial_t h + \frac{1}{\epsilon} v \cdot \nabla_x h = \frac{1}{\epsilon^2}\mathcal{L}(h) + \frac{1}{\epsilon}\mathcal{F}(h,h).$$



Why it works: hypocoercivity decay of the linaar part dominates the bounded (weaker) nonlinear part

- Also relevant contributions from Y. Zhu and R. Shu

The Boltzmann equation with uncertain initial data

hypocoercivity

$$\langle h,\, \mathcal{L}(h)\rangle_{L^2_v} \le -\lambda\, ||h^\perp||_{\Lambda^2_v}$$

$$h^\perp = h - \Pi_{\mathcal{L}}(h)$$

$\Pi_{\mathcal{L}}(h)$ is the orthogonal projection in L^2_v on $N(\mathcal{L})$

$$||h||_{\Lambda_v} = ||h(1+|v|)^{\gamma/2}||_{L^2}$$

$$||\cdot||_{\Lambda} := ||\, ||\cdot||_{\Lambda_v}\, ||_{L^2_x}.$$

$$N(\mathcal{L}) = \mathrm{Span}\{M,\, v_1 M,\, \cdots, v_d M,\, |v|^2 M\} := \mathrm{Span}\{\varphi_1, \cdots, \varphi_n\}, \qquad n = d+2$$

Boundedness of the nonlinear term

$$\left|\langle \partial^m \partial_l^j \mathcal{F}(h,h),\ f\rangle_{L^2_{x,v}}\right| \le \begin{cases} \mathcal{G}^{s,m}_{x,v,z}(h,h)\,||f||_\Lambda\,, & \text{if } j \neq 0, \\ \mathcal{G}^{s,m}_{x,z}(h,h)\,||f||_\Lambda\,, & \text{if } j = 0. \end{cases}$$

there exists a z-independent $C_{\mathcal{F}} > 0$ such that

$$\sum_{|m|\le r} (\mathcal{G}^{s,m}_{x,v,z}(h,h))^2 \le C_{\mathcal{F}}\,||h||^2_{H^{s,r}_{x,v}}\,||h||^2_{H^{s,r}_\Lambda}\,,$$

$$\sum_{|m|\le r} (\mathcal{G}^{s,m}_{x,z}(h,h))^2 \le C_{\mathcal{F}}\,||h||^2_{H^{s,r}_x L^2_v}\,||h||^2_{H^{s,r}_\Lambda}\,.$$

$$||h||^2_{H^{s,r}_{x,v}} = \sum_{|m|\le r} ||\partial^m h||^2_{H^s_{x,v}}\,, \qquad ||h||^2_{H^{s,r}_\Lambda} = \sum_{|m|\le r} ||\partial^m h||^2_{H^s_\Lambda}$$

$$||h||^2_{H^{s,r}_x L^2_v} = \sum_{|m|\le r} ||\partial^m h||^2_{H^s_x L^2_v} \qquad ||h||^2_{H^{s,r}_\Lambda} = \sum_{|m|\le r} ||\partial^m h||^2_{H^s_\Lambda}$$

$$||h||^2_{H^s_\Lambda} = \sum_{|j|+|l|\le s} ||\partial^j_l h||^2_\Lambda$$

Convergence to global equilibrium (random initial data)

$$Assume\ ||h_{in}||_{H^s_{x,v}L^\infty_z} \le C_I,\ then$$

- For incompressible N-S scaling:

$$||h_\epsilon||_{H^{s;r}_{x,v}L^\infty_z} \le C_I\, e^{-\tau_s t}\,, \qquad ||h_\epsilon||_{H^s_{x,v}H^r_z} \le C_I\, e^{-\tau_s t}\,.$$

- For Euler (acoustic) scaling:

$$||h_\epsilon||_{\mathcal{H}^{s,r}_\epsilon L^\infty_z} \le \delta_s\, e^{-\epsilon\,\tau_s t}\,, \qquad ||h_\epsilon||_{\mathcal{H}^s_\epsilon H^r_z} \le \delta_s\, e^{-\epsilon\,\tau_s t}$$

A Lyapunov functional (following Briant)

$$|| \cdot ||^2_{\mathcal{H}^s_{\epsilon_\perp}} = \sum_{|j|+|l|\leq s,\, |j|\geq 1} b^{(s)}_{j,l}\, ||\partial^j_l (\mathbb{I} - \Pi_{\mathcal{L}}) \cdot ||^2_{L^2_{x,v}} + \sum_{|l|\leq s} \alpha^{(s)}_l\, ||\partial^0_l \cdot ||^2_{L^2_{x,v}}$$

$$+ \sum_{|l|\leq s,\, i, c_i(l)>0} \epsilon\, a^{(s)}_{i,l}\, \langle \partial^{\delta_i}_{l-\delta_i} \cdot, \partial^0_l \cdot \rangle_{L^2_{x,v}},$$

$$|| \cdot ||_{\mathcal{H}^s_{\epsilon_\perp}} \sim || \cdot ||_{H^s_{x,v}}$$

Random collision kernel

$$B(|v - v_*|, \cos\theta, z) = \phi(|v - v_*|)\, b(\cos\theta, z), \qquad \phi(\xi) = C_\phi\, \xi^\gamma, \text{ with } \gamma \in [0, 1],$$

$$\forall \eta \in [-1, 1], \qquad |b(\eta, z)| \le C_b,\ |\partial_\eta b(\eta, z)| \le C_b, \text{ and } |\partial_z^k b(\eta, z)| \le C_b^*,\ \forall\, 0 \le k \le r\,.$$

- Need to use a weighted Sobolev norm in random space as in Jin-Ma-J.G. Liu

$$||g||_{L^{2,r*}_{x,v}} := \sum_{m=0}^{r} \widetilde{C}_{m,r+1}\, ||\partial^m g||_{L^2_{x,v}},$$

- Similar decay rates can be obtained

gPC-SG approximation

$$f(t,x,v,z) \approx \sum_{|\mathbf{k}|=1}^{K} f_{\mathbf{k}}(t,x,v)\psi_{\mathbf{k}}(z) := f^K(t,x,v,z),$$

$$h(t,x,v,z) \approx \sum_{|\mathbf{k}|=1}^{K} h_{\mathbf{k}}(t,x,v)\psi_{\mathbf{k}}(z) := h^K(t,x,v,z).$$

- Perturbative setting

$$f_{\mathbf{k}} = \mathcal{M} + \epsilon\, M h_{\mathbf{k}}$$

$$\begin{cases} \partial_t h_{\mathbf{k}} + \frac{1}{\epsilon} v \cdot \nabla_x h_{\mathbf{k}} = \frac{1}{\epsilon^2}\mathcal{L}_{\mathbf{k}}(h^K) + \frac{1}{\epsilon}\mathcal{F}_{\mathbf{k}}(h^K, h^K), \\ h_{\mathbf{k}}(0,x,v) = h_{\mathbf{k}}^0(x,v), \qquad x \in \Omega \subset \mathbb{T}^d,\ v \in \mathbb{R}^d, \end{cases}$$

Spectral accuracy for gPC-sG

$$\partial_t f + v \cdot \nabla_x f = L(f) + \Gamma(f, f)$$

$$\partial_t f_k + v \cdot \nabla_x f_k = L(f_k) + \sum_{i,j=1}^{K} S_{ijk} \Gamma(f_i, f_j)$$

$$\partial_t \frac{1}{2} \sum_{k=1}^{K} \|f_k\|^2 \leq \boxed{-c \sum_{k=1}^{K} \|f_k\|^2} + \boxed{C \sum_{i,j,k=1}^{K} |S_{ijk}| \|f_i\| \|f_j\| \|f_k\|}$$

K good terms

K^3 bad terms

This will require the small data assumption
depending on K
Not good, since K is a numerical parameters

$$S_{ijk} = \int \phi_i \phi_j \phi_k \pi(z) \, \mathrm{d}z.$$

$$\partial_t f_k + v \cdot \nabla_x f_k = L(f_k) + \sum_{i,j=1}^{K} S_{ijk} \Gamma(f_i, f_j)$$

$$\partial_t \frac{1}{2} \sum_{k=1}^{K} \|k^q f_k\|^2 \leq -c \sum_{k=1}^{K} \|k^q f_k\|^2 + C \sum_{i,j,k=1}^{K} |S_{ijk}| \frac{k^q}{i^q j^q} \|i^q f_i\| \|j^q f_j\| \|k^q f_k\|$$

Weighted sum

gain

- With the technical assumption $|S_{ijk}| \leq k^p, \quad q > p + 2$, we showed that the bad terms can be controlled, with small data assumptions independent of K.
- This holds for most cases with bounded random domain

Shu-Jin (2017)

For small random perturbation

- Assumptions: z bounded

$$|\partial_z b| \leq O(\epsilon).$$

(following R. Shu-Jin) $||\psi_k||_{L^\infty} \leq C k^p, \qquad \forall\, k,$

Let $q > p + 2$, *define the energy* E^K *by*

$$E^K(t) = E^K_{s,q}(t) = \sum_{k=1}^{K} ||k^q h_k||^2_{H^s_{x,v}},$$

Regularity and exponential decay

(i) Under the incompressible Navier-Stokes scaling,

$$E^K(t) \leq \eta\, e^{-\tau t} \qquad ||h^K||_{H^s_{x,v} L^\infty_z} \leq \eta\, e^{-\tau t}$$

(ii) Under the acoustic scaling,

$$E^K(t) \leq \eta\, e^{-\epsilon \tau t}, \qquad ||h^K||_{H^s_{x,v} L^\infty_z} \leq \eta\, e^{-\epsilon\, \tau t}$$

gPC-SG error

Theorem 5.3. *Suppose the assumptions on the collision kernel and basis functions in Theorem 5.1 are satisfied, then*

(i) Under the incompressible Navier-Stokes scaling,

$$||h^e||_{H_z^s} \leq C_e \frac{e^{-\lambda t}}{K^r}, \tag{5.22}$$

(ii) Under the acoustic scaling,

$$||h^e||_{H_z^s} \leq C_e \frac{e^{-\epsilon\lambda t}}{K^r}, \tag{5.23}$$

with the constants C_e, $\lambda > 0$ independent of K and ϵ.

$$||h(x,v,\cdot)||_{H_z^s}^2 = \int_{I_z} ||h||_{H_{x,v}^s}^2 \, \pi(z)\, dz \,.$$

What about variance?

$$f(t, \mathbf{v}, z) = \sum_{k=0}^{\infty} \hat{f}_k(t, \mathbf{v}) \Phi_k(z),$$

$$\mathbb{E}[f] \approx f_{\mathbf{0}}, \quad \mathrm{Var}[f] \approx \sum_{|\mathbf{k}|=1}^{K} f_{\mathbf{k}}^2,$$

- By Parseval's identity, our results directly imply the same accuracy and decay rate for the variance, since the global equilibrium is deterministic

A general framework

- This framework works for general linear and nonlinear collisional kinetic equations
- Linear and nonlinear Boltzmann, Landau, relaxation-type quantum Boltzmann, etc.
- Also works for non-collision kinetic equation: Vlasov-Poisson-Fokker-Planck system (J-Y. Zhu)

Vlasov-Poisson-Fokker-Planck system (J., & Y. Zhu, SIMA '18)

$$\begin{cases} \partial_t f + \frac{1}{\delta}\mathbf{v}\cdot\nabla_{\mathbf{x}} f - \frac{1}{\epsilon}\nabla_{\mathbf{x}}\phi\cdot\nabla_{\mathbf{v}} f = \frac{1}{\delta\epsilon}\mathcal{F}f, \\ -\Delta_{\mathbf{x}}\phi = \rho - 1, \quad t>0, \quad \mathbf{x}\in\Omega\subset\mathbb{R}^N,\ \mathbf{v}\in\mathbb{R}^N,\ \mathbf{z}\in I_{\mathbf{z}}, \end{cases}$$

$$\mathcal{F}f = \nabla\cdot\left(M\nabla\left(\frac{f}{M}\right)\right) \qquad M = \frac{1}{(2\pi)^{\frac{N}{2}}}e^{-\frac{|\mathbf{v}|^2}{2}}$$

$$f(0,\mathbf{x},\mathbf{v},\mathbf{z}) = f^0(\mathbf{x},\mathbf{v},\mathbf{z}), \quad \mathbf{x}\in\Omega,\ \mathbf{v}\in\mathbb{R}^N,\ \mathbf{z}\in I_{\mathbf{z}}.$$

Asymptotic regimes

- High field regime: $\delta = 1$

$$\begin{cases} \partial_t \rho + \nabla \cdot (\rho \nabla_{\mathbf{x}} \phi) = 0, \\ -\triangle_{\mathbf{x}} \phi = \rho - 1, \end{cases}$$

- Parabolic regime: $\delta = \epsilon$

$$\begin{cases} \partial_t \rho - \nabla \cdot (\nabla_{\mathbf{x}} \rho - \rho \nabla_{\mathbf{x}} \phi) = 0, \\ -\triangle_{\mathbf{x}} \phi = \rho - 1. \end{cases}$$

Norms and energies

$$d\mu = d\mu(\mathbf{x}, \mathbf{v}, \mathbf{z}) = \pi(\mathbf{z}) d\mathbf{x} d\mathbf{v} d\mathbf{z}.$$

$$\langle f, g \rangle = \int_\Omega \int_{\mathbb{R}^N} \int_{I_\mathbf{z}} f g \, d\mu(\mathbf{x}, \mathbf{v}, \mathbf{z}), \quad \text{or,} \quad \langle \rho, j \rangle = \int_\Omega \int_{I_\mathbf{z}} \rho j \, d\mu(\mathbf{x}, \mathbf{z}), \quad \text{with norm } \|f\|^2 = \langle f, f \rangle.$$

$$h = \frac{f - M}{\sqrt{M}}, \quad \sigma = \int_\mathbb{R} h \sqrt{M} \, dv, \quad u = \int_\mathbb{R} h \, v \sqrt{M} \, dv,$$

$$\Pi_1 h = \sigma \sqrt{M}, \quad \Pi_2 h = u v \sqrt{M}, \quad \Pi h = \Pi_1 h + \Pi_2 h.$$

$$\|f\|_{H^m}^2 = \sum_{l=0}^m \left\|\partial_z^l f\right\|^2$$

hypocoercivity

- Linearized Fokker-Planck operator

$$\mathcal{L}h = \frac{1}{\sqrt{M}}\mathcal{F}\left(M + \sqrt{M}h\right) = \frac{1}{\sqrt{M}}\partial_v\left(M\partial_v\left(\frac{h}{\sqrt{M}}\right)\right)$$

- Duan-Fornaiser-Toscani ‘10

(a) $-\langle \mathcal{L}h, h\rangle = -\langle L(1-\Pi)h, (1-\Pi)h\rangle + \|u\|^2$;

(b) $-\langle \mathcal{L}(1-\Pi)h, (1-\Pi)h\rangle = \|\partial_v(1-\Pi)h\|^2 + \frac{1}{4}\|v(1-\Pi)h\|^2 - \frac{1}{2}\|(1-\Pi)h\|^2$;

(c) $-\langle \mathcal{L}(1-\Pi)h, (1-\Pi)h\rangle \geq \|(1-\Pi)h\|^2$;

(d) *There exists a constant* $\lambda_0 > 0$, *such that the following hypocoercivity holds,*

$$-\langle \mathcal{L}h, h\rangle \geq \lambda_0 \|(1-\Pi)h\|_v^2 + \|u\|^2,$$

and the largest $\lambda_0 = \frac{1}{7}$ *in one dimension.*

- Energy terms:
  - $E_h^m = \|h\|_{H^m}^2 + \|\partial_x h\|_{H^{m-1}}^2$, $E_\phi^m = \|\partial_x \phi\|_{H^m}^2 + \left\|\partial_x^2 \phi\right\|_{H^{m-1}}^2$;
- Dissipation terms:
  - $D_h^m = \|(1-\Pi)h\|_{H^m}^2 + \|(1-\Pi)\partial_x h\|_{H^{m-1}}^2$, $D_\phi^m = E_\phi^m$,
  - $D_u^m = \|u\|_{H^m}^2 + \|\partial_x u\|_{H^{m-1}}^2$, $D_\sigma^m = \|\sigma\|_{H^m}^2 + \|\partial_x \sigma\|_{H^{m-1}}^2$.

Uniform regularity and convergence to the global equilibrium

Theorem 3.4. *For the high field regime ($\delta = 1$), if*

$$E_h^m(0) + \frac{1}{\epsilon^2} E_\phi^m(0) \le \frac{2\lambda_0^3}{(80AC_1)^2},$$

then,

$$E_h^m(t) + \frac{1}{\epsilon^2} E_\phi^m(t) \le \frac{3}{\lambda_0} e^{-\frac{\lambda_0}{3} t} \left(E_h^m(0) + \frac{1}{\epsilon^2} E_\phi^m(0) \right).$$

For the parabolic regime ($\delta = \epsilon$), if

$$E_h^m(0) + \frac{1}{\epsilon} E_\phi^m(0) \le \frac{2\lambda_0^3}{(80AC_1)^2},$$

then,

$$E_h^m(t) + \frac{1}{\epsilon} E_\phi^m(t) \le \frac{3}{\lambda_0} e^{-\frac{\lambda_0}{2} t} \left(E_h^m(0) + \frac{1}{\epsilon} E_\phi^m(0) \right).$$

Here A and C_1 are the same as in Lemma 3.2.

- Initial data larger than those obtained by Hwang-Jang ('13)

$$E_h^m(0) + \frac{1}{\epsilon^2} E_\phi^m(0) \lesssim O(\epsilon)$$

gPC-SG for many different kinetic equations

- Boltzmann: a fast algorithm for collision operator (J. Hu-Jin, JCP '16), sparse grid for high dimensional random space (J. Hu-Jin-R. Shu '16): initial regularity in the random space is preserved in time; but not clear whether it is unifornly stable in the fluid dynamics limit (s-AP?): gPC-SG for nonlinear hyperbolic system is not globally hpperbolic! (APUQ is open)
- Landau equation (J. Hu-Jin-R. Shu, '16): not able to prove regularity result in the random space (APUQ is open)
- Semiconductor Boltzmann-drift diffusion limit (uniform regularity. Jin-L. Liu MMS 17, Uniform spectral convergence is open)
- Radiative heat transfer (APUQ OK: Jin-H. Lu JCP'17): proof of regularity in random space for linearized problem (nonlinear? Open)
- Kinetic-incompresssible fluid couple models for disperse two phase flow: (efficient algorithm in multi-D: Jin-Shu.)