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Connecting Caplets And Swaptions In The Displaced Diffusion Market Model

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Declaration

The work contained in this thesis is my own work unless otherwise stated.

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Date: 7th September 2021

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Abstract

With the evolution of the economies and the unforeseeable changes and crisis that happened over the recent decades, negative interest rates have become more and more relevant to the financial industry and economies nowadays. Interest rate models that have once been viewed as disadvantaged due to the fact that they can model negative rates have now become popular and are acting as useful foundations of interest rate modelling.

Taking the well-established market models, the Libor Market Model (LMM) and the Swap Markt Model (SMM) as the starting point, in this project we explored the effectiveness of an extension of LMM and SMM, the Displaced Diffusion Market Model (DDM). DDM has the advantage over LMM and SMM of being capable of negative rates modelling. DDM can take negative forward rates and swap rates up to the value of the chosen shift in the model. Our testing approach is comparing the close form analytical swaption pricing formula result coming from the Displaced Diffusion Swap Model (extension of SMM) with the simulated swaption pricing results from the Displaced Diffusion Forward Model (extension of LMM) with various settlement dates, maturity dates and strike prices. The results are promising and sound, demonstrating a strong connection between the DDF and DDS model. The results also testified the effectiveness of the Displaced Diffusion Model to model interest rates.

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1 Motivation

1.1 Negative rates

1.1.1 Monetary Policy

Monetary policy is a macroeconomic tool to influence nations' economic activities and is essential for saving economies from crisis. It is always used to facilitate the goal from the government and most of the time in conjunction with fiscal policy to make the changes in fiscal policy more effective. Fiscal policy refers to the government policies regarding taxation, government spending and budget management. Both fiscal and monetary policy were once controlled together by government, however, many nations made the decision to separate monetary policy from government entities and let central banks take control of it. Many experience have also confirmed that central banks controlling monetary policy independently leads to better economics outcomes because of the minimisation of political influence in decision making.

For example, in the United States, Federal Reserve has the responsibility to monitor monetary policy to achieve desirable economics goals. The goals mainly include inflation goals and unemployment goals, and are usually set by the Congress but the government has no impact on how the Federal Reserve actually implements the policy (2).

Monitoring interest rates and the quantitative easing tools are two main aspects of monetary policy. The former tool is used to control inflation, usually to keep it within a certain target range; the later tool is to control money circulation in the economy and the level of liquidity in the economy. An expansionary monetary policy refers to a decrease in interest rate and an action of quantitative easing. This is due to the fact that when interest rate decreases, there is less incentive for economic agents to deposit their money in a bank. Instead, they may alternatively choose to spend their money in the economy. This in turn boosts GDP, lowers unemployment rate, and eventually increase inflation in the economy, preferably to the desired target level of the government and the central bank. In addition, quantitative easing actions inject liquidity into the economy potentially through purchasing corporate bonds, and this can lead to an increased level of money circulation. This also has the same effect on the economy as a decrease in interest rate discussed previously. Contractionary Monetary Policy works in the opposite way as the expansionary Monetary Policy. Monetary policy in the history had helped nations overcome crisis, for instance, Fed implemented an aggressively monetary policy to lead the nation out of Great Depression in the 30s.

1.1.2 The role of interest rates in the economy

Interest rate can be considered as the premium to be paid to the lender from the borrower to compensate the risk of lending and the time value of money. In particular, if the borrower is considered 'risky', then this compensation is higher, i.e. the borrower will be charged a larger interest rate compared to a less risky counterparty (3).

Interest rates are usually in percentage terms. For example, the interest rates on a loan is quoted as a percentage of the loan principle. The earnings from banks or credit unions on one's deposit account can also be considered in percentage terms, i.e. a proportion of the money one deposited with the bank. Principles of interest rates can be cash or other forms of asset which are tangible, such as vehicles, land and houses (3).

There are many kinds of interest rates, and the two main ones are simple interest rate and compound interest rate. Simple interest rate is calculated as follows: *principle* x *interest rate* x *time*. Time is in the form of 'fraction of year'. Compound interest rate is calculated as follows: *principle* x $[(1 + interest)^n - 1]$.

The evolution of interest rates is an interesting history. Interest rate typically can be viewed as two categories: long-term rates and short-term rates. Bank rates are very short-term loans since it is the rate at which central bank lends to commercial banks. According to Bank of England's database on official bank rate history, the bank rate had long been above 10% for over 15 years in the last century. The official bank rate was at 10.5% on 17^{th} February 1975 and it was above 10% until 4^{th} September 1991 at 10.38%. The bank rate subsequently went down all the way from above 10% to the current rate of 0.1% (1).

To get a clearer idea on the evolution of interest rates, I explored the evolutionary trend of interest rate on OECD Data source. Below are two insightful plots produced by OECD: short-term rate evolution and short-term rate forecast. The timeline for the former plot is Jan 2007 - July 2021 and for the later is Q4 2021 - Q4 2022.

We discovered that the evolution of interest rates are similar for almost all nations, although the actual interest rates differ from nation to nation. We selected three nations for illustration purposes: Japan, United Kingdom and United States. In Figure 1.1 it shows the general trend of short-term interest: it decreased from pre-2008 level to the current level which is around 0%. Japan (blue) has a low interest rate throughout this selected period, and it has negative interest currently (in August 2021) at around -0.1%. United Kingdom (Red) has a base rate of 0.1%, and it has dropped sharply from around 6% in 2007 and 2008. Although interesting rates of United States has a peak in interest rate around 2019, the overall trend is similar to the other two. From here we can see

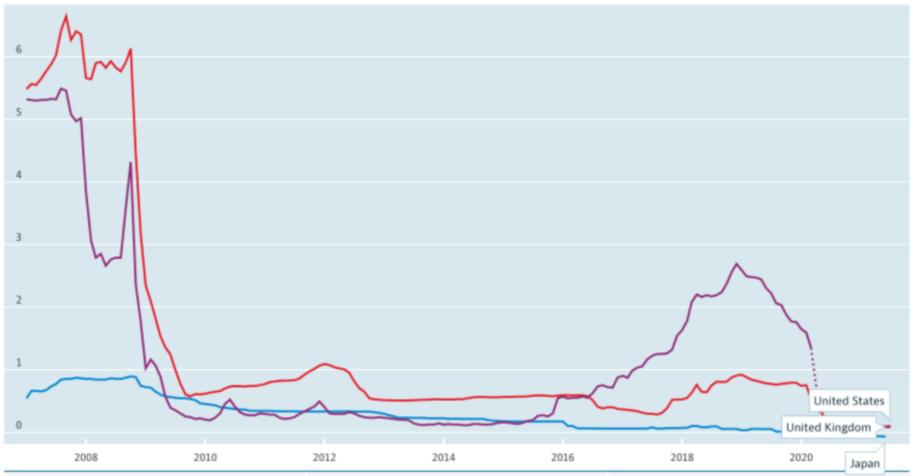


Figure 1.1: Short term interest rate from Jan 2007 to July 2021.
Red: United Kingdom; Purple: United States; Blue: Japan.
Source: OECD (2021), Short-term interest rates (indicator). doi: 10.1787/2cc37d77-en (Accessed on 07 August 2021)(4)

that the economies now have much lower interest rates, and even negative rates.

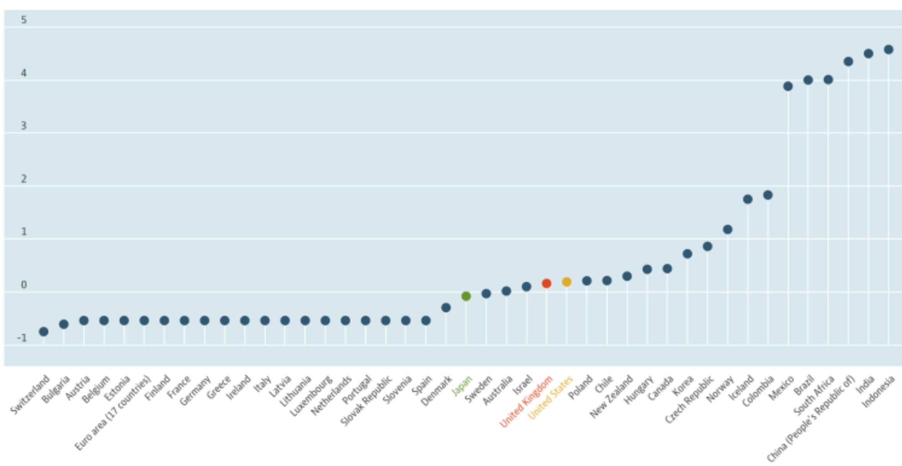


Figure 1.2: Short term interest rate forecast from Q4 2021 to Q4 2022
Source: OECD (2021), Short-term interest rates forecast (indicator). doi: 10.1787/9446e151-en (Accessed on 07 August 2021)(5)

In addition, according to OECD data on short-term rate forecast, as shown in Figure 1.2, more than 22 countries and nations will have a negative interest rate during Q4 of 2021 and Q4 of 2022. Couple of decades back, people view interest rate models that allow negative rates to be ineffective, since economies back then did not have a negative interest rate or any sign of having one in the future. However, nowadays, having an interest model that can properly capture the current interest rate level and negative rates is important. There are many existing models which does not have this ability of capturing a negative interest rate: interest rates are typically bounded below at zero in these models. Again, this is because these models are created a while back when interest rates were significantly different from zero.

On the other hand, an interest rate models that allows negative interest rates does not necessarily mean that it is effective and useful – this motivates us to explore the effectiveness of one of the interest rate models that allows negative rates and test its accuracy of pricing derivatives, in this case, swaptions.

1.2 Derivatives and derivatives pricing

There are many financial instruments and derivatives in the financial world to allow investors to choose from based on their taste of the economy, risk appetite as well as investment goals. Some of the most common derivatives are vanilla options, forward contract, future contract and swaps. In addition, these popular derivatives are mostly written on underlyings including commodities, currencies, interest rates, stocks and bonds ([7]).

In comparison to investing in the actual underlyings, derivatives give investors an opportunity to 'leverage' up their positions with their capital, and be exposed to a better porfolio PnL movement when there are changes in the underlying prices. Apart from this beneficial feature, derivatives are also a great hedging tool for corporations and investors to protect themselves from large losses. For example, airline companies may enter a call option for fuels to protect themselves from a sudden rise of fuel prices; an investor who holds large amount of Apple stock and speculates that the stock price may go down can by put option on Apple stock to hedge his portfolio position. Lastly, derivatives provide investors a chance to enter a position with a non-tradable underlying: for instance, buying an option that is written on carbon emission or weather.

In this thesis, we focus on one type of derivatives, namely swaptions. We are motivated to figure out how Displaced Diffusion Market Model can connect caplet market (DDF model) and swaption market (DDS model) and if it can price swaptions reasonably well.

Below is a brief overview of three important and popular derivatives in the financial world.

- **Swaps** are derivatives that allows two parties to exchange cash flows of the financial instruments they are holding. In particular, interest rate swaps are derivatives for two parties to exchange the interest rates (cash flows) that one of the parties is paying between two differently indexed legs with a fixed swap rate, starting from a agreed future point in time.

For example, the two parties of the Interest Rate Swap (IRS) agree at time t that they are going to exchange cash flows at a swap rate K on future dates $T_{\alpha+1}, T_{\alpha+2}, \ldots, T_{\beta}$, with $T_i - T_{i-1} = \tau_i$. Then one party is going to pay the 'fixed leg payment' $\tau_i K$ and the other party is paying the 'floating rate payment' which is $\tau_i L(T_{i-1}, T_i)$ at time T_i. After taking expectation on the floating rate payment, this payment becomes $\mathbb{E}^{\alpha}\Big(\tau_i L(T_{i-1}, T_i)\Big) = \tau_i F(T_{\alpha}, T_{i-1}, T_i)$. Note that in this case the first reset date will be T_{α}.

From perspective of the party that is receiving the fixed leg payment, the swap is called the 'receiver swap'. It is called 'payer swap' for the other party.

We learnt in the lectures (10) that there are three possible formulas for the forward swap rate:

$$\begin{aligned} S_{\alpha,\beta} &= \frac{P(t,T_{\alpha}) - P(t,T_{\beta})}{\sum_{i=\alpha+1}^{\beta} \tau_i P(t,T_i)} \\ S_{\alpha,\beta} &= \sum_{i=\alpha+1}^{\beta} w_i(t) F_i(t), \quad w_i(t) = \frac{\tau_i P(t,T_i)}{\sum_{j=\alpha+1}^{\beta} \tau_j P(t,T_j)} \\ S_{\alpha,\beta} &= \frac{1 - \prod_{i=\alpha+1}^{\beta} \frac{1}{1+\tau_j F_j(t)}}{\sum_{i=\alpha+1}^{\beta} \tau_i \prod_{i=\alpha+1}^{\beta} \frac{1}{1+\tau_j F_j(t)}} \end{aligned} \tag{1.1}$$

Swap spread refers to the difference between the fixed rate in the swap and the corresponding floating rate in the swap. One interesting fact was pointed out in one of the recent studies by N Boyarchenko, P Gupta, N Steele and J Yen: there could be a negative swap spread in some trades. A negative swap spread could lead to arbitrage opportunities if interest rate volatility is the only source of risk in the trade (8). According to N Boyarchenko, P Gupta, N Steele and J Yen (8), there are many other factors that affect swap spread, such as counterparty risk and regulatory changes.

- **Caps** have the following discounted payoff:

$$\sum_{i=\alpha+1}^{\beta} D(t,T_i)\tau_i\Big(L(T_{i-1},T_i) - K\Big)^{+} = \sum_{i=\alpha+1}^{\beta} D(t,T_i)\tau_i\Big(F_i(T_{i-1}) - K\Big)^{+} \tag{1.2}$$

Relating to the swaps that are described above, caps can be seen as a payer IRS where each exchange payment is executed only if it has positive value from the long party's perspective (10). Similarly, floors follow a similar logic but instead of mimicking the payer IRS, it is the positive payoffs of the receiver IRS.

Caplet refers to each of the individual components in a cap, i.e. the payoff is $D(t,T_i)\tau_i\Big(L(T_{i-1},T_i)-K\Big)^+ = D(t,T_i)\tau_i\Big(F_i(T_{i-1})-K\Big)^+$ for each of them. Therefore, if we know the whole distribution of future rates, we can evaluate each caplet separately and add all the corresponding values together to obtain the final cap price.

- **Swaptions** are options to enter a payer or receiver IRS at a pre-agreed future point in time. In the payer swaption case, the payoff of this derivative is:

$$\begin{aligned} &D(t,T_\alpha)C_{\alpha,\beta}(T_\alpha)(S_{\alpha,\beta}(T_\alpha)-K)^+ \\ &\qquad = D(t,T_\alpha)\left(\sum_{i=\alpha+1}^{\beta} P(T_\alpha,T_i)\tau_i\left(F(T_\alpha,T_{i-1},T_i)-K\right)\right)^+ \end{aligned} \tag{1.3}$$

  However, we can see from the above formula that the swaption price is not additive: it cannot to decomposed in to separate parts and sum them all together. Instead, in order to accurately price swaptions one needs to obtain the correlation between future rates (10).

1.3 Choosing interest rate models

We have discussed in the last section the importance of the dynamics of future rates when pricing derivatives like caplets and swaptions. We now move on to discuss some possible choices of interest rate model that we can use to model the dynamics of those future rates.

There are a few choices that could be considered.

- **Firstly, we can choose to model short rates r_t.**
  Short rates are rates at which one can borrow or lend money for a very short period of time. This is a valid modelling approach due to the fact that the zero coupon bond curve can be derived completely from the characteristics of the dynamic of the short rate r_t, i.e. the zero coupon curve formula is: $T \to P(t,T) = \mathbb{E}_t^{\mathbb{Q}} \exp\left(-\int_t^T r_s ds\right)$ that contains r_t as an input. Some famous short rate models include

  1. Endogenous Models
     (a) **Vasicek Model (1977)**: $dx_t = k(\theta - x_t)dt + \sigma dW_t, \quad \alpha = (k,\theta,\sigma)$;
     (b) **Cox-Ingersoll-Ross model (CIR,1985)**: $dx_t = k(\theta - x_t)dt + \sigma\sqrt{x_t}dW_t, \quad \alpha = (k,\theta,\sigma), \quad 2k\theta > \sigma^2$.
  2. Exogenous Models
     (a) **Hull-White (Extended Vasicek)**: $dx_t = k(\theta(t) - x_t)dt + \sigma dW_t$;

(b) **Hull-White (Extended CIR)**: $dx_t = k(\theta(t) - x_t)dt + \sigma\sqrt{x_t}dW_t$.

In general, Endogenous Models take zero curve and model volatility as outputs, whereas exogenous models take the zero curve as an input of the model . The calibration of endogeous models are usually poor since it does not produce the inverted shape of the zero curve as an output. On the other hand, exogenous model have a perfect inverted zero curve since it is given as an input of the model. Therefore, exogenous models give better fit of market rates (10). As mentioned in lectures (10), the basic strategy to transform an endogenous model into an exogenous model is to turn the parameters in the endogenous models into time-varying parameters. For example, in the example models given above, we can see that the Extended Vasicek model uses a time-varying parameter $\theta(t)$ instead of the fixed parameter used in the Vasicek model θ.

Despite the high level of analytical tractability , short rate models have several disadvantages. Some disadvantages of the Vasicek Model (1977) include that it assumes constant volatility, thus there is no term structure of volatility; and it assumes short rates to be perfectly correlated (13). The CIR model (and also the extended CIR model) was also challenged by Pearson and Sun (1994) (22) for its large errors in pricing of the Treasury market (bills). In general, short rate models give poor calibration results for large number of caplets and swaptions, although this can be improved by using multi-factor models for r. In addition, market views and quotes cannot be fully and accurately expressed by these short rate models due to the limited number of the model parameters (10). More importantly, short rate models lack consistency in the pricing of some basic financial derivatives like caplets and swaptions with the universally used prices derived from Black's formula (10). Thus, we want to consider several other models for interest rates.

- **We can alternatively choose to model forward rates $F(t, T, S)$.**
  The most significant practical advantage of Market Models compared to short-rate models is that they specify specific arbitrage-free dynamics on forward rates and swap rates, although those models are of high level of computational difficulty due to high dimensionality of the dynamics (14).

  1. **Libor Market Model (LMM)**
     LMM is the most popular market model in the academia and was developed by Brace, Gatarek and Musiela (1997) (9), Musiela and Rutkowski (1997) (21), Goldys (1997) (15) and Miltersen, Jamshidian (1997) (18), Sandmann and Sondermann (1997) (20). LMM is also called 'log-normal forward-Libor or Linear model'. This is due to the fact that the dynamics of F follows a geometric Brownian Motion under the corresponding measure $\mathbb{Q}^k$, which is

the forward arbitrage free measure (9). The dynamic of forward rates is:

$$dF_k(t) = \sigma_k(t)F_k(t)dW_k(t), \quad F_k(0), \quad \mathbb{Q}^k \tag{1.4}$$

The advantage of LMM is that it is consistent with the Black formula for caplets.

However, according to many recent studies and market data, it was found that the basic assumption of the 'log-normality' of forward rates of LMM is violated due to the volatility skew that is observed by the market of caps/floors and swaps (6). Moreover, the 'log-normality' characteristics assumed in LMM is not at all necessary due to the fact that forward rates are not tradable in the market - the dynamics of forward rate do not have to have zero drift under numeraire pairs (17). Another main disadvantage of LMM is that since $F_k's$ are log-normal, they cannot be negative.

There are some market models under which the dynamics of forward rates are not geometric Brownian Motion martingale. Apart from the Displaced Diffusion Market Model that we are going to discuss shortly, there is one other useful model that is worth mentioning - if we take a Brownian Motion martingale instead, we will arrive at the Bachelier market model (BMM) which has the following dynamics:

$$dF_k(t) = \sigma_k(t)dW_k(t), \quad F_k(0), \quad \mathbb{Q}^k \tag{1.5}$$

This model implies normally distributed forward rates that can take negative values.

2. **Swap Market Model**

   As mentioned above, LMM is consistent with Black's formula when pricing caplets and floorlets. On the other hand, swap market model (SMM) is consistent with Black's formula for swaptions. Although LMM is, up until now, the most popular market model and is studied the most, some authors hold an alternative view about the usefulness of LMM: for example, Huang and Scaillet (2003) claimed that Swap market model (SMM) is a theoretically and practically better model (16). Similar to LMM, SMM also assumes log-normality of the swap rates, and the dynamics of swap rates are described as:

$$dS_{\alpha,\beta}(t) = \sigma_{\alpha,\beta}(t)S_{\alpha,\beta}(t)dW_t^{\alpha,\beta}, \quad \mathbb{Q}^{\alpha,\beta} \quad \text{(SMM)} \tag{1.6}$$

   A more comprehensive introduction of the SMM is written in Section 4.1.

3. **Displaced Diffusion Market Model**

Displace diffusion model is useful due to the fact that it allows us to obtain models that have properties ranging from Libor market model (LMM) to Bachelier market model (BMM) (10).

Displace diffusion model is decribed as follows (Note that W_i is a standard brownian motion under measure $\mathbb{Q}^i$):

$$dF_i(t) = \sigma_i(t)(F_i(t) - \alpha_F)dW_i(t), \quad \text{where } \alpha_F \text{ is a constant deterministic shift.} \tag{1.7}$$

When taking limits on F_i, we can recover the LMM and BMM.

(a) For very small F_i, i.e. $F_i \to 0$, (1.7) becomes

$$dF_i(t) = \sigma_2(t)\left(F_i(t) - \alpha_F\right)dW_i(t) \approx -\sigma_2(t)\alpha_F dW_i(t) \tag{1.8}$$

which corresponds to BMM.

(b) And when F_i is very large and positive, i.e. $F_i \to +\infty$, (1.7) becomes

$$\begin{aligned} dF_i(t) &= \sigma_i(t)\left(F_i(t) - \alpha_F\right)dW_i(t) \\ &= \sigma_i(t)F_i(t)\left(1 - \frac{\alpha_F}{F_i(t)}\right)dW_i(t) \\ &\approx \sigma_i(t)F_i(t)dW_i(t) \end{aligned} \tag{1.9}$$

which corresponds to LMM. Please note that here we have used the fact that $\frac{\alpha}{F_i(t)} \to 0$ as $F_i \to +\infty$.

This model allows negative rates by choosing $\alpha_F < 0$. In fact, the model can attain negative rates down to level α_F. Since α_F is a shift on the forward rate, intuitively it should be not be too big in absolute value. One can interpret α_F as a constant real number which is bigger than -0.5.

1.4 Structure of the thesis

The thesis is written in the following structure: Chapter 2 summarises all the useful notations that appear in the subsequent Chapters. Chapter 3 is the derivation of forward rate dynamics under different measures in the Displaced Diffusion Forward model (DDF). Chapter 4 derives the analytical formula of swaptions in the Displaced Diffusion Swaption model (DDS). Chapter 5 is the procedure of Monte Carlos simulation of the DDF model and it describes how the DDF model (aka.Caplet model) can be connected with the DDS model (aka.Swaption model). The pricing results of swaptions under DDF and DDS models are summerised in Chapter 6, and the analysis of the discrepancies between the results are written in Chapter 7. **This thesis is written based heavily on the concepts in lecture notes (10) written by Prof. Damiano Brigo.**

2 Notations

In this section some useful notations are summarised for convenience.

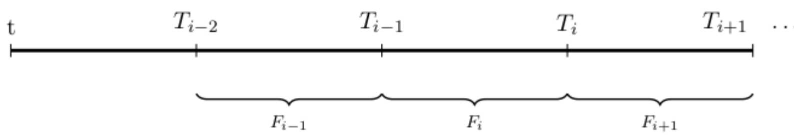
- **Forward rate** $F_k(t) = F(t, T_{k-1}, T_k)$ describes the forward rate between time $T_{k-1} \to T_k$ seen at time t, with $F(t, T_{k-1}, T_k) = \frac{1}{T_k - T_{k-1}}(\frac{P(t,T_{k-1})}{P(t,T_k)} - 1)$.
- **Volatilities in different models** $\sigma_{\alpha,\beta}$ refers to the volatility in the Displaced Diffusion model for swaption, and σ^F refers to the volatility in the Displaced Diffusion model for Forward Rates.
- **Measure** $\mathbb{Q}^k$ denotes the T_k-forward measure.
- **Standard brownian motion** W_k^i refers to the standard brownian motion under measure $\mathbb{Q}^i$ at time T_k.
- **Chosen constant shift** α_F denotes the chosen constant shift in the forward rate Displaced Diffusion model and α_S denotes the chosen constant shift in the swap rate Displaced Diffusion model
- **Different Models** DDF model represents the Displaced Diffusion Forward model and DDS model represents the Displaced Diffusion Swap model.

3 Deriving the Displaced Diffusion Forward Model: Dynamics of forward rates under different measures

Similar to the LMM, we expect the Displaced Diffusion Forward Model (DDF Model) introduced in this chapter to be able to price caplets/floorlets accurately as in LMM. This is because DDF Model can produce a closed-form formula for caplets/floorlets prices consistent when the Black's formula just as LMM. As the focus of this thesis is on pricing swaptions, we are not going to go into details of the caplet pricing under the DDF Model.

3.1 Introduction to Forward Rates under different measures

Timeline Illustration of the dynamic of forward rates:



As illustrated in the timeline above, when using measure $\mathbb{Q}^i$, the forward rate F_i is a martingale. This means that between time T_{i-1} and T_i, we have $dF_i(t) = \sigma_i(t)\,(F_i(t) - \alpha_F)\,dW_i(t)$, where W_i is a standard brownian motion. However, the forward rate F is not a martingale in any other cases under this measure. Instead, we have the following dynamics

$$dF_{i-1} = \mu^i_{i-1}\sigma_{i-1}\,(F_{i-1}(t) - \alpha_F)\,dW^i_{i-1} \tag{3.1}$$

$$dF_{i+1} = \mu^i_{i+1}\sigma_{i+1}\,(F_{i+1}(t) - \alpha_F)\,dW^i_{i+1} \tag{3.2}$$

for time $T_{i-2} \to T_{i-1}$ and $T_i \to T_{i+1}$ respectively. In summary, we have a drift term μ in the dynamic of forward rate F when the forward rate F_k is not measured under $\mathbb{Q}^k$. Thus, we want to use the change of numeraire method to work out the dynamic of F_k under measure $\mathbb{Q}^i$ for $k \neq i$. For simplicity, we assume the shift is the same for all forward rates, and denote the shift by α_F. Of course, this assumption is a very strong one and may cause biases or inaccuracies in the model results.

3.2 Change of numeraire method to determine the dynamic of F_k

Now we try to work out the dynamics of F_k under T_i-forward measure $\mathbb{Q}^i$. As stated above, the dynamics under the T_k-forward measure has no drift. Therefore, all we need to do is to figure out the dynamic of F_k for $k \neq i$. Here we follow the steps described in Lecture Notes (10).

Suppose first that $i < k$**.** We use the change of numeraire toolkit which provides a formula to relate standard Brownian Motions under two different numeraires (say U

and S). We have the following formula:

$$dZ_t^S = dZ_t^U - \rho \left(\frac{DC(S)}{S_t} - \frac{DC(U)}{U_t} \right)^T dt \tag{3.3}$$

Note that 'DC' stands for 'Vector Diffusion Coefficient'. For clarity, we can view DC as a linear operator of the diffusion part of a stochastic process. For example, if we have a process X_t which has the following dynamic in differentiated form

$$dX_t = adt + \boldsymbol{v} dZ_t \tag{3.4}$$

where a is some function of X or some constant, then the row vector $\boldsymbol{v}$ is the 'DC' term of X_t. In other words, we can view dZ_t as a column vector of Brownian Motion and DC is the linear operator that attach the corresponding diffusion term to each dZ_t^i (10). For instance, if at time one we have $dX_1 = \sigma_1 X_1 dZ_1^1$ as the column vector dX_1, then the DC term is

$$DC(X_1) = [\sigma_1 X_1, 0, 0, \ldots, 0] = \sigma_1 F_1 \boldsymbol{e_1} \tag{3.5}$$

With the above, now we see that (3.3) can be re-written as

$$\begin{aligned} dZ_t^S &= dZ_t^U - \rho(DC(\ln(\frac{S}{U})))^T dt \\ \iff \frac{DC(S)}{S_t} - \frac{DC(U)}{U_t} &= DC(\ln(S)) - DC(\ln(U)) \\ &= DC(\ln(S) - \ln(U)) \\ &= DC(\ln(\frac{S}{U})) \end{aligned} \tag{3.6}$$

Here we use $S = P(\cdot, T_k)$ and $U = P(\cdot, T_i)$ to obtain the following relationship:

$$dZ_t^k = dZ_t^i - \rho DC \left(\ln \left(\frac{P(\cdot, T_k)}{P(\cdot, T_i)} \right) \right)^T dt \tag{3.7}$$

Now we can see that

$$\begin{aligned} \ln \left(\frac{P(t, T_k)}{P(t, T_i)} \right) &= \ln \left(\frac{P(t, T_k)}{P(t, T_{k-1})} \frac{P(t, T_{k-1})}{P(t, T_{k-2})} \cdots \frac{P(t, T_{i+1})}{P(t, T_i)} \right) \\ &= \ln \left(\frac{1}{1 + \tau_k F_k(t)} \cdot \frac{1}{1 + \tau_{k-1} F_{k-1}(t)} \cdots \frac{1}{1 + \tau_{i+1} F_{i+1}(t)} \right) \\ &= \ln \left(\frac{1}{\prod_{j=i+1}^{k} (1 + \tau_j F_j(t))} \right) \\ &= - \sum_{j=i+1}^{k} \ln(1 + \tau_j F_j(t)) \end{aligned} \tag{3.8}$$

where $\tau_k = T_k - T_{k-1}$ denotes the time lag between the two bond maturities. In the second equality, we used the definition of forward rate $F(t,T,S) = F_S(t) = \frac{1}{S-T}\left(\frac{P(t,T)}{P(t,S)} - 1\right) \Rightarrow \frac{P(t,T)}{P(t,S)} = (S-T)F_S(T)+1$ for $T < S$. Note that this relationship between forward rates and zero-coupon bond prices is model-independent. Therefore, by the linearity of 'DC', we have the following result

$$\begin{aligned} DC\ln\left(\frac{P(t,T_k)}{P(t,T_i)}\right) &= -DC\sum_{j=i+1}^{k}\ln(1+\tau_j F_j(t)) \\ &= -\sum_{j=i+1}^{k} DC\ln(1+\tau_j F_j(t)) \\ &= -\sum_{j=i+1}^{k}\frac{DC(1+\tau_j F_j(t))}{1+\tau_j F_j(t)} \\ &= -\sum_{j=i+1}^{k}\tau_j\frac{DC(F_j(t))}{1+\tau_j F_j(t)} \\ &= -\sum_{j=i+1}^{k}\tau_j\frac{\sigma_j(t)(F_j(t)-\alpha_F)e_j}{1+\tau_j F_j(t)} \end{aligned} \tag{3.9}$$

where e_j refers to a vector of the form $[0 \quad 0\cdots1\cdots0]^T$, with 1 in the j-th entry and 0 in all other entries (10). Note that the vector diffusion coefficient 'DC' of F_j is $\sigma_j(t)(F_j(t)-\alpha_F)e_j$ by definition of the dynamic of F_j under DDM. Substituting the above result into (3.7) we obtain the following expression:

$$dZ_t^k = dZ_t^i + \rho\sum_{j=i+1}^{k}\tau_j\frac{\sigma_j(t)(F_j(t)-\alpha_F)e_j^T}{1+\tau_j F_j(t)}dt \tag{3.10}$$

In order to obtain the $k-th$ row of dZ_t, i.e. the dynamic of Z_t at time T_k, we pre-multiply both sides of (3.10) by e_k^T. As a result, we obtain the following dynamic of dZ^k:

$$\begin{aligned} dZ_k^k &= dZ_k^i + [\rho_{k,1} \ \ \rho_{k,2}\ldots\rho_{k,n}]\sum_{j=i+1}^{k}\tau_j\frac{\sigma_j(t)(F_j(t)-\alpha_F)e_j}{1+\tau_j F_j(t)}dt \\ &= dZ_k^i + \sum_{j=i+1}^{k}\tau_j\frac{\sigma_j(t)(F_j(t)-\alpha_F)\rho_{k,j}}{1+\tau_j F_j(t)}dt \end{aligned} \tag{3.11}$$

Finally, we substitute (3.11) into the original equation of the model (model definition) $dF_k = \sigma_k\,(F_k-\alpha_F)\,dZ_k^k$, we get

$$dF_k = \sigma_k\,(F_k-\alpha_F)\left(dZ_k^i + \sum_{j=i+1}^{k}\tau_j\frac{\sigma_j(t)(F_j(t)-\alpha_F)\rho_{k,j}}{1+\tau_j F_j(t)}dt\right) \tag{3.12}$$

which is the dynamic of the forward rate with maturity T_k under the forward measure

$\mathbb{Q}^i$ when $i < k$.

Suppose now that $i > k$**.** Analogously to the above case where $i < k$, now with $S = P(\cdot, T_i)$ and $U = P(\cdot, T_k)$:

$$dZ_t^i = dZ_t^k - \rho DC\left(\ln\left(\frac{P(\cdot,T_i)}{P(\cdot,T_k)}\right)\right)^T dt \tag{3.13}$$

Then, by following the same reasoning as (3.8), we have

$$\begin{aligned}\ln\left(\frac{P(t,T_i)}{P(t,T_k)}\right) &= -\sum_{j=k+1}^{i}\ln(1+\tau_j F_j(t))\\ \Rightarrow DC\ln\left(\frac{P(t,T_i)}{P(t,T_k)}\right) &= -\sum_{j=k+1}^{i}\tau_j\frac{\sigma_j(t)(F_j(t)-\alpha_F)e_j}{1+\tau_j F_j(t)}\end{aligned} \tag{3.14}$$

Substituting the above result into (3.13), we obtain the following expression:

$$\begin{aligned}dZ_t^i &= dZ_t^k + \rho\sum_{j=k+1}^{i}\tau_j\frac{\sigma_j(t)(F_j(t)-\alpha_F)e_j^T}{1+\tau_j F_j(t)}dt\\ \Rightarrow dZ_t^k &= dZ_t^i - \rho\sum_{j=k+1}^{i}\tau_j\frac{\sigma_j(t)(F_j(t)-\alpha_F)e_j^T}{1+\tau_j F_j(t)}dt\end{aligned} \tag{3.15}$$

Finally, by calculating the $k-th$ row of dZ^k, we obtain the following dynamic of dZ^k:

$$\begin{aligned}dZ_k^k &= dZ_k^i - \sum_{j=i+1}^{k}\tau_j\frac{\sigma_j(t)(F_j(t)-\alpha_F)\rho_{k,j}}{1+\tau_j F_j(t)}dt\\ \Rightarrow dF_k &= \sigma_k\left(F_k-\alpha_F\right)\left(dZ_k^i - \sum_{j=i+1}^{k}\tau_j\frac{\sigma_j(t)(F_j(t)-\alpha_F)\rho_{k,j}}{1+\tau_j F_j(t)}dt\right)\end{aligned} \tag{3.16}$$

In summary, the dynamics of F_k under forward measure $\mathbb{Q}^i$ is:

$$\begin{cases} dF_k(t) = \mu_i^k\left(t,F(t)\right)\sigma_k(F_k-\alpha_F)dt + \sigma_k(t)(F_k(t)-\alpha_F)dZ_k^i(t) & \text{for } i<k\\ dF_k(t) = \sigma_k(t)\left(F_k(t)-\alpha_F\right)dZ_k^k(t) & \text{for } i=k\\ dF_k(t) = -\mu_k^i\left(t,F(t)\right)\sigma_k(F_k-\alpha)dt + \sigma_k(t)(F_k(t)-\alpha)dZ_k^i(t) & \text{for } i>k\end{cases} \tag{3.17}$$

where we set $\mu_n^m = \sum_{j=n+1}^{m}\tau_j\frac{\sigma_j(t)(F_j(t)-\alpha_F)\rho_{m,j}}{1+\tau_j F_j(t)}$

3.3 Existence and uniqueness of solutions

Now we set $X(t) = F(t) - \alpha_F$. Then the dynamic of $dX_k(t)$ is therefore the following:

$$\begin{cases} dX_k(t) = \mu_i^k\left(t, X(t)\right)\sigma_k X_k dt + \sigma_k(t)X_k(t)dZ_k^i(t) & \text{for } i < k \\ dX_k(t) = \sigma_k(t)X_k(t)dZ_k^k(t) & \text{for } i = k \\ dX_k(t) = -\mu_k^i\left(t, X(t)\right)\sigma_k X_k dt + \sigma_k(t)X_k(t)dZ_k^i(t) & \text{for } i > k \end{cases} \tag{3.18}$$

Under this substitution, $X_k(t)$ is the same as the process under the LIBOR Marker Model (LMM).

Case 1: $i = k$. The existence and uniqueness of solution is guaranteed since the dynamic is a GBM process with zero drift.

Case 2: $i < k$. We apply Ito's formula on $\ln X_k(t)$ to get

$$d\ln X_k(t) = \sigma_k(t)\sum_{j=i+1}^{k}\frac{\rho_{k,j}\tau_j\sigma_j(t)X_j(t)}{1+\tau_j F_j(t)}dt - \frac{\sigma_k(t)^2}{2}dt + \sigma_k(t)dZ_k(t) \tag{3.19}$$

The diffusion coefficient $\sigma_k(t)$ is deterministic and bounded, and $\frac{\sigma_k(t)^2}{2}$ is uniformly bounded from below. In addition, for the first term in (3.19), we have two cases to consider:

- If $\alpha_F > -1/\tau$, $0 < \tau_j X_j(t)/(1+\tau_j F_j(t)) = \tau_j(F_j(t) - \alpha_F)/(1+\tau_j F_j(t)) < 1$, so the drift term is also bounded. Since τ is a time period that is a proportion of one year, $\tau \leq 1$. Thus, $\frac{1}{\tau} > 1$ and in turn $-\frac{1}{\tau} < -1$. This means that $\alpha_F > -1$ in the upper bound case and $\alpha_F > -\infty$ in the lower bound case, and it is satisfied by the definition of α_F which is a shift on the forward rate F.

- If the condition of $\alpha_F > -1/\tau$ is not satisfied, the drift is still bounded if α_F and τ are finite, which are centainly the case by the construction of α_F and τ. In reality, since α_F is the shift applied on forward rates, and τ is a year fraction which is typically less than 1, $\alpha_F > -1/\tau$ is satisfied almost with certainty.

Case 3: $i > k$. We apply Ito's formula on $\ln X_k(t)$ again to get

$$d\ln X_k(t) = -\sigma_k(t)\sum_{j=k+1}^{i}\frac{\rho_{i,j}\tau_j\sigma_j(t)X_j(t)}{1+\tau_j F_j(t)}dt - \frac{\sigma_k(t)^2}{2}dt + \sigma_k(t)dZ_k(t) \tag{3.20}$$

Following the same reasoning as in case 2, we can see that the drift and diffusion terms are bounded.

In summary, the existence and uniqueness of a strong solution to the SDE in (3.18) is guaranteed.

4 Displaced Diffusion Swap Model: Pricing swaptions through the analytical Black's formula

4.1 The Black formula under Swap Market Model

In Section 1.3, we have mentioned that LMM is a useful market model to price caplets and floorlets. However, since market options on interest rates are divided into two different markets – the cap/floor market and the swap market, LMM is not the only market model that has been largely used. To obtain swaption pricing results using LMM, one needs to use techniques such as drift freezing to obtain an approximation. On the other hand, the Swap Market Model (SMM) is considered as a better market model to manage swaptions.

As in the lecture notes (10), the SMM works as follows:
we consider a payer swaption with first reset in T_α and paying at time $T_{\alpha+1}$, $T_{\alpha+2}$,...,T_β and a fixed rate K. The payoff can be written as

$$(S_{\alpha,\beta}(T_\alpha) - K)^+ \sum_{i=\alpha+1}^{\beta} \tau_i P(T_\alpha, T_i) \tag{4.1}$$

Then, when we take $C_{\alpha,\beta}(t) = \sum_{i=\alpha+1}^{\beta} \tau_i P(t, T_i)$ as the numeraire, the forward swap rate $S_{\alpha,\beta}$ is a martingale under this numeraire and the correponding measure $\mathbb{Q}^{\alpha,\beta}$. For completeness, the numeraire $C_{\alpha,\beta}(t)$ is called Present Value per Basis Point (PVPBP), PV01 or DV01. Therefore, we have

$$S_{\alpha,\beta}(t) = \frac{P(t,T_\alpha) - P(t,T_\beta)}{\sum_{i=\alpha+1}^{\beta} \tau_i P(t,T_i)} = \frac{P(t,T_\alpha) - P(t,T_\beta)}{C_{\alpha,\beta}(t)} \tag{4.2}$$

In turn we get

$$dS_{\alpha,\beta}(t) = \sigma_{\alpha,\beta}(t) S_{\alpha,\beta}(t) dW_t^{\alpha,\beta}, \quad \mathbb{Q}^{\alpha,\beta} \quad \text{(SMM)} \tag{4.3}$$

which is the swap market model. In this model, we can obtain a closed-form analytical pricing formula for swaptions:

$$\begin{aligned}
\text{Swaption Price} &= \mathbb{E}^{\mathbb{Q}}\left[\frac{B(0)}{B(T_\alpha)} \sum_{i=\alpha+1}^{\beta} \tau_i P(T_\alpha, T_i) \left(S_{\alpha,\beta}(T_\alpha) - K\right)^+\right] \\
&= \mathbb{E}^{\alpha,\beta}\left[\frac{C_{\alpha,\beta}(0)}{C_{\alpha,\beta}(T_\alpha)} C_{\alpha,\beta}(T_\alpha) \left(S_{\alpha,\beta}(T_\alpha) - K\right)^+\right] \\
&= C_{\alpha,\beta}(0) \mathbb{E}^{\alpha,\beta}\left[(S_{\alpha,\beta}(T_\alpha) - K)^+\right] \\
&= C_{\alpha,\beta}(0) \left[S_{\alpha,\beta}(0)\Phi(d_1) - K\Phi(d_2)\right]
\end{aligned} \tag{4.4}$$

where $d_{1,2} = \frac{\ln \frac{S_{\alpha,\beta}(0)}{K} \pm \frac{1}{2} T_\alpha v^2_{\alpha,\beta}(T_\alpha)}{\sqrt{T_\alpha} v_{\alpha,\beta}(T_\alpha)}$ and $v^2_{\alpha,\beta}(T) = \frac{1}{T}\int_0^T (\sigma_{\alpha,\beta}(t))^2 \, dt$.

4.2 Displaced Diffusion Swap (DDS) Model

As mentioned in Chapter 3, the DDF model should be a safe choice for the caplet market. We now want to link the DDS model, which should be effective for the swaption pricing, to the DDF model. We want to see if the pricing results are consistent in DDF model and DDS model for swaptions.

4.2.1 The dynamic of $S_{\alpha,\beta}$

Now we fix α and β and consider a swaption with underlying $S_{\alpha,\beta}$ and try to work out the dynamic of $S_{\alpha,\beta}$ under the Displaced Diffusion model. Firstly, we set $S_{\alpha,\beta} = X_{\alpha,\beta} + \alpha_S$, where α_S is a constant shift that we choose for the model. Since under our assumption, $X_{\alpha,\beta}$ is log-normal, it follows a GBM with zero drift, which is identical to the dynamic of $S_{\alpha,\beta}$ in (4.3), i.e.

$$dX_{\alpha,\beta} = \sigma_{\alpha,\beta} X_{\alpha,\beta} dW, \quad \mathbb{Q}^{\alpha,\beta} \tag{4.5}$$

Then, since α_S is a constant shift, we observe that $dS = dX$, which gives

$$\begin{aligned} dS_{\alpha,\beta} &= \sigma_{\alpha,\beta} X_{\alpha,\beta} dW \\ &= \sigma_{\alpha,\beta} \left(S_{\alpha,\beta} - \alpha_S\right) dW, \quad \mathbb{Q}^{\alpha,\beta} \end{aligned} \tag{4.6}$$

and this is the dynamic of $S_{\alpha,\beta}$ in the Displaced Diffusion model under the swap measure $\mathbb{Q}^{\alpha,\beta}$.

4.2.2 Swaptions Pricing

Now we go on to price swaptions under the Displaced Diffusion model using the dynamics described in (4.6). We apply the change of numeraire toolkit, where we set $\sum_{i=\alpha+1}^{\beta} \tau_i P(T_\alpha, T_i) = C_{\alpha,\beta}(T_\alpha)$ as the numeraire:

$$\begin{aligned} \text{Swaption Price} &= \mathbb{E}^{\mathbb{Q}} \left[\frac{B(0)}{B(T_\alpha)} \sum_{i=\alpha+1}^{\beta} \tau_i P(T_\alpha, T_i) \left(S_{\alpha,\beta}(T_\alpha) - K\right)^+ \right] \\ &= \mathbb{E}^{\alpha,\beta} \left[\frac{C_{\alpha,\beta}(0)}{C_{\alpha,\beta}(T_\alpha)} C_{\alpha,\beta}(T_\alpha) \left(S_{\alpha,\beta}(T_\alpha) - K\right)^+ \right] \\ &= C_{\alpha,\beta}(0) \mathbb{E}^{\alpha,\beta} \left[\left(S_{\alpha,\beta}(T_\alpha) - K\right)^+ \right] \\ &= C_{\alpha,\beta}(0) \mathbb{E}^{\alpha,\beta} \left[\left(S_{\alpha,\beta}(T_\alpha) - \alpha_S - K + \alpha_S\right)^+ \right] \\ &= C_{\alpha,\beta}(0) \mathbb{E}^{\alpha,\beta} \left[\left((S_{\alpha,\beta}(T_\alpha) - \alpha_S) - (K - \alpha_S)\right)^+ \right] \\ &= C_{\alpha,\beta}(0) \mathbb{E}^{\alpha,\beta} \left[\left(X_{\alpha,\beta}(T_\alpha) - (K - \alpha_S)\right)^+ \right] \\ &= C_{\alpha,\beta}(0) \left[X_{\alpha,\beta}(0) \Phi(d_1) - (K - \alpha_S) \Phi(d_2) \right] \end{aligned} \tag{4.7}$$

where $d_{1,2} = \frac{\ln \frac{X_{\alpha,\beta}(0)}{K-\alpha_S} \pm \frac{1}{2} T_\alpha v^2_{\alpha,\beta}(T_\alpha)}{\sqrt{T_\alpha} v_{\alpha,\beta}(T_\alpha)}$ and $v^2_{\alpha,\beta}(T) = \frac{1}{T} \int_0^T (\sigma_{\alpha,\beta}(t))^2 \, dt$. Again, we are using the assumption that $X_{\alpha,\beta}$ is log-normal in the calculation.

Therefore, from (4.7) we can conclude that the swaption price under the Displaced Diffusion model has a closed form formula, and it can be interpreted as $BlackFormulaForSwaption$ $(T_\alpha, \sigma_{\alpha,\beta}, K - \alpha_S, X_{\alpha,\beta}(0) = S_{\alpha,\beta}(0) - \alpha_S)$.

There is a special case of the analytical pricing formula (4.7): there could be situations in which the strike $K - \alpha_S$ is less than the zero. In this case, the expectation is no longer an option since it is always positive: $X_{\alpha,\beta} > 0$ and $-(K - \alpha_S) > 0$. Instead, it becomes a forward contract. To express it numerically, we have, when $K - \alpha_S < 0$:

$$\begin{aligned} \text{Swaption Price} &= C_{\alpha,\beta}(0)\mathbb{E}^{\alpha,\beta}\left[(X_{\alpha,\beta}(T_\alpha) - (K - \alpha_S))^+\right] \\ &= C_{\alpha,\beta}(0)\mathbb{E}^{\alpha,\beta}\left[X_{\alpha,\beta}(T_\alpha) - (K - \alpha_S)\right] \\ &= C_{\alpha,\beta}(0)\left[X_{\alpha,\beta}(0) - (K - \alpha_S)\right] \end{aligned} \tag{4.8}$$

Since in this case there is no 'option' embedded in the swaption anymore, it makes the situation less interesting from this thesis's perspective. Therefore, we tried to avoid this situation by choosing the parameters wisely.

Subsequently we are going to compare the swaption price results calculated from this formula with the Monte Carlo simulation results based on the DD-Forward model described in the next chapter. Note that prior to the comparison of results, we tested the accuracy of this closed-form swaption formula by setting volatility very close to zero and computing the intrinsic value as well as the value of this pricing formula (4.7). We found the results to be accurate up to the 11th decimal place, which is a very high level of accuracy.

4.3 Connecting Forward rate F and Swap rate S in the Displaced Diffusion Model through volatility $v^2_{\alpha,\beta}(T_\alpha)$ and shift α_S

4.3.1 Volatility under DDM

If we do an integration on the Black-Scholes Volatility component in the Libor Market Model (LMM) as in the lecture notes (10), we can see that:

$$\int_0^{T_\alpha} \sigma^2_{\alpha,\beta}(t)dt = \int_0^{T_\alpha} \sigma_{\alpha,\beta}(t)dW_{\alpha,\beta}(t)\sigma_{\alpha,\beta}(t)dW_{\alpha,\beta}(t) = \int_0^{T_\alpha} (d\ln X_{\alpha,\beta}(t))\,(d\ln X_{\alpha,\beta}(t)) \tag{4.9}$$

The same relationship holds under the Displaced Diffusion model due to the dynamics of $X_{\alpha,\beta}$ stated in (4.5). $X_{\alpha,\beta}$ in the Displaced Diffusion Model is also log-normal, just as $S_{\alpha,\beta}$ in the Libor Market Model.

Under the Displaced Diffusion Model, we can do something analogously, where $S_{\alpha,\beta}$ is approximated by weighted average of forward rates.

$$
\begin{aligned}
S_{\alpha,\beta}(t) &= \sum_{i=\alpha+1}^{\beta} w_i(t)F_i(t) \\
w_i(t) &= w_i(F_{\alpha+1}(t), F_{\alpha+2}(t), \ldots, F_\beta(t)) \\
&= \frac{\tau_i \prod_{j=\alpha+1}^{i} \frac{1}{1+\tau_j F_j(t)}}{\sum_{k=\alpha+1}^{\beta} \tau_k \prod_{j=\alpha+1}^{k} \frac{1}{1+\tau_j F_j(t)}}
\end{aligned}
\tag{4.10}
$$

We then use the 'Freezing the w's at time 0' technique to get

$$
S_{\alpha,\beta}(t) = \sum_{i=\alpha+1}^{\beta} w_i(t)F_i(t) \approx \sum_{i=\alpha+1}^{\beta} w_i(0)F_i(t) \tag{4.11}
$$

the reasoning behind this approximation is that we can view the variability of the $w's$ to be much smaller than the variability of the $F's$. Note that the swap rate computed from (4.11) is the same at time 0 with the swap rate computed from (5.2). This quantity is going to be our choice for the strike K since we want to price at-the-money swaptions due to the paratemerisation we choose for the volatilities (described in Section 5.2.1).

Now we differentiate $S_{\alpha,\beta}(t)$ in (4.11) to obtain the dynamic of $dS_{\alpha,\beta}$

$$
dS_{\alpha,\beta} \approx \sum_{i=\alpha+1}^{\beta} w_i(0)dF_i = (\ldots)dt + \sum_{i=\alpha+1}^{\beta} w_i(0)\sigma_i(t)\left(F_i(t) - \alpha_F\right)dZ_i(t) \tag{4.12}
$$

We can then observe that $dX_{\alpha,\beta}(t)$ is the same as $dS_{\alpha,\beta}$ in (4.12), due to the fact that $X_{\alpha,\beta} = S_{\alpha,\beta} - \alpha_S$ and α_S is a constant so $d\alpha_S = 0$. Thus, we have

$$
\begin{aligned}
dX_{\alpha,\beta} = d(S_{\alpha,\beta} - \alpha_S) \approx \sum_{i=\alpha+1}^{\beta} w_i(0)dF_i &= (\ldots)dt + \sum_{i=\alpha+1}^{\beta} w_i(0)\sigma_i^F(t)\left(F_i(t) - \alpha_F\right)dZ_i(t) \\
= \sum_{i=\alpha+1}^{\beta} w_i(0)dF_i &= (\ldots)dt + \sum_{i=\alpha+1}^{\beta} w_i(0)\sigma_i^F(t)X^F dZ_i(t)
\end{aligned}
\tag{4.13}
$$

where we used the fact that $X^F = F_i(t) - \alpha_F$. From (4.13) we get the relationship between the two rates X^F (Forward rate) and $X_{\alpha,\beta}$ (Swap rate).

The differentiated form of the quadratic variation of $X_{\alpha,\beta}$ therefore is

$$\begin{aligned}
dX_{\alpha,\beta}(t)dX_{\alpha,\beta}(t) &\approx \sum_{i,j=\alpha+1}^{\beta} w_i(0)\sigma_i(t)\left(F_i(t)-\alpha_F\right)dZ_i w_j(0)\sigma_j(t)\left(F_j(t)-\alpha_F\right)dZ_j \\
&= \sum_{i,j=\alpha+1}^{\beta} w_i(0)w_j(0)(F_i(t)-\alpha_F)(F_j(t)-\alpha_F)\rho_{i,j}\sigma_i(t)\sigma_j(t)dt \\
&= \sum_{i,j=\alpha+1}^{\beta} w_i(0)w_j(0)X_i^F(t)X_j^F(t)\rho_{i,j}\sigma_i(t)\sigma_j(t)dt
\end{aligned} \tag{4.14}$$

Here, we have used the fact that $dZ_i dZ_j = \rho_{i,j}dt$ in calculating the last equality. By applying the Ito's formula, we can get the quadratic covariation of $d\ln X_{\alpha,\beta}(t)$ is

$$\begin{aligned}
\left(d\ln X_{\alpha,\beta}(t)\right)\left(d\ln X_{\alpha,\beta}(t)\right) &= \frac{dX_{\alpha,\beta}(t)}{X_{\alpha,\beta}(t)}\frac{dX_{\alpha,\beta}(t)}{X_{\alpha,\beta}(t)} \\
&\approx \frac{\sum_{i,j=\alpha+1}^{\beta} w_i(0)w_j(0)(F_i(t)-\alpha_F)(F_j(t)-\alpha_F)\rho_{i,j}\sigma_i(t)\sigma_j(t)}{X_{\alpha,\beta}^2}dt \\
&= \frac{\sum_{i,j=\alpha+1}^{\beta} w_i(0)w_j(0)X_i^F(t)X_j^F(t)\rho_{i,j}\sigma_i(t)\sigma_j(t)}{X_{\alpha,\beta}^2}dt
\end{aligned} \tag{4.15}$$

Now we can introduce another approximation which is freezing the $F_i(t)$'s and $F_j(t)$'s to time zero:

$$\begin{aligned}
\left(d\ln X_{\alpha,\beta}(t)\right)\left(d\ln X_{\alpha,\beta}(t)\right) &\approx \frac{\sum_{i,j=\alpha+1}^{\beta} w_i(0)w_j(0)(F_i(0)-\alpha_F)(F_j(0)-\alpha_F)\rho_{i,j}}{X_{\alpha,\beta}^2}\sigma_i(t)\sigma_j(t)dt \\
&= \frac{\sum_{i,j=\alpha+1}^{\beta} w_i(0)w_j(0)X_i^F(0)X_j^F(0)\rho_{i,j}}{X_{\alpha,\beta}^2}\sigma_i(t)\sigma_j(t)dt
\end{aligned} \tag{4.16}$$

and now we can see that the σ's are the only time-dependent functions in the formula. We are in a great position to compute the time-averaged percentage variance of $X_{\alpha,\beta}$ under the Displaced Diffusion model (DDM).

Volatility in DDM Swaption Formula

$$\begin{aligned}
(\sigma_{\alpha,\beta}^{DDM})^2 &= \frac{1}{T_\alpha}\int_0^{T_\alpha}\left(d\ln X_{\alpha,\beta}(t)\right)\left(d\ln X_{\alpha,\beta}(t)\right) \\
&= \sum_{i,j=\alpha+1}^{\beta}\frac{w_i(0)w_j(0)X_i^F(0)X_j^F(0)\rho_{i,j}}{T_\alpha X_{\alpha,\beta}(0)^2}\int_0^{T_\alpha}\sigma_i^F(t)\sigma_j^F(t)dt \\
&= \sum_{i=\alpha+1}^{\beta}\sum_{j=\alpha+1}^{\beta}\frac{w_i(0)w_j(0)X_i^F(0)X_j^F(0)\rho_{i,j}}{T_\alpha X_{\alpha,\beta}(0)^2}\int_0^{T_\alpha}\sigma_i^F(t)\sigma_j^F(t)dt
\end{aligned} \tag{4.17}$$

Similar to the Libor Market Model (LMM), $\sigma_{\alpha,\beta}^{DMM}$ can be viewed as an approximation for the Black volatility $v_{\alpha,\beta}(T_\alpha)$ (10). As a result, we can substitute the DMM volatility approximation $\sigma_{\alpha,\beta}^{DMM}$ into the Black's formula for Swaptions to price swaptions which is calculated in (4.7). To make things clearer, now we have a close form formula for DDM swaption pricing if we subsititute the $\sigma_{\alpha,\beta}^{DDM}$ derived in (4.17) into (4.7):

$$\text{Swaption Price Under DDM} = C_{\alpha,\beta}(0)\left[X_{\alpha,\beta}(0)\Phi(d_1) - (K-\alpha_S)\Phi(d_2)\right] \tag{4.18}$$

where $d_{1,2} = \frac{\ln \frac{X_{\alpha,\beta}(0)}{K-\alpha_S} \pm \frac{1}{2}T_\alpha v_{\alpha,\beta}^2(T_\alpha)}{\sqrt{T_\alpha} v_{\alpha,\beta}(T_\alpha)}$ and, with substitution,

$$v_{\alpha,\beta}^2(T) = (\sigma_{\alpha,\beta}^{DDM})^2 \quad = \sum_{i,j=\alpha+1}^{\beta} \frac{w_i(0)w_j(0)X_i^F(0)X_j^F(0)\rho_{i,j}}{T_\alpha X_{\alpha,\beta}(0)^2} \int_0^{T_\alpha} \sigma_i^F(t)\sigma_j^F(t)dt \tag{4.19}$$

Note that for the calculation of $\sigma_{\alpha,\beta}^{DMM}$ we used the technique described in (4.9).

In the actual computation of $\sigma_{\alpha,\beta}^{DMM}$, we need to turn the integral in (4.19) into a sum. Precisely, we are going to use

$$\begin{aligned}\sum_{i,j=\alpha+1}^{\beta} & \frac{w_i(0)w_j(0)X_i^F(0)X_j^F(0)\rho_{i,j}}{T_\alpha X_{\alpha,\beta}(0)^2} \int_0^{T_\alpha} \sigma_i^F(t)\sigma_j^F(t)dt \\ &= \sum_{i,j=\alpha+1}^{\beta} \frac{w_i(0)w_j(0)X_i^F(0)X_j^F(0)\rho_{i,j}}{T_\alpha X_{\alpha,\beta}(0)^2} \sum_{t=0}^{T_\alpha} \sigma_i^F(t)\sigma_j^F(t)\tau \end{aligned} \tag{4.20}$$

In our setting, we have a constant τ which is set to be 0.25 (3 months).

4.3.2 Shift α_S under DDS model

The only question left is about swaption price under DDM (4.18) is to decide the shift α_S in DDM. One method can be tried out: trying to match the integrated quadratic variations of dS in both models (DD-Forward model and DD-Swaption model).

- In the DD-Swaption model, the dynamic of $S_{\alpha,\beta}$ is given by (4.6). The quadratic variation of $S_{\alpha,\beta}$ therefore is

$$\begin{aligned}(dS_{\alpha,\beta})(dS_{\alpha,\beta}) &= \sigma_{\alpha,\beta}(t)(S_{\alpha,\beta}-\alpha_S)\,dW\,\sigma_{\alpha,\beta}(S_{\alpha,\beta}-\alpha_S)\,dW \\ &= \sigma_{\alpha,\beta}^2 (S_{\alpha,\beta}-\alpha_S)^2\,dt\end{aligned} \tag{4.21}$$

  by doing integration on (4.21), we found the quadratic variation to be $\sigma_{\alpha,\beta}^2(t)\,(S_{\alpha,\beta}(t)-\alpha_S)^2$.

- In the DD-Forward model, the dynamic of $S_{\alpha,\beta}$ is given by (4.12). Following the same logic as above, without freezing the time for w to zero, we get:

$$
\begin{aligned}
(dS_{\alpha,\beta})(dS_{\alpha,\beta}) &= \sum_{i=\alpha+1}^{\beta} w_i(t)\sigma_i(t)\left(F_i(t)-\alpha_F\right)dZ_i(t)\sum_{j=\alpha+1}^{\beta} w_j(t)\sigma_j(t)\left(F_j(t)-\alpha_F\right)dZ_j(t) \\
&= \sum_i\sum_j w_i(t)w_j(t)\sigma_i^F(t)\sigma_j^F(t)\rho_{i,j}\left(F_i(t)-\alpha_F\right)\left(F_j(t)-\alpha_F\right)dt
\end{aligned}
\tag{4.22}
$$

by doing integration on (4.22), we found the quadratic variation to be $\sum_i\sum_j w_i(t)w_j(t)\sigma_i(t)\sigma_j(t)\rho_{i,j}\left(F_i(t)-\alpha_F\right)\left(F_j(t)-\alpha_F\right)$. Note that the sum here goes from $\alpha+1$ to β.

Now we apply the same 'freezing to time zero' technique as before on the quadratic variations calculated from the two displaced diffusion models. Then the quadratic variation in DD-Swaption model becomes $\sigma^2_{\alpha,\beta}(t)\left(S_{\alpha,\beta}(0)-\alpha_S\right)^2$, and the quadratic variation in DD-Forward model becomes $\sum_i\sum_j w_i(0)w_j(0)\sigma_i^F(t)\sigma_j^F(t)\rho_{i,j}\left(F_i(0)-\alpha_F\right)\left(F_j(0)-\alpha_F\right)$. Now we can integrate both sides of (4.21) with time freeze to 0 and (4.22) from 0 to T_α, and we equate both sides to obtain

$$
\int_0^{T_\alpha}\sigma^2_{\alpha,\beta}\left(S_{\alpha,\beta}(0)-\alpha_S\right)^2dt=\int_0^{T_\alpha}\sum_i\sum_j w_i(0)w_j(0)\sigma_i(0)\sigma_j(0)\rho_{i,j}\left(F_i(0)-\alpha_F\right)\left(F_j(0)-\alpha_F\right)dt
\tag{4.23}
$$

Now we assume that everything apart from α_S is known, and assume $F_i(0)$ and $F_j(0)$ are constants. We try to solve (4.23) in α_S and get an expression of α_S in terms of α_F. For left hand side of the equation we have

$$
\begin{aligned}
LHS &= \int_0^{T_\alpha}\sigma^2_{\alpha,\beta}\left(S^2_{\alpha,\beta}(0)-2S_{\alpha,\beta}(0)\alpha_S+\alpha_S^2\right)dt \\
&= \int_0^{T_\alpha}\sigma^2_{\alpha,\beta}S^2_{\alpha,\beta}(0)dt-2\alpha_S\int_0^{T_\alpha}S_{\alpha,\beta}(0)dt+\alpha_S^2T_\alpha \\
&= S^2_{\alpha,\beta}(0)\int_0^{T_\alpha}\sigma^2_{\alpha,\beta}dt-2\alpha_S S_{\alpha,\beta}(0)T_\alpha+\alpha_S^2T_\alpha
\end{aligned}
\tag{4.24}
$$

And for the right hand side we have

$$
RHS=\sum_i\sum_j w_i(0)w_j(0)\rho_{i,j}\left(F_i(0)-\alpha_F\right)\left(F_j(0)-\alpha_F\right)\int_0^{T_\alpha}\sigma_i^F(t)\sigma_j^F(t)dt
\tag{4.25}
$$

However, by equating LHS and RHS, we got an identity equation which leads to no solution of α_S.

As a result, we decided to try another method: **we can just set $\alpha_S=\alpha_F$, i.e. setting the shift in the DD-Forward model and the shift in the DD-Swaption equal.** This is the method we are using the the Monte Carlo Simulation described in the next Chapter.

4.4 Computation of the analytical formula for swaption price under DDS model

To see the connection between the DDF model and the DDS model, we want to compare the swaption price result computed from the above analytical formula (under DDS model) and the formula under DDF described in the next Chapter. To compute the swaption price from DDS model using the analytical formula, all we need to do is to select all the inputs and compute individual components that are used in the formula. Below is how we choose or compute the components of the formula:

1. **Weights** $w_i(0)$

   We computed the weights used in the volatility component of the analytical formula by using the chosen forward rates $F_k(0)'s$ for $k = \alpha+1, \alpha+2, \ldots, \beta$.

2. **Integral in** $v^2_{\alpha,\beta}(T_\alpha)$

   Since we choose to model instantaneous volatility $\sigma_k(t)'s$ using the SPC (GPC) method (described in the next chapter) and since τ_i is a fixed constant which is set to be 0.25, the integral involved in the volatility formula (4.17) can be turned into a sum: $\int_0^{T_\alpha} \sigma_i^F(t)\sigma_j^F(t)dt = \sum_{i=0}^{T_\alpha} \sigma_i^F(t)\sigma_j^F(t)\tau$.

3. **Correlation matrix** ρ

   In the analytical pricing formula of DDS, the correlation that we need is the instantaneous correlation matrix instead of the terminal correlations. Here we are using the same instantaneous correlation matrix as in the Monte Carlo Simulation. The method is called **'Full rank, classical, two-parameters, exponentially decreasing' parameterisation'**. We are also using the same sets of α and ρ_∞ as in Table 5 to compute this correlation matrix (the parametrisation is explained in the next chapter).

   For completeness, we can apply the method described in Brigo and Mercurio (2001) (12) to compute the terminal correlations in this case if needed. The method is:

$$\frac{\exp\left(\int_0^{T_\alpha}\sigma_i(t)\sigma_j(t)\rho_{i,j}dt\right)-1}{\sqrt{\exp\left(\int_0^{T_\alpha}\sigma_i^2(t)dt\right)-1}\sqrt{\exp\left(\int_0^{T_\alpha}\sigma_j^2(t)dt\right)-1}} \approx \rho_{i,j}\frac{\int_0^{T_\alpha}\sigma_i(t)\sigma_j(t)dt}{\sqrt{\int_0^{T_\alpha}\sigma_i^2(t)dt}\sqrt{\int_0^{T_\alpha}\sigma_j^2(t)dt}} \tag{4.26}$$

   Note that $\rho_{i,j}$ in (4.26) refers to the instantaneous correlations.

4. **Shift** α_S

   As described in Section 4.3.2, we are using $\alpha_S = \alpha_F$ to perform our computations. Note that α_F is the corresponding shift in the DDF model which is also the shift used for the Monte Carlo simulation in the next chapter.

5. **Discount factor** $C_{\alpha,\beta}(0)$
To compute $C_{\alpha,\beta}(0)$, we need to know zero-coupon bond prices $P(0,T_i)$ for $i = \alpha + 1, \ldots, \beta$. These can be computed recursively using the formula for the relationship between zero-coupon bond prices and forward rates:

$$P(0,T_i) = \frac{P(0,T_{i-1})}{(T_i - T_{i-1})F_i(0) + 1} \tag{4.27}$$

Also note that τ in the discount factor $C_{\alpha,\beta}(0)$ is corresponding to $T_i - T_{i-1}$ not Δt. Now, the only quantity that is left for us to choose freely is $P(0,T_\alpha)$ which is also the starting point of the recursive formula (4.27). We choose $P(0,T_\alpha) = 0.01$ in our computation.

5 Displaced Diffusion Forward Model: Testing using Monte Carlo simulation methods

5.1 Pricing Swaption under the Displaced Diffusion Forward Model

Now we want to compare the above close form formula for swaption price under Displaced Diffusion Swap Model and the actual swaption price obtained from Monte Carlo Simulation using the Displaced Diffusion Forward model.

First of all, let's recall the swaption pricing formula:

$$\begin{aligned}\text{Swaption Price} &= E^B\left(\frac{B(0)}{B(T_\alpha)}\big(S_{\alpha,\beta}(T_\alpha)-K\big)^+\sum_{i=\alpha+1}^{\beta}\tau P(T_\alpha,T_i)\right)\\ &= E^\alpha\left(\frac{P(0,T_\alpha)}{P(T_\alpha,T_\alpha)}\big(S_{\alpha,\beta}(T_\alpha)-K\big)^+\sum_{i=\alpha+1}^{\beta}\tau P(T_\alpha,T_i)\right)\\ &= P(0,T_\alpha)E^\alpha\left(\big(S_{\alpha,\beta}(T_\alpha)-K\big)^+\sum_{i=\alpha+1}^{\beta}\tau P(T_\alpha,T_i)\right)\end{aligned} \tag{5.1}$$

We can now see that due to the fact that

$$S_{\alpha,\beta}(T_\alpha)=\frac{1-\prod_{j=\alpha+1}^{\beta}\frac{1}{1+\tau_j F_j(T_\alpha)}}{\sum_{i=\alpha+1}^{\beta}\tau_i\prod_{j=\alpha+1}^{i}\frac{1}{1+\tau_j F_j(T_\alpha)}} \tag{5.2}$$

the above expectation in (5.1) depends on the joint distribution of the forward rates $F_{\alpha+1}(T_\alpha)$, $F_{\alpha+2}(T_\alpha)$,..., $F_\beta(T_\alpha)$ under the measure $\mathbb{Q}^\alpha$. In addition, we know that the dynamic of forward rates under $\mathbb{Q}^\alpha$ is as described in (3.17). Please note that here by construction we have forward rates at time between $\alpha+1$ and β inclusive, which are all bigger than the measure time α, thus we are using the dynamic we derived in chapter 3:

$$dF_k=\sigma_k\left(F_k-\alpha_F\right)\left(dZ_k^\alpha+\sum_{j=\alpha+1}^{k}\tau_j\frac{\sigma_j(t)(F_j(t)-\alpha_F)\rho_{k,j}}{1+\tau_j F_j(t)}dt\right) \tag{5.3}$$

where $k=\alpha+1,\alpha+2,\ldots,\beta$.

We need to simulate the $F's$ under the measure $\mathbb{Q}^\alpha$ and use the simulation result to price the swaption by simulation. One method that we can use is the Milstein Scheme. We are going to use $X^F=F-\alpha_F$ in this simulation, as it gives a log-normal variable.

Milstein Scheme for $\ln X$:

$$\ln X_k^{\Delta t}(t+\Delta t) = \ln X_k^{\Delta t}(t) + \sigma_k(t) \sum_{j=\alpha+1}^{k} \frac{\rho_{k,j}\tau_j\sigma_j(t)X_j^{\Delta t}}{1+\tau_j F_j^{\Delta t}}\Delta t - \frac{\sigma_k^2(t)}{2}\Delta t + \sigma_k(t)\left(Z_k(t+\Delta t) - Z_k(t)\right) \quad (5.4)$$

which is a recursive generation of forward rates within a given period of time with properly chosen parameters.

From lecture notes (10), we know that this is a approximation such that there exists a δ_0 with

$$\mathbb{E}^{\alpha}\{|\ln X_k^{\Delta t}(T_\alpha) - \ln X_k(T_\alpha)|\} \leq C(T_\alpha)(\Delta t)^1, \quad \text{for all } \Delta t \leq \delta_0 \quad (5.5)$$

Note that here $C(T_\alpha)$ is a constant so that means the above approximation has strong convergence of order 1, and $(Z_k(t+\Delta t) - Z_k(t))$ is a normally distributed known object that we can simulate (10).

5.2 Choosing Parameters

In order to perform the Monte Carlo simulation of the forward rate path, we need to choose several parameters to be the (initial) inputs of the recursively generated paths. If we take a close look at the equation (5.4), we can list out the parameters that need to be chosen which are written below. We need to choose them wisely in order to achieve the desired combinations of the scenarios that we want to simulate and test.

Parameters to choose:

1. The volatility of the swaption at each time point time t and for each forward rate F_k, $\sigma_k(t)$;
2. The instantaneous correlation matrix ρ for all F_k for $k = \alpha+1, \dots, \beta$;
3. The shift α_F (which is the same as α_S);
4. The initial forward rate $F_k(0)$ for $k = \alpha+1, \dots, \beta$.

We summarise below the method used for choosing each of the inputs mentioned above. The DDM is completely specified once we specify $\sigma_k(t)$, $\rho_{i,j}$ and $F_k(0)$ for all i, j, k in the tenor structure. We select the methods from the lecture notes (10).

5.2.1 Volatility σ_k

In the Monte Carlo simulation, we decided to choose $\sigma_k(t)'s$ using the **Separable Piecewise Contant (SPC)** method. SPC is a special case under the General Piecewise

Contant (GPC) method and GPC is the richest parametrization and can fit all the at-the-money options (10). The numerical illustration of SPC $\sigma's$ is $\sigma_k(t) = \phi_k \psi_{k-(\beta(t)-1)}$ with $T_{\beta(t)-2} < t \leq T_{\beta(t)-1}$. With the SPC parametrisations we get M parameters ϕ and M parameters ψ. Thus, there is a total of $2M$ volatility parameters with M being the number of forward rates that we are generating.

	t$\in (0, T_0]$	$(T_0, T_1]$	$(T_1, T_2]$	$(T_2, T_3]$	...	$(T_{M-2}, T_{M-1}]$
$F_1(t)$	$\sigma_{1,1}$			...		
$F_2(t)$	$\sigma_{2,1}$	$\sigma_{2,2}$		...		
$F_3(t)$	$\sigma_{3,1}$	$\sigma_{3,2}$	$\sigma_{3,3}$	...		
$\vdots$	...	...	...	...	...	...
$F_M(t)$	$\sigma_{M,1}$	$\sigma_{M,2}$	$\sigma_{M,3}$	$\sigma_{M,4}$	...	$\sigma_{M,M}$

Table 1: Ziggurat matrix for intantaneous volatilities of forward rates under GPC method.

Table 1 is an illustration of the idea behind the GPC method and Table 2 is an illustration of the idea behind the SPC method which is the one that we will focus on.

	t$\in (0, T_0]$	$(T_0, T_1]$	$(T_1, T_2]$	...	$(T_{M-2}, T_{M-1}]$
$F_1(t)$	$\phi_1\psi_1$	Expired	Expired	...	Expired
$F_2(t)$	$\phi_2\psi_2$	$\phi_2\psi_1$	Expired	...	Expired
$\vdots$	...	...	...	...	...
$F_M(t)$	$\phi_M\psi_M$	$\phi_M\psi_{M-1}$	$\phi_M\psi_{M-2}$	...	$\phi_M\psi_1$

Table 2: Ziggurat matrix for intantaneous volatilities of forward rates under SPC method.

Two choices of SPC

1. **Homogeneous in the time-to-expiry:** $\sigma_k(t) = \psi_{k-(\beta(t)-1)}$, and $\sigma_k(T_{j-}) = \psi_{k-j}$. This is achieved by setting all $\phi's$ to 1. The idea is shown in Table 3. In real life, this assumption is used when traders have no view on future volatility term structures and they prefer a stationary model (10). We are using this method for the Monte Carlo simulations. Please note that the subscript of forward rates refer to the corresponding number of time point: for example, $F_1(t)$ is referring to $F_{\alpha+1}(t)$ in our Monte Carlo Simulation.

2. **Homogeneous in the time:** $\sigma_k(t) = \phi_k$.This is achieved by setting all $\psi's$ to 1. The idea is shown in Table 4. With $\psi = 1$, the volatility term structure shows no humped shape at the tail when time is further away from the 'current time', and it converges to a flat line as time goes on (10). This assumption makes calculation

	t$\in (0,T_0]$	$(T_0,T_1]$	$(T_1,T_2]$	$\dots$	$(T_{M-2},T_{M-1}]$
$F_1(t)$	ψ_1	Expired	Expired	$\dots$	Expired
$F_2(t)$	ψ_2	ψ_1	Expired	$\dots$	Expired
$\vdots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$
$F_M(t)$	ψ_M	ψ_{M-1}	ψ_{M-2}	$\dots$	ψ_1

Table 3: Volatility Matrix under SPC method and $\phi = 1$.

and calibration easier by making terminal and instantaneous correlations the same, however, it makes a strong assumption that future volatility will be lower than the current ones which is not justified in some economic situations (10). Therefore, this method should be avoided as long as we can.

	t$\in (0,T_0]$	$(T_0,T_1]$	$(T_1,T_2]$	$\dots$	$(T_{M-2},T_{M-1}]$
$F_1(t)$	ϕ_1	Expired	Expired	$\dots$	Expired
$F_2(t)$	ϕ_2	ϕ_2	Expired	$\dots$	Expired
$\vdots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$
$F_M(t)$	ϕ_M	ϕ_M	ϕ_M	$\dots$	ϕ_M

Table 4: Volatility Matrix under SPC method and $\psi = 1$.

5.2.2 Instantaneous and Terminal Correlations ρ

Swaption price depends on the terminal correlations between forward rates. These terminal correlations in turn depend on the corresponding instantaneous correlations $\rho_{i,j}$ and the way the associated caplet volatilities are decomposed in instantaneous volatilities for t (10).

Optimally, we want the **Instantaneous Correlations Matrix** ρ to have two notable features that are explained below.

- Firstly, because we expect the movements of rates that have underlying times closer to each other be more correlated, we want to have a decrease in the value of the entries in the matrix when we are moving away from the diagonal (both vertically and horizontally) where the correlation entries are 1. For example, we expect that the 6m-1y rate is more correlated with the 1y-1y6m rate compared to the 9y-9y6m rate since the later is very far away from the 6m-1y time period.
- Secondly, we expect an increase in the value of the entries along sub-diagonals. This is because that rates tend to be more correlated for long maturities compared to shorter maturities. For example, the correlation between 6m-1y rate and 1y-1y6m rate will be lower than the correlation between 2y6m-3y rate and 3y-3y6m

rate.

There are several parameterizations discovered in previous studies can achieve those desirable features. Many effective choices have been summerised in “A note on correlation and rank reduction” (11). One example is the **‘Improved, stable, full rank, two-parameters, increasing along sub-diagonals’ parameterisation**(S&C2) (23):

$$\rho_{i,j} = \exp\left[-\frac{|i-j|}{M-1}\left(-\ln \rho_\infty + \eta \frac{i^2 + j^2 + ij - 3Mi - 3Mj + 3i + 3j + 2M^2 - M - 4}{(M-2)(M-3)}\right)\right] \tag{5.6}$$

However, due to computational difficulties, we are going to use a methods that compromises on the second point and gives constant sub-diagonal entries instead. In this thesis we are using the **‘Full rank, classical, two-parameters, exponentially decreasing’ parameterisation**. This parameterisation is characterised as follows:

$$\rho_{i,j} = \rho_\infty + (1 - \rho_\infty)\exp[-\alpha|i-j|], \quad \alpha \geq 0. \tag{5.7}$$

In this formula, ρ_∞ represents asymptotically the correlation between the furthest forward rates pair in the forward rate family considered, i.e. $\rho_{1,M}$ in our case (10).

5.2.3 Initial Forward Rate $F_k(0)$

In our Monte Carlo simulation we assumed $P(0, T_\alpha)$ to be 0.01. In addition, we choose $F_k(0)$ for all $k = \alpha + 1, \ldots, \beta$ based on the scenarios that we want to explore. With T_α and T_β fixed, each $F_k(0)$ is responsible for the generation of the forward rate path $F_k(0 + \Delta t), \ldots, F_k(T_\alpha - \Delta t), F_k(T_\alpha)$ for all $k \in \alpha + 1, \ldots, \beta$.

5.2.4 Zero-coupon bond prices $P(T_\alpha, T_i)$ and $P(0, T_\alpha)$

To calculate the final swaption price given by formula (5.1), we also need to calculate the zero-coupon bond prices $P(T_\alpha, T_i)$ for $i = \alpha + 1, \ldots, \beta$. In order to calculate them, we apply the relationship between zero-coupon bond and forward rates again:

$$\begin{aligned} F(T_\alpha, T_{k-1}, T_k) &= \frac{1}{T_k - T_{k-1}}\left(\frac{P(T_\alpha, T_{k-1})}{P(T_\alpha, T_k)} - 1\right) \\ \Rightarrow P(T_\alpha, T_k) &= \frac{P(T_\alpha, T_{k-1})}{(T_k - T_{k-1})F(T_\alpha, T_{k-1}, T_k) + 1} \end{aligned} \tag{5.8}$$

With the relationship given in (5.8), we can derive all the $P(T_\alpha, T_i)$ needed for the swaption price by starting with $P(T_\alpha, T_\alpha)$. This quantity is equal to 1 by definition. Then we can get all the other bond prices recursively: for instance, for $P(T_\alpha, T_{\alpha+1})$, we can simply do $P(T_\alpha, T_{\alpha+1}) = \frac{P(T_\alpha, T_\alpha)}{(T_{\alpha+1} - T_\alpha)F(T_\alpha, T_\alpha, T_{\alpha+1}) + 1} = \frac{1}{\tau F_{\alpha+1}(T_\alpha) + 1}$.

For the discounting factor $P(0, T_{\alpha+1})$, we apply the formula

$$P(0, T_\alpha) = \frac{1}{(T_k - T_{k-1})F(0, 0, T_\alpha) + 1} \tag{5.9}$$

to get its value. Here we assume $F(0, 0, T_\alpha) = 0.01$.

5.2.5 Brownian Motion Z

Since we have that $dZ_i dZ_j = \rho_{i,j} dt$ and ρ being an MxM full rank correlation matrix, then by Lemma A.1.1, we know that there exists an MxM matrix A and a vector of standard Brownian Motion W whose individual components are independent of each other, i.e. $dW_i dW_j = 0$, such that:

$$AA^T = \rho, \quad dZ = AdW \tag{5.10}$$

Thus, the method we are using to generate dZ is by generating dW and multiply the vector by the matrix A. In other word, in our formula (5.4), we simulate standard normal random variables $W_k(t)$ for $k = \alpha + 1, \ldots, \beta$ and $t = 0, 0.025, \ldots, 5$. Then we multiply the simulated standard normal vectors with matrix A which comes from the cholesky factorization as described in Definition A.4.1 to get a matrix of correlated normal variables $Z_k(t)$. Then we take the difference $(Z_k(t + \Delta t) - Z_k(t))$ where W to get the desired Brownian Motion quantities.

To simulate a vector of standard normal random variables at each chosen time point t, we choose to use the Marsaglia polar method (19) (aka acception-rejection method) as it is faster that the Box-Muller method (Definition A.5.1) since it does not involve calculations of trigonometry.

Marsaglia Polar Method

1. Let U_1 and U_2 be two independent Uniform(0,1) random variables. Set $V_1 = 2U_1 - 1$ and $V_2 = 2U_2 - 1$.

2. Let $W = V_1^2 + V_2^2$.

3. If $W > 1$, return to step 1. Otherwise, return $N_1 = \sqrt{\frac{(-2\ln W)}{W}} V_1$ and $N_2 = \sqrt{\frac{(-2\ln W)}{W}} V_2$, where N_1 and N_2 are two independent standard normal variables.

To generate a larger vector of standard normal variables at each time point t, we simply generate multiple pairs of standard normal variables using the Marsaglia Polar Method described above and put all the pairs in the same vector.

Finally, the only issue left to consider is the generation of independent standard uniform random variables. We choose to use the **Linear Congruential Generator (LCG)**. This generator works as follows (described in (25)):

1. Choose a prime number $m > 0$ and ab integer $a > 0$ such that $a^{m-1} - 1$ is divisible by m, and $a^j - 1$ is not divisible by m for $j = 1, \ldots, m-2$. In our simulation, we choose $a = 7^5 = 16807$ and $m = 2^{31} - 1 = 2147483647$.

2. Start with an integer seed $n_0 > 0$ randomly and generate a sequence of integers n_i recursively by $n_i = (an_{i-1}) \mod m$ for $i = 1, 2, \ldots, N$.

3. Then we perform the division $x_i = \frac{n_i}{m}$ to get a sequence of numbers between 0 and 1. These $x_i's$ are called pseudo-uniform random numbers that follow independent Uniform(0,1) distributions.

Now we have all the components needed for the Milstein Scheme of Monte Carlo simulation. We are going to discuss our results in the next chapter.

6 Results discussions

6.1 Correlation matrices, Volatility matrices, Initial forward rates and Tenor

Correlation Matrices ρ

We have used mainly 4 correlation matrices in our simulation and computation of the analytical formula. Two of them represent high correlation between forward rates, and the other two represent low correlation between forward rates. The choices of α and ρ_∞ are shown in Table 5.

Classification	α	ρ_∞	Highest Correlation	Lowest Correlation	Range
1. High Correlation	0.02	0.9	0.99802	0.98025	0.01777
2. High Correlation	0.04	0.8	0.99216	0.90976	0.0824
3. Low Correlation	0.15	0.5	0.93035	0.52489	0.40546
4. Low Correlation	0.15	0.4	0.91642	0.51523	0.40119

Table 5: Correlation Matrix parameters used in computations and simulations.

Volatility Matrices $\sigma(t)$

We have used mainly 4 sets of volatility matrices by setting the upper and lower bounds of the $\sigma's$ we choose. Again, two of them correspond to high volatility and two of them correspond to low volatility. The choices are shown in Table 6.

Classification	Lower Bound	Upper Bound
1. High Volatility	0.15	0.2
2. High Volatility	0.25	0.3
3. Low Volatility	0.05	0.08
4. Low Volatility	0.08	0.1

Table 6: Volatility Matrix parameters used in computations and simulations.

Initial shifted forward rates $X_k(0)$

We have used mainly 6 sets of initial shifted forward rates $X_k(0)$, corresponding to 3 different scenarios: large rates, near zero rates and negative rates. The choices are shown in Table 7. Table 8 shows the corresponding initial forward rate $F_k(0)$.

Tenor t

In our simulation and analytical price computations, we used $\tau = T_i - T_{i-1} = 0.25$. In addition, in the Monte Carlo simulation, we choose $\Delta t = 0.025$ for our simulation paths.

Classification	Lower Bound	Upper Bound
1. Normal Rates	0.04	0.06
2. Normal Rates	0.05	0.08
3. Near Zero Rates	0.008	0.01
4. Near Zero Rates	0.007	0.009
5. Negative Rates	0.003	0.004
6. Negative Rates	0.002	0.003

Table 7: Initial shifted forward rates $X_k(0)$ parameters used in computations and simulations.

Classification	Lower Bound	Upper Bound
1. Normal Rates	0.035	0.055
2. Normal Rates	0.045	0.075
3. Near Zero Rates	0.003	0.005
4. Near Zero Rates	0.002	0.004
5. Negative Rates	-0.002	-0.001
6. Negative Rates	-0.003	-0.002

Table 8: Initial shifted forward rates $F_k(0)$ parameters used in computations and simulations.

6.2 Absolute, Percentage and Standard Errors

In the Tables of results listed below in Section 6.4, we analysed the discrepancies between the two pricing results using both the absolute errors and the percentage errors. We also listed the rounded standard errors in the table to ensure the accuracy of the simulation results.

1. **Absolute Errors**
   Absolute Errors are calculated as (Analytical Formula Price - Monte Carlo Price).

2. **Percentage Errors**
   Percentage Errors are calculated as $\frac{100(\text{Analytical Formula Price}-\text{Monte Carlo Price})}{\text{Monte Carlo Price}}$.

3. **Standard Error**
   Following the description in Lecture Notes (10), the calculation is as follows:
   Let n_p be the number of paths in each simulation, and let $\Pi(T_\alpha)$ be the swaption payoff. The logic behind the Monte Carlo simulation is:

$$\mathbb{E}\left[D(0,T_\alpha)\Pi(T_\alpha)\right] = P(0,T_\alpha)\sum_{j=1}^{n_p}\frac{\Pi^j(T_\alpha)}{n_p} \tag{6.1}$$

   Note that each Π^j is coming from the one set of simulation of forward rates $F_k^j(T_\alpha)$ for $k = \alpha+1,\dots,\beta$.

We want to get the error between the true expectation value $\mathbb{E}[\Pi(T_\alpha)]$ and the Monte Carlo estimation of this expectation $\sum_{j=1}^{n_p} \frac{\Pi^j}{n_p}$. Here we can view the $(\Pi^j)_j$ as a sequence of independent and identically distributed random variable, all coming from the same distribution as $\Pi(T_\alpha)$.

By Central Limit Theorem (CLT), under suitable assumption we have:

$$\frac{\sum_{j=1}^{n_p}(\Pi(T_\alpha)^j - \mathbb{E}(\Pi(T_\alpha))}{\sqrt{n_p}Std(\Pi(T_\alpha))} \overset{n_p\to\infty}{\longrightarrow} N(0,1) \tag{6.2}$$

From (6.2) we can derive that

$$\begin{aligned} \sum_{j=1}^{n_p}(\Pi(T_\alpha)^j - \mathbb{E}(\Pi(T_\alpha)) &\approx \sqrt{n_p}Std(\Pi(T_\alpha))N(0,1) \\ \Rightarrow \frac{\sum_{j=1}^{n_p}(\Pi(T_\alpha)^j)}{n_p} - \mathbb{E}(\Pi(T_\alpha)) &\approx \frac{Std(\Pi(T_\alpha))}{\sqrt{n_p}}N(0,1) \end{aligned} \tag{6.3}$$

We want to find the ϵ such that

$$\begin{aligned} \mathbb{Q}^{T_\alpha}\left(\left|\frac{\sum_{j=1}^{n_p}(\Pi(T_\alpha)^j)}{n_p} - \mathbb{E}(\Pi(T_\alpha))\right| < \epsilon\right) = 2\phi\left(\epsilon\frac{\sqrt{n_p}}{Std(\Pi(T_\alpha))}\right) - 1 = 0.98 \\ \Rightarrow \epsilon = 2.33\frac{Std(\Pi(T_\alpha))}{\sqrt{n_p}} \end{aligned} \tag{6.4}$$

By doing the above, we guarantee that the true value of $\mathbb{E}(\Pi(T_\alpha))$ is in the 98% confidence interval:

$$\left[\frac{\sum_{j=1}^{n_p}(\Pi(T_\alpha)^j)}{n_p} - 2.33\frac{Std(\Pi(T_\alpha))}{\sqrt{n_p}},\quad \frac{\sum_{j=1}^{n_p}(\Pi(T_\alpha)^j)}{n_p} + 2.33\frac{Std(\Pi(T_\alpha))}{\sqrt{n_p}}\right] \tag{6.5}$$

Finally, since the standard deviation $Std(\Pi(T_\alpha))$ is unknown in our case, we follow in same method as Lecture Notes (10) by setting

$$(\widehat{Std}(\Pi(T_\alpha);n_p)^2 := \frac{\sum_{j=1}^{n_p}(\Pi^j)^2}{n_p} - \left(\frac{\sum_{j=1}^{n_p}\Pi^j}{n_p}\right)^2 \tag{6.6}$$

This means that the actual 98% confidence interval of the Monte Carlo Simulation is:

$$\left[\frac{\sum_{j=1}^{n_p}(\Pi(T_\alpha)^j)}{n_p} - 2.33\frac{\widehat{Std}(\Pi(T_\alpha);n_p)}{\sqrt{n_p}},\quad \frac{\sum_{j=1}^{n_p}(\Pi(T_\alpha)^j)}{n_p} + 2.33\frac{\widehat{Std}(\Pi(T_\alpha);n_p)}{\sqrt{n_p}}\right] \tag{6.7}$$

and the standard error is therefore $2.33\frac{\widehat{Std}(\Pi(T_\alpha);n_p)}{\sqrt{n_p}} = 2.33\sqrt{\frac{\frac{\sum_{j=1}^{n_p}(\Pi^j)^2}{n_p} - \left(\frac{\sum_{j=1}^{n_p}\Pi^j}{n_p}\right)^2}{n_p}}$.

Note: if the standard error is too big to be accepted, the control variate method described in Section A.3 should be applied.

6.3 Results: at-the-money and in-the-money swaptions

To make sure that our model and testing procedure code are well-established, we first looked at results from the corresponding LMM ($\alpha_S = \alpha_F = 0$), which has been proved to have little discrepancies between the two pricing methods. We found that the result from my code testifies this.

The swaptions are priced at-the-money, except for cases that the at-the-money swaption prices are far too small (e.g. $3^{*}10^{-4}$). In those cases, we price in-the-money swaptions instead to obtain more meaningful results.

For each swaption underlying $S_{\alpha,\beta}$, we computed the price under 12 different cases. The cases descriptions are summerised below:

1. **Case 1: High correlation, large rates, high volatility.**
2. **Case 2: High correlation, large rates, low volatility.**
3. **Case 3: High correlation, near zero rates, high volatility.**
4. **Case 4: High correlation, near zero rates, low volatility.**
5. **Case 5: High correlation, negative rates, high volatility.**
6. **Case 6: High correlation, negative rates, low volatility.**
7. **Case 7: Low correlation, large rates, high volatility.**
8. **Case 8: Low correlation, large rates, low volatility.**
9. **Case 9: Low correlation, near zero rates, high volatility.**
10. **Case 10: Low correlation, near zero rates, low volatility.**
11. **Case 11: Low correlation, negative rates, high volatility.**
12. **Case 12: Low correlation, negative rates, low volatility.**

Due to the limitation in computational power, we only tested the results for swaptions which have $T_\beta \leq 10$. In reality, swaptions can have maturity dates T_β up to 30 years.

$S_{1,10}$ Table 10	$S_{1,8}$ Table 11	$S_{1,5}$ Table 12	$S_{1,3}$ Table 13	$S_{1,2}$ Table 14
$S_{2,10}$ Table 15	$S_{2,5}$ Table 16	$S_{3,10}$ Table 17	$S_{3,8}$ Table 18	$S_{3,5}$ Table 19
$S_{4,10}$ Table 20	$S_{4,7}$ Table 21	$S_{5,10}$ Table 22	$S_{5,8}$ Table 23	$S_{5,7}$ Table 24
$S_{5,6}$ Table 25	$S_{6,10}$ Table 26	$S_{6,8}$ Table 27	$S_{8,10}$ Table 28	$S_{8,9}$ Table 29

Table 9: swaptions pricing results table summary

Table 9 shows all the swaptions that we tested.

Below are the results from Monte Carlo simulation (DDF model) and from direct computation of the close form Black's formula for swaption (DDS model), subject to rounding errors. Note that the numbers in brackets represent negative numbers.

$S_{1,10}$	Analytical	MC Simulation	Abs. Error	Per. Error(%)	Std. Error
Case 1	0.03929	0.04015	(0.0009)	(2.134)	0.0015
Case 2	0.01021	0.01037	(0.0001)	(1.505)	0.0003
Case 3	0.02769	0.02713	0.0005	2.084	0.0002
Case 4	0.02855	0.02857	(2^*10^{-5})	(0.073)	0.0001
Case 5	0.00233	0.00229	3^*10^{-5}	1.339	8.7^*10^{-5}
Case 6	0.00100	0.00100	1^*10^{-6}	0.104	3.2^*10^{-5}
Case 7	0.01944	0.01897	0.0005	2.496	0.0007
Case 8	0.01024	0.01008	0.0001	1.580	0.0003
Case 9	0.03211	0.03162	0.0005	1.548	0.0009
Case 10	0.02910	0.02926	(0.0001)	(0.545)	0.0004
Case 11	0.00187	0.00187	(3^*10^{-7})	(0.016)	6.7^*10^{-5}
Case 12	0.00033	0.00033	2^*10^{-6}	0.794	1.6^*10^{-5}

Table 10: At-the-money and in-the-money $S_{1,10}$ pricing results under DDM

From the results we can conclude that the DDF and DDS model have high level of consistency in terms of swaption pricing, even when the underlying forward rates are negative.

$S_{1,8}$	Analytical	MC Simulation	Abs. Error	Per. Error(%)	Std. Error
Case 1	0.05263	0.05232	0.0003	0.593	0.0003
Case 2	0.02479	0.02466	0.0001	0.529	0.0004
Case 3	0.02066	0.02049	0.00016	0.812	0.0003
Case 4	0.01967	0.01972	($4.0*10^{-5}$)	(0.234)	0.0001
Case 5	0.00511	0.00502	$8.8*10^{-5}$	1.749	0.0001
Case 6	0.00333	0.00331	$1.2*10^{-5}$	0.351	$5.1*10^{-5}$
Case 7	0.08963	0.08982	(0.0002)	(0.213)	0.0001
Case 8	0.08329	0.08295	0.0003	0.418	0.0004
Case 9	0.01005	0.00990	0.0001	1.527	0.0002
Case 10	0.00709	0.00722	(0.0001)	(1.650)	$8.2*10^{-5}$
Case 11	0.00386	0.00392	($6.7*10^{-5}$)	(1.710)	0.0001
Case 12	0.00375	0.00374	$9.8*10^{-6}$	0.261	$3.3*10^{-5}$

Table 11: At-the-money and in-the-money $S_{1,8}$ pricing results under DDM

$S_{1,5}$	Analytical	MC Simulation	Abs. Error	Per. Error(%)	Std. Error
Case 1	0.02173	0.02142	0.0003	1.445	0.0008
Case 2	0.01095	0.01085	0.0001	0.952	0.0002
Case 3	0.00871	0.00856	0.0001	1.861	0.0002
Case 4	0.01186	0.01184	$2.4*10^{-5}$	0.201	$6.5*10^{-5}$
Case 5	0.00282	0.00277	$5.4*10^{-5}$	1.965	$7.7*10^{-5}$
Case 6	0.00227	0.00226	$5.1*10^{-6}$	0.224	$2.8*10^{-5}$
Case 7	0.02300	0.02290	$9.9*10^{-5}$	0.433	0.0007
Case 8	0.02248	0.02242	$5.9*10^{-5}$	0.267	0.0002
Case 9	0.01158	0.01161	($3.4*10^{-5}$)	(0.290)	0.0001
Case 10	0.01153	0.11510	$2.3*10^{-5}$	0.200	$4.9*10^{-5}$
Case 11	0.00220	0.00222	($1.9*10^{-5}$)	(0.869)	$6.2*10^{-5}$
Case 12	0.00205	0.00206	($5.3*10^{-6}$)	(0.258)	$2.2*10^{-5}$

Table 12: At-the-money and in-the-money $S_{1,5}$ pricing results under DDM

$S_{1,3}$	Analytical	MC Simulation	Abs. Error	Per. Error(%)	Std. Error
Case 1	0.06175	0.06156	0.0004	0.682	$7.0*10^{-5}$
Case 2	0.06532	0.06535	($2.3*10^{-5}$)	(0.359)	$6.1*10^{-5}$
Case 3	0.00464	0.00464	$2.9*10^{-6}$	0.006	$3.9*10^{-5}$
Case 4	0.00395	0.00395	$3.7*10^{-7}$	0.009	$1.2*10^{-5}$
Case 5	0.00139	0.00139	$1.8*10^{-6}$	0.131	$1.2*10^{-5}$
Case 6	0.00103	0.00103	($5.34*10^{-7}$)	(0.052)	$3.6*10^{-6}$
Case 7	0.04649	0.04644	$4.7*10^{-5}$	0.102	0.0002
Case 8	0.04887	0.04884	$3.0*10^{-7}$	0.062	$5.2*10^{-5}$
Case 9	0.00997	0.00997	($2.9*10^{-6}$)	(0.029)	$4.1*10^{-5}$
Case 10	0.01022	0.01023	($1.1*10^{-5}$)	(0.116)	$1.2*10^{-5}$
Case 11	0.00160	0.00159	$2.8*10^{-6}$	0.175	$1.2*10^{-5}$
Case 12	0.00075	0.00074	$1.5*10^{-6}$	0.196	$3.4*10^{-6}$

Table 13: At-the-money and in-the-money $S_{1,3}$ pricing results under DDM

$S_{1,2}$	Analytical	MC Simulation	Abs. Error	Per. Error(%)	Std. Error
Case 1	0.02035	0.02051	(0.0001)	(0.778)	0.0004
Case 2	0.00849	0.00851	($1.8*10^{-5}$)	(0.216)	$2.5*10^{-5}$
Case 3	0.00272	0.00271	$2.9*10^{-6}$	0.106	$1.2*10^{-5}$
Case 4	0.00276	0.00276	$3.6*10^{-6}$	0.132	$5.8*10^{-6}$
Case 5	0.00236	0.00236	($4.6*10^{-7}$)	(0.019)	$4.0*10^{-6}$
Case 6	0.00276	0.00276	$8.9*10^{-7}$	0.032	$1.8*10^{-6}$
Case 7	0.01746	0.01748	($2.1*10^{-5}$)	(0.120)	$6.4*10^{-5}$
Case 8	0.01485	0.01484	$5.0*10^{-6}$	0.034	$2.2*10^{-5}$
Case 9	0.00142	0.00141	$8.3*10^{-6}$	0.589	$9.7*10^{-6}$
Case 10	0.00188	0.00188	$9.5*10^{-7}$	0.051	$4.1*10^{-6}$
Case 11	0.00268	0.00268	($1.3*10^{-6}$)	(0.047)	$5.7*10^{-6}$
Case 12	0.00242	0.00242	$1.5*10^{-7}$	0.006	$2.0*10^{-6}$

Table 14: At-the-money and in-the-money $S_{1,2}$ pricing results under DDM

$S_{2,10}$	Analytical	MC Simulation	Abs. Error	Per. Error(%)	Std. Error
Case 1	0.07398	0.07426	(0.0003)	(0.382)	0.0002
Case 2	0.03091	0.03035	0.0006	1.828	0.0007
Case 3	0.02812	0.02822	($9.6*10^{-5}$)	(0.341)	0.0007
Case 4	0.02443	0.2440	$2.9*10^{-5}$	0.118	0.0002
Case 5	0.01994	0.2004	(0.0001)	(0.516)	0.0002
Case 6	0.02074	0.02074	($3.4*10^{-6}$)	(0.016)	$7.1*10^{-5}$
Case 7	0.09939	0.09952	(0.0001)	(0.131)	0.0002
Case 8	0.10495	0.10550	(0.0005)	(0.516)	0.0006
Case 9	0.01828	0.01845	(0.0002)	(0.936)	0.0005
Case 10	0.01677	0.01662	0.0001	0.875	0.0001
Case 11	0.02020	0.2018	$2.0*10^{-5}$	0.099	0.0002
Case 12	0.01944	0.01955	$2.5*10^{-6}$	0.013	$5.2*10^{-5}$

Table 15: At-the-money and in-the-money $S_{2,10}$ pricing results under DDM

$S_{2,5}$	Analytical	MC Simulation	Abs. Error	Per. Error(%)	Std. Error
Case 1	0.07202	0.07187	0.0001	0.209	0.0006
Case 2	0.00970	0.00973	($3.8*10^{-5}$)	(0.395)	0.00001
Case 3	0.01023	0.01028	($5.0*10^{-5}$)	(0.489)	0.0001
Case 4	0.00955	0.00953	$1.9*10^{-5}$	0.207	$3.5*10^{-5}$
Case 5	0.00791	0.00788	$3.0*10^{-5}$	0.379	$4.5*10^{-5}$
Case 6	0.00754	0.00754	$6.9*10^{-7}$	0.009	$1.2*10^{-5}$
Case 7	0.07227	0.07217	0.0001	0.0139	0.0005
Case 8	0.06411	0.06412	($7.7*10^{-6}$)	(0.012)	0.0001
Case 9	0.01057	0.01069	(0.0001)	(1.083)	0.0001
Case 10	0.00903	0.00905	(1.541)	(0.170)	$3.5*10^{-5}$
Case 11	0.00778	0.00776	$1.9*10^{-5}$	0.243	$3.9*10^{-5}$
Case 12	0.00733	0.00733	($2.0*10^{-7}$)	(0.003)	$9.8*10^{-6}$

Table 16: At-the-money and in-the-money $S_{2,5}$ pricing results under DDM

$S_{3,10}$	Analytical	MC Simulation	Abs. Error	Per. Error(%)	Std. Error
Case 1	0.06322	0.06389	(0.0007)	(1.050)	0.0002
Case 2	0.08272	0.08294	(0.0002)	(0.268)	0.0001
Case 3	0.03523	0.03539	(0.0002)	(0.467)	0.0001
Case 4	0.03396	0.03387	$8.5*10^{-5}$	0.252	0.0003
Case 5	0.00878	0.00872	$5.4*10^{-5}$	0.622	$1.0*10^{-5}$
Case 6	0.00863	0.00862	$1.0^{8}10^{-5}$	0.119	$5.1*10^{-5}$
Case 7	0.07867	0.07742	0.0012	1.619	0.0002
Case 8	0.05589	0.05637	(0.0005)	(0.855)	0.0008
Case 9	0.01376	0.01417	(0.0004)	(2.898)	0.0005
Case 10	0.00504	0.00507	$(2.7*10^{-5})$	(0.530)	0.0001
Case 11	0.01740	0.01754	(0.0001)	(0.745)	0.0002
Case 12	0.01647	0.01651	$(3.7*10^{-5})$	(0.221)	$5.6*10^{-5}$

Table 17: At-the-money and in-the-money $S_{3,10}$ pricing results under DDM

$S_{3,8}$	Analytical	MC Simulation	Abs. Error	Per. Error(%)	Std. Error
Case 1	0.05944	0.05727	0.0022	3.781	0.0002
Case 2	0.02934	0.02912	0.00022	0.776	0.0006
Case 3	0.01808	0.01860	(0.0005)	(2.769)	0.0007
Case 4	0.00976	0.00970	$6.1*10^{-5}$	0.627	0.0001
Case 5	0.00478	0.00485	$(6.5*10^{-5})$	(1.346)	$8.8*10^{-5}$
Case 6	0.00683	0.00682	$8.5*10^{-6}$	0.125	$2.7*10^{-5}$
Case 7	0.06805	0.06878	(0.0007)	(1.068)	0.0001
Case 8	0.05846	0.05834	0.0001	0.207	0.0002
Case 9	0.01301	0.01304	$(2.5*10^{-5})$	(0.191)	0.0001
Case 10	0.00938	0.00940	$(2.6*10^{-5})$	(0.276)	0.0001
Case 11	0.00804	0.00804	$1.1*10^{-6}$	0.014	$2.0*10^{-5}$
Case 12	0.00790	0.00790	$(3.0*10^{-7})$	(0.004)	$4.5*10^{-5}$

Table 18: At-the-money and in-the-money $S_{3,8}$ pricing results under DDM

$S_{3,5}$	Analytical	MC Simulation	Abs. Error	Per. Error(%)	Std. Error
Case 1	0.02127	0.02165	(0.0004)	(1.761)	0.0004
Case 2	0.01207	0.01211	$(3.4*10^{-5})$	(0.285)	$9.7*10^{-5}$
Case 3	0.00587	0.00579	$7.4*10^{-5}$	1.276	$6.7*10^{-5}$
Case 4	0.00360	0.00360	$2.4*10^{-6}$	0.067	$6.3*10^{-5}$
Case 5	0.00393	0.00394	$(6.5*10^{-6})$	(0.122)	$8.0*10^{-5}$
Case 6	0.00405	0.00404	$4.9*10^{-6}$	0.122	$2.2*10^{-5}$
Case 7	0.03219	0.03235	(0.0001)	(0.354)	0.0003
Case 8	0.02859	0.02870	(0.0001)	(0.354)	0.0003
Case 9	0.00731	0.00728	$3.5*10^{-5}$	0.477	0.0002
Case 10	0.00537	0.00538	$(7.2*10^{-6})$	(0.133)	$2.4*10^{-5}$
Case 11	0.00493	0.00493	$3.8*10^{-7}$	0.008	$3.3*10^{-5}$
Case 12	0.00478	0.00478	$(2.3*10^{-6})$	(0.047)	$8.7*10^{-6}$

Table 19: At-the-money and in-the-money $S_{3,5}$ pricing results under DDM

$S_{4,10}$	Analytical	MC Simulation	Abs. Error	Per. Error(%)	Std. Error
Case 1	0.07868	0.07632	0.0023	3.090	0.0003
Case 2	0.02723	0.02793	(0.0007)	(2.531)	0.0001
Case 3	0.02289	0.02315	(0.0002)	(2.531)	0.0001
Case 4	0.01696	0.01701	($5.4*10^{-5}$)	(0.317)	0.0002
Case 5	0.01474	0.01511	(0.0003)	(2.371)	0.0004
Case 6	0.01491	0.01483	$7.1*10^{-5}$	0.479	$9.9*10^{-5}$
Case 7	0.08705	0.08719	(0.0001)	(0.156)	0.0003
Case 8	0.06527	0.06492	0.0003	0.539	0.0001
Case 9	0.02016	0.01944	0.0007	3.719	0.0008
Case 10	0.01757	0.01757	$1.2*10^{-6}$	0.007	0.0002
Case 11	0.01506	0.01504	$1.5*10^{-5}$	0.102	0.0003
Case 12	0.01466	0.01469	($3.5*10^{-5}$)	(0.236)	$7.8*10^{-5}$

Table 20: At-the-money and in-the-money $S_{4,10}$ pricing results under DDM

$S_{4,7}$	Analytical	MC Simulation	Abs. Error	Per. Error(%)	Std. Error
Case 1	0.05919	0.05882	0.0004	0.631	0.001
Case 2	0.04231	0.04235	($3.6*10^{-5}$)	(0.085)	0.0002
Case 3	0.00942	0.00946	($4.2*10^{-5}$)	(0.444)	0.0002
Case 4	0.00584	0.00583	$3.7*10^{-6}$	0.064	$4.5*10^{-5}$
Case 5	0.00771	0.00774	($2.5*10^{-5}$)	(0.319)	$6.4*10^{-5}$
Case 6	0.00774	0.00773	$3.9*10^{-6}$	0.050	$1.7*10^{-5}$
Case 7	0.07421	0.07461	(0.0004)	(0.531)	0.0008
Case 8	0.07135	0.07134	$4.4*10^{-6}$	0.006	0.0002
Case 9	0.01107	0.01106	$1.2*10^{-5}$	0.109	0.0001
Case 10	0.00897	0.00894	$2.3*10^{-5}$	0.254	0.0006
Case 11	0.00751	0.00752	($1.1*10^{-5}$)	(0.140)	$5.5*10^{-5}$
Case 12	0.00734	0.00734	($1.5*10^{-6}$)	(0.021)	$1.4*10^{-5}$

Table 21: At-the-money and in-the-money $S_{4,7}$ pricing results under DDM

$S_{5,10}$	Analytical	MC Simulation	Abs. Error	Per. Error(%)	Std. Error
Case 1	0.06470	0.06390	0.0009	1.365	0.0034
Case 2	0.03620	0.03680	(0.0006)	(1.684)	0.0024
Case 3	0.01102	0.01101	$1.6*10^{-5}$	0.142	0.0011
Case 4	0.00861	0.00862	($1.6*10^{-5}$)	(0.187)	0.0001
Case 5	0.00378	0.00383	($5.3*10^{-5}$)	(1.370)	0.0001
Case 6	0.00273	0.00275	($1.4*10^{-5}$)	(0.523)	$5.5*10^{-5}$
Case 7	0.02877	0.02919	(0.0004)	(1.426)	0.0001
Case 8	0.01220	0.01207	0.0001	1.112	0.0004
Case 9	0.00571	0.00571	($5.9*10^{-7}$)	(0.010)	0.0002
Case 10	0.00125	0.00124	$7.0*10^{-6}$	0.559	$5.8*10^{-5}$
Case 11	0.00222	0.00218	$3.4*10^{-5}$	1.578	$9.2*10^{-5}$
Case 12	0.00087	0.00087	$1.0*10^{-5}$	1.191	$3.2*10^{-6}$

Table 22: At-the-money and in-the-money $S_{5,10}$ pricing results under DDM

$S_{5,8}$	Analytical	MC Simulation	Abs. Error	Per. Error(%)	Std. Error
Case 1	0.04537	0.04535	$1.9*10^{-5}$	0.042	0.0003
Case 2	0.01252	0.01234	0.0002	1.503	0.0004
Case 3	0.01178	0.01187	$(8.8*10^{-5})$	(0.745)	0.0004
Case 4	0.00866	0.00865	$1.3*10^{-5}$	0.147	0.0001
Case 5	0.00318	0.00320	$(2.2*10^{-5})$	(0.696)	0.0001
Case 6	0.00149	0.00149	$1.1*10^{-6}$	0.071	$3.4*10^{-5}$
Case 7	0.04143	0.04058	0.0008	2.088	0.0001
Case 8	0.01328	0.01339	(0.0001)	(0.854)	0.0004
Case 9	0.01015	0.01018	$(2.6*10^{-5})$	(0.254)	0.0003
Case 10	0.00622	0.00628	$(6.3*10^{-5})$	(0.999)	0.0001
Case 11	0.00329	0.00330	$(1.4*10^{-5})$	(0.430)	$6.6*10^{-5}$
Case 12	0.00238	0.00239	$(1.1*10^{-5})$	(0.462)	$1.9*10^{-5}$

Table 23: At-the-money and in-the-money $S_{5,8}$ pricing results under DDM

$S_{5,7}$	Analytical	MC Simulation	Abs. Error	Per. Error(%)	Std. Error
Case 1	0.02874	0.02867	$7.7*10^{-5}$	0.267	0.0001
Case 2	0.01423	0.01432	$(9.4*10^{-5})$	(0.653)	0.0001
Case 3	0.00941	0.00943	$(1.8*10^{-5})$	(0.187)	0.0001
Case 4	0.00825	0.00826	$(9.8*10^{-6})$	(0.119)	$3.8*10^{-5}$
Case 5	0.00281	0.00281	$5.1*10^{-6}$	0.180	$2.7*10^{-5}$
Case 6	0.00173	0.00173	$5.9*10^{-8}$	0.003	$4.8*10^{-6}$
Case 7	0.03876	0.03905	(0.0003)	(0.730)	0.0005
Case 8	0.03447	0.03441	$6.2*10^{-5}$	0.181	0.0002
Case 9	0.00692	0.00693	$(1.2*10^{-5})$	(0.175)	$1.0*10^{-5}$
Case 10	0.00405	0.00404	$1.7*10^{-6}$	0.042	$3.3*10^{-5}$
Case 11	0.00507	0.00510	$(2.4*10^{-5})$	(0.472)	$4.5*10^{-5}$
Case 12	0.00508	0.00507	$7.7*10^{-6}$	0.152	$1.2*10^{-5}$

Table 24: At-the-money and in-the-money $S_{5,7}$ pricing results under DDM

$S_{5,6}$	Analytical	MC Simulation	Abs. Error	Per. Error(%)	Std. Error
Case 1	0.01832	0.01813	0.0002	1.010	0.0008
Case 2	0.00821	0.00817	$3.5*10^{-5}$	0.432	$9.0*10^{-5}$
Case 3	0.00343	0.00342	$1.3*10^{-5}$	0.376	$6.4*10^{-5}$
Case 4	0.00175	0.00174	$6.3*10^{-6}$	0.357	$1.6*10^{-5}$
Case 5	0.00230	0.00229	$1.9*10^{-5}$	0.815	$2.3*10^{-5}$
Case 6	0.00233	0.00233	$(5.4*10^{-8})$	(0.002)	$5.6*10^{-6}$
Case 7	0.01550	0.01551	$(8.8*10^{-6})$	(0.057)	0.0006
Case 8	0.01609	0.01613	$(3.7*10^{-5})$	(0.227)	$9.5*10^{-5}$
Case 9	0.00342	0.00348	$(5.9*10^{-5})$	(1.705)	$6.3*10^{-5}$
Case 10	0.00259	0.00259	$(3.3*10^{-6})$	(0.128)	$1.9*10^{-5}$
Case 11	0.00132	0.00132	$(1.2*10^{-6})$	(0.090)	$1.2*10^{-5}$
Case 12	0.00160	0.00161	$(3.7*10^{-6})$	(0.231)	$3.6*10^{-6}$

Table 25: At-the-money and in-the-money $S_{5,6}$ pricing results under DDM

$S_{6,10}$	Analytical	MC Simulation	Abs. Error	Per. Error(%)	Std. Error
Case 1	0.07019	0.06815	0.0020	2.995	0.004
Case 2	0.02498	0.02497	$1.6*10^{-5}$	0.064	0.0007
Case 3	0.01896	0.01905	$(8.6*10^{-5})$	(0.452)	0.0007
Case 4	0.01566	0.01559	$7.0*10^{-5}$	0.451	0.0002
Case 5	0.01009	0.00995	0.00014	1.431	0.0002
Case 6	0.01006	0.01012	$(5.5^{8}10^{-5})$	(0.545)	$5.9*10^{-5}$
Case 7	0.07096	0.06985	0.0011	1.594	0.0002
Case 8	0.05423	0.05366	0.0006	1.074	0.0007
Case 9	0.01327	0.01350	(0.0002)	(1.741)	0.0005
Case 10	0.00747	0.00749	$(2.7*10^{-5})$	(0.367)	0.0001
Case 11	0.01056	0.01066	$(9.8*10^{-5})$	(0.917)	0.0002
Case 12	0.01017	0.01016	$5.2*10^{-6}$	0.051	$5.1*10^{-5}$

Table 26: At-the-money and in-the-money $S_{6,10}$ pricing results under DDM

$S_{6,8}$	Analytical	MC Simulation	Abs. Error	Per. Error(%)	Std. Error
Case 1	0.03062	0.03114	(0.0005)	(1.681)	0.001
Case 2	0.01416	0.01405	0.0001	0.781	0.0005
Case 3	0.00838	0.00840	$(2.7*10^{-5})$	(0.317)	0.0001
Case 4	0.00499	0.00496	$3.7*10^{-5}$	0.738	$3.7*10^{-5}$
Case 5	0.00481	0.00482	$(1.3*10^{-5})$	(0.264)	$5.4*10^{-5}$
Case 6	0.00473	0.00473	$(2.2*10^{-6})$	(0.047)	$1.3*10^{-5}$
Case 7	0.04949	0.04912	0.0003	0.749	0.0006
Case 8	0.05446	0.05452	$(6.0*10^{-5})$	(0.111)	0.0002
Case 9	0.00822	0.00822	$5.8*10^{-6}$	0.070	0.0001
Case 10	0.00565	0.00566	$(1.2*10^{-5})$	(0.214)	$3.8*10^{-5}$
Case 11	0.00512	0.00513	$(4.2*10^{-6})$	(0.082)	$4.9*10^{-5}$
Case 12	0.00493	0.00494	$(7.6*10^{-6})$	(0.154)	$1.2*10^{-5}$

Table 27: At-the-money and in-the-money $S_{6,8}$ pricing results under DDM

$S_{8,10}$	Analytical	MC Simulation	Abs. Error	Per. Error(%)	Std. Error
Case 1	0.03532	0.03597	(0.0006)	(1.805)	0.0018
Case 2	0.01804	0.01776	0.0003	1.587	0.0005
Case 3	0.00679	0.00668	0.0001	1.634	0.0003
Case 4	0.00313	0.00309	$4.6*10^{-5}$	1.486	$8.4*10^{-5}$
Case 5	0.00514	0.00510	$4.2*10^{-5}$	0.824	0.0001
Case 6	0.00505	0.00506	$(3.7*10^{-6})$	(0.073)	$3.5*10^{-5}$
Case 7	0.03493	0.03474	0.0002	0.554	0.0001
Case 8	0.01817	0.01816	$1.7*10^{-5}$	0.824	0.0001
Case 9	0.00683	0.00663	0.0002	3.130	$3.0*10^{-5}$
Case 10	0.00282	0.00289	$(6.5*10^{-5})$	(2.230)	$8.1*10^{-5}$
Case 11	0.00477	0.00470	$7.3*10^{-5}$	1.556	$1.0*10^{-5}$
Case 12	0.00507	0.00508	$(1.1*10^{-6})$	(0.021)	$3.1*10^{-5}$

Table 28: At-the-money and in-the-money $S_{8,10}$ pricing results under DDM

$S_{8,9}$	Analytical	MC Simulation	Abs. Error	Per. Error(%)	Std. Error
Case 1	0.01935	0.01911	0.0002	1.222	0.0004
Case 2	0.00887	0.00895	($8.6*10^{-5}$)	(0.995)	0.0002
Case 3	0.00443	0.00444	($1.0*10^{-5}$)	(0.228)	$8.9*10^{-5}$
Case 4	0.00286	0.00287	($1.1*10^{-5}$)	(0.377)	$2.2*10^{-5}$
Case 5	0.00163	0.00164	($5.4*10^{-6}$)	(0.332)	$2.8*10^{-5}$
Case 6	0.00134	0.00134	$1.6*10^{-6}$	0.120	$7.2*10^{-6}$
Case 7	0.02399	0.02405	($5.6*10^{-5}$)	(0.533)	$9.1*10^{-5}$
Case 8	0.00983	0.00988	($5.3*10^{-5}$)	(0.233)	0.0004
Case 9	0.00476	0.00475	$1.3*10^{-5}$	0.279	$9.2*10^{-5}$
Case 10	0.00329	0.00330	($1.0*10^{-5}$)	(0.312)	$2.4*10^{-5}$
Case 11	0.00132	0.00133	($7.8*10^{-6}$)	(0.539)	$1.6*10^{-5}$
Case 12	0.00166	0.00166	($7.5*10^{-7}$)	(0.045)	$5.0*10^{-6}$

Table 29: At-the-money and in-the-money $S_{8,9}$ pricing results under DDM

7 Analysis of the discrepancies between results of MC simulation and the analytical pricing formula

Although the results from the previous chapter were sound, there are still some biases in it that should be and analysed. Below are several aspects that could be improved if we have higher computational power or better analytical tools.

7.1 Shift α_F in DDF Model

In Section 3.2, it was mentioned that the shift in the DDF model in this thesis is chosen to be the same for all forward rates $F_k' s$. This choice may not be the best choice for the DDF model. To improve, one can choose different shifts α_F for different forward rates by developing a relationship between all the shifts.

7.2 Relationship between shift α_F and α_S

In Section 4.3.2, we discussed the ways to choose the shift α_S in DDS model. Due to the failure to apply the method we once thought, we decided to use $\alpha_F = \alpha_S$ which is an unjustifiable method of choosing the shift in the DDS and DDF models. Thus, we would interpret the majority of the errors in the two pricing methods to be coming from this choice of the shifts. In further studies and investigations, one could try to develop a relationship between the two shifts and to test the results from the two pricing methods described in this thesis by applying the relationship of the shifts.

7.3 The choice of α_F

In this thesis, the pricing results are based on the assumption of $\alpha_F = \alpha_S = -0.005$. This means that the model allows forward rates to be as low as 0.5% at extreme. This value may not be the best choice for the shift of the Displaced Diffusion market model.

We also used $\alpha_F = \alpha_S = -0.05$ in the model and checked the pricing results. The results were also sound and had little absolute and percentage errors. We would expect that the results obtained using other value of shift to have high level of consistencies as well.

7.4 Volatility Matrix $\sigma_k(t)$

In Section 5.2.1, we discussed the ways to choose the instantaneous volatilities σ_k. We mentioned that GPC is the richest parametrization that can essentially fit all the at-the-money options. However, we used a special case of GPC which is SPC. We also then subsequently set $\phi' s$ to 1 to get a homogeneous volatility matrix in the time-to-expiry. This could again cause biases and inaccuracies in the pricing results since we applied many strong assumptions when modelling volatility $\sigma' s$. To overcome this problem, one can implement the general version of GPC method as a way to choose the volatility matrix or fit the volatility matrix from historical data. In addition, we can choose the volatility such that it allows humped shape as desired.

7.5 Correlation Matrix ρ

In Section 5.2.2, we discussed the ways to choose the instantaneous correlations $\rho'_{i,j}s$. As mentioned in Section 5.2.2 and also learnt in lectures (10), there are many potential ways to choose the instantaneous correlation parametrisation. The parametrisation method that we used, which is the **'Full rank, classical, two-parameters, exponentially decreasing' parameterisation**, may not be the best one to choose. In fact, the main drawback of this choice is that the resulting correlation matrix is not increasing along its sub-diagonals. One can improve the model by choosing a better parameterisation for ρ: for example, in lectures we studied several other parameterisations such as the **Stable, full-rank, 3-parameters,' increasing along sub-diagonals' parametrization SC3** and the **'Improved, stable, full-rank, two-parameters, increasing along sub-diagonals' paraterisation**.

7.6 The choices of Δt in the Monte Carlo Simulation

In our simulation described in Chapter 5 and 6, we originally used $\Delta t = 0.005$ in the Milstein Scheme formula. This is approximately equal to 0.005 x 365=1.825 days. However, for large T_α (i.e. $T_\alpha \geq 5$), $\Delta t = 0.005$ implies that we have more than 1000 steps in each generation path in the Milstein Scheme. Therefore, the generating process was extremely slow even for small number of paths. Therefore, we needed to alter it to be $\Delta t = 0.025$, which implies approximately 0.025 x 365=9.125days. However, as mentioned in (5.5), larger Δt implies larger biases in the Monte Carlo Simulations. After this alternation, the simulation process became faster, but we still can only use up to $10,000$ paths in some of the generating processes due to limited computational power. If one has higher computational power, one can choose the number of path and the value of Δt such that the standard errors can be kept small.

7.7 Displaced Diffusion Model and other market models for forward rates

Finally, the reason of the discrepancies could be that the Displaced Diffusion Model has some disadvantage and incompleteness in itself. Therefore, there will always be discrepancies in the pricing results no matter how we improve the above mentioned aspects. Nonetheless, the discrepancies in the pricing results are small enough for us to conclude that the DDM is an effective interest rate model to use.

8 Conclusion

The aim of this thesis is to test the effectiveness of Displaced Diffusion Market Model by examining the discrepancies between the pricing results under Displaced Diffusion Forward Model and Displaced Diffusion Swap Model. Previous studies have shown that the connection between the Libor Market Model (for caplets/floorlets) and Swap Market Model (for swaptions) is a valid one, and our results in this thesis testified the connection between the extensions of LMM (DDF model) and SMM (DDS model).

In Chapter 3, we first derived the dynamics of forward rates in the Displaced Diffusion Market Model under different forward measures $\mathbb{Q}^i$. We know that under LMM, the forward rate dynamics are assumed to be log-normal. Therefore, we introduced shifted forward rates X_k^F's, which are log-normal, in the DDM in order to utilise all the properties that the forward rates under LMM have. These (shifted) forward rate dynamics are used in Chapter 5 in order to simulate the swap rates and the swaption prices under the Displaced Diffusion Forward Model (DDF Model) through Milstein Scheme. Then we developed the Displaced Diffusion Swap Model (DDS Model) in Chapter 4. Similar to the forward rate cases, we introduced shifted swap rates $X_k^{\alpha,\beta}$'s to benefit from the convenience of log-normal property. We derived the closed-form Black's formula for swaptions under the DDS Model which we used as the 'analytical pricing formula' for the Displaced Diffusion Market Model.

From our results of 20 different Swaptions, we conclude that the DDF and DDS model have high level of consistency for swaption pricing. The absolute errors were between 10^{-4} and 10^{-6} in absolute magnitude for most of the time, and percentage errors were below 1% for most of the cases. More importantly, this consistency is retained even in the cases where we were modelling negative forward rates, which is an advantage over LMM and SMM. This proves that the DDM is an effective model for interest rate modelling for all levels of volatility, correlation and rates, and for both in-the-money and at-the-money swaptions. In addition, it shows a valid and significant connection between the market of caplets and the market of swaptions under DDM.

Despite the high level consistency between DDF and DDS model, there were still some small discrepancies between the results. We analysed these discrepancies in Chapter 7 and proposed ways to potentially minimise the discrepancies. For example, one can develop a relationship between the shifts α_S and α_F to potentially mitigate the errors in pricing. One can also try to use another parametrisation for volatility σ and correlation ρ to obtain improved consistency.

Overall, we conclude that the DDM is a useful foundation and effective market model for negative interest rate modelling as well as normal positive interest rate modelling. In addition, we conclude that the connection between the caplet market and the swap market is a valid and strong one under DDM.

A Appendix

A.1 Eigenvalues zeroing an rescaling

Lemma A.1.1. *Let ρ be a positive definite symmetric matrix, then ρ satisfied the following equation $\rho = PHP^T$, where P is a real orthogonal matrix such that $P^T P = PP^T = I_M$, and H is a diagonal matrix of the positive eigenvalues of the matrix ρ. The columns of P are the eigenvectors of ρ, associated to the eigenvalues located in the corresponding position in H.*

Let Λ be the diagonal matrix whose entries are the square roots of the corresponding entries of H, so that if we set $A := P$, we have both:

$$
\begin{gathered}
AA^T = \rho, \quad A^T A = H. \\
\rho = PHP^T, \quad \Lambda = \sqrt{H}, \quad A := P\Lambda, \quad AA^T = \rho, \quad A^T A = H.
\end{gathered}
\tag{A.1}
$$

A.2 Facts

Below are two important lemmas described in the Lecture Notes (10).

Lemma A.2.1 (Fact One)**.** *The price of any asset divided by a reference asset (called numeraire) is a martingale (no drift) under the measure associated with that numeraire.*

For instance, let $\mathbb{Q}^2$ be the measure associated with the numeraire $P(\cdot, T_2)$, then by FACT ONE $F_2(t)$ is a martingale under the numeraire pair $(\mathbb{Q}^2, P(\cdot, T_2))$.

Lemma A.2.2 (Fact Two)**.** *The prices of the time-t risk neutral price for an option are the same under all suitably chosen numeraire pairs.*

For example, let (S, S_t) and (B, B_t) be two numeraire pairs. Then we have:

$$
\begin{aligned}
Price_t &= \mathbb{E}^B \left[B(t) \frac{\text{Payoff}(T)}{B(T)} \right] \\
&= \mathbb{E}^S \left[S_t \frac{\text{Payoff}(T)}{S_T} \right]
\end{aligned}
\tag{A.2}
$$

A.3 Using Control Variate Estimators to minimise the variance of Monte Carlo Simulations

To make the simulation more efficient, one can apply the control variate estimator method described in the Lecture Notes (10). Below is a brief summary of the method.

Consider a new random variable $\Pi_c(\gamma) := \Pi + \gamma(\Pi^{an} - \pi^{an})$, where γ is a constant to be determined, Π^{an} is the known. simulated analytical payoff and $\pi^{an} = \mathbb{E}(\Pi^{an})$. This is an unbiased estimator of $\mathbb{E}(\Pi)$ since $\mathbb{E}(\Pi + \gamma(\Pi^{an} - \pi^{an})) = \mathbb{E}(\Pi) + 0 = \mathbb{E}(\Pi)$. Thus, define the contral variate estimator to be:

$$\widehat{\Pi_c}(\gamma; n_p) := \frac{\sum_{j=1}^{n_p} \Pi^j}{n_p} + \gamma \left(\frac{\sum_{j=1}^{n_p} \Pi^{an,j}}{n_p} - \pi^{an} \right) \tag{A.3}$$

In this situation, $\pi^{an} = 0$ due to the martingale property. As we want to minimise the variance of (A.3), it follows that

$$\gamma^* := -Corr(\Pi, \Pi^{an}) Std(\Pi) / Std(\Pi^{an}) \tag{A.4}$$

Finally, plug (A.4) back into (A.3) and use

$$\begin{aligned}
\left(\widehat{Std}(\Pi_c(\gamma^*); n_p)\right)^2 &= \frac{\sum_{j=1}^{n_p} (\Pi^{an,j})^2}{n_p} - \left(\frac{\sum_{j=1}^{n_p} \Pi^{an,j}}{n_p} \right)^2 \\
\widehat{Cov}(\Pi, \Pi^{an}; n_p) &= \frac{\sum_{j=1}^{n_p} \Pi^j \Pi^{an,j}}{n_p} - \frac{\left(\sum_{j=1}^{n_p} \Pi^j\right)\left(\sum_{j=1}^{n_p} \Pi^{an,j}\right)}{n_p^2} \\
\widehat{Corr}(\Pi, \Pi^{an}; n_p) &= \frac{\widehat{Cov}(\Pi, \Pi^{an}; n_p)}{\widehat{Std}(\Pi; n_p)\widehat{Std}(\Pi^{an}; n_p)}
\end{aligned} \tag{A.5}$$

to get an estimated simulation result which has a lower variance. Note that in those cases that use the Control Variate Estimator method, the 98% confidence interval of the true value of the payoff now becomes

$$\left[\frac{\sum_{j=1}^{n_p} (\Pi(T_\alpha)^j)}{n_p} - 2.33 \frac{\widehat{Std}(\Pi_c(T_\alpha; \gamma^*); n_p)}{\sqrt{n_p}}, \quad \frac{\sum_{j=1}^{n_p} (\Pi(T_\alpha)^j)}{n_p} + 2.33 \frac{\widehat{Std}(\Pi_c(T_\alpha; \gamma^*))}{\sqrt{n_p}} \right] \tag{A.6}$$

and the standard error is now calculated using

$$2.33 \frac{\widehat{Std}(\Pi_c(T_\alpha; \gamma^*))}{\sqrt{n_p}} = 2.33 \frac{\widehat{Std}(\Pi; n_p)\left(1 - \widehat{Corr}(\Pi, \Pi^{an}; n_p)^2\right)^{\frac{1}{2}}}{\sqrt{n_p}} \tag{A.7}$$

A.4 Cholesky Factorisation

Definition A.4.1 (Cholesky Factorisation)**.** *The Cholesky factorisation of a symmetric matrix ρ is a representation*

$$\rho = AA^T \tag{A.8}$$

where A is a lower triangular matrix A.
If ρ is a symmetric matrix which is positive definite, then a Cholesky factorisation exists ([24]).

Since A is a lower triangular matrix, AA^T can be illustrated as

$$\begin{aligned} AA^T &= \begin{bmatrix} A_{11} & & & \\ A_{21} & A_{22} & & \\ \vdots & \vdots & \ddots & \\ A_{d1} & A_{d2} & \cdots & A_{dd} \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{d1} \\ & A_{22} & \cdots & A_{d2} \\ & \ddots & \vdots & \vdots \\ & & & A_{dd} \end{bmatrix} \\ &= \begin{bmatrix} A_{11}^2 & A_{11}A_{21} & \dots & A_{11}A_{d1} \\ A_{21}A_{11} & A_{21}^2 + A_{22}^2 & \dots & A_{21}A_{d1} + A_{22}A_{d2} \\ \vdots & \vdots & & \vdots \\ A_{d1}A_{11} & A_{d1}A_{21} + A_{d2}A_{22} & \dots & A_{d1}^2 + A_{d2}^2 + \cdots + A_{dd}^2 \end{bmatrix} \end{aligned} \tag{A.9}$$

We subsequently equate (A.9) with the matrix ρ to obtain the value of A_{11} to A_{dd} in the lower triangular matrix A.

A.5 Box Muller method for generating standard normal variables

Definition A.5.1. *The Box-Muller method works as follows (described in [24]):*

1. *Generate independent* U_1, $U_2 \sim \mathcal{U}(0,1)$; $SetR = \text{-}2\ln(U_1)$;

3. *Set* $\theta = 2_2$;

4. *Set* $X_1 = \sqrt{R}\cos(\theta)$;

5. *Set* $X_1 = \sqrt{R}\sin(\theta)$;

6. *Return* X_1, X_2.

References

[1] Bank of England Official Bank Rate history. https://www.bankofengland.co.uk/boeapps/database/Bank-Rate.asp, 2021. [Online; accessed 07-August-2021].

[2] Board of Governors of the Federal Reserve System, 2021. [Online; accessed 07-August-2021].

[3] Interest Rate, Investopedia. https://www.investopedia.com/terms/i/interestrate.asp, 2021. [Online; accessed 10-August-2021].

[4] the OECD: Interest rates - Short-term interest rates - OECD Data. https://data.oecd.org/interest/short-term-interest-rates.htm?context=OECD, 2021. [Online; accessed 07-August-2021].

[5] the OECD: Interest rates - Short-term interest rates forecast - OECD Data. https://data.oecd.org/interest/short-term-interest-rates-forecast.htm?context=OECD, 2021. [Online; accessed 07-August-2021].

[6] Andersen, L., and Andreasen, J. Volatility skews and extensions of the libor market model. *Applied Mathematical Finance 7*, 1 (2000), 1–32.

[7] Boundless. Boundless finance. https://courses.lumenlearning.com/boundless-finance/chapter/derivatives/. [Online; accessed 10-August-2021].

[8] Boyarchenko, N., Gupta, P., Steele, N., Yen, J., et al. Negative swap spreads. *Economic Policy Review*, 24-2 (2018), 1–14.

[9] Brace, A., Gątarek, D., and Musiela, M. The market model of interest rate dynamics. *Mathematical finance 7*, 2 (1997), 127–155.

[10] Brigo, D. Interest rate models with credit risk, collateral, funding liquidity risk and multiple curves. *MSc course lecture notes, Imperial College London.*.

[11] Brigo, D. A note on correlation and rank reduction. *Disponibile sul sito www. damianobrigo. it* (2002).

[12] Brigo, D., and Mercurio, F. A deterministic–shift extension of analytically–tractable and time–homogeneous short–rate models. *Finance and Stochastics 5*, 3 (2001), 369–387.

[13] Burgess, N. An overview of the vasicek short rate model. *Available at SSRN 2479671* (2014).

[14] Galluccio, S., Ly, J.-M., Huang, Z., and Scaillet, O. Theory and calibration of swap market models. *Mathematical Finance 17*, 1 (2007), 111–141.

[15] Goldys, B. A note on pricing interest rate derivatives when forward libor rates are lognormal. *Finance and Stochastics 1*, 4 (1997), 345–352.

[16] Huang, Z., and Scaillet, O. Modelling and calibration of swap market models.

[17] Hunter, C. J., Jäckel, P., and Joshi, M. S. Drift approximations in a forward-rate-based libor market model. *Getting the Drift* (2001), 81–84.

[18] Jamshidian, F. Libor and swap market models and measures. *Finance and Stochastics 1*, 4 (1997), 293–330.

[19] Marsaglia, G., and Bray, T. A. A convenient method for generating normal variables. *SIAM review 6*, 3 (1964), 260–264.

[20] Miltersen, K. R., Sandmann, K., and Sondermann, D. Closed form solutions for term structure derivatives with log-normal interest rates. *The Journal of Finance 52*, 1 (1997), 409–430.

[21] Musiela, M., and Rutkowski, M. Continuous-time term structure models: Forward measure approach. *Finance and Stochastics 1*, 4 (1997), 261–291.

[22] Pearson, N. D., and Sun, T.-S. Exploiting the conditional density in estimating the term structure: An application to the cox, ingersoll, and ross model. *The Journal of Finance 49*, 4 (1994), 1279–1304.

[23] Schoenmakers, J., and Coffey, B. Systematic generation of parametric correlation structures for the libor market model. *International Journal of Theoretical and Applied Finance 6*, 05 (2003), 507–519.

[24] Veraart, L. A. M. Computational methods in financial mathematics. *MA323 Lecture Notes, The London School of Economics and Political Science, Lent Term 2020*.

[25] Zheng, H. Simulation methods for finance. *MSc course lecture notes, Imperial College London.*.

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