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Dual control method for tight bounds of value function when the drift following the Ornstein-Uhlenbeck process

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Abstract

One of the aims of this paper is to find the feedback control in constrained quadratic minimization problem under two-dimensional cases. Another aim is to study the upper and lower bounds on the primal value function for utility maximization problem when the drift following the Ornstein-Uhlenbeck(OU) process with power utility and non-HARA utility functions. We can find a closed-form solution for power utility using both primal HJB and dual HJB due to its homothetic property when the drift μ_t following the OU process. However, it is impossible to get an exact solution for general utilities. In this paper, by the duality relationship, we construct the upper bound from a dual problem and then construct a feasible control to find the lower bound under the OU process with power and non-HARA utilities and estimate the gap of upper and lower bounds. The dual control Monte-Carlo method is used to calculate the tight upper and lower bounds of the value function. Finally, we perform some numerical tests to check the robustness, efficiency and accuracy of the upper and lower bounds method.

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Introduction

Stochastic optimal control and convex duality methods are widely used to solve dynamic portfolio optimization problems. In the optimization theory, the duality principle is a principle problem for optimizing problems from the primal problem's angles or the dual problem. The relationship of the primal problem and the dual problem is that the solution of the dual problem produces a lower bound to the solution of the primal problem.[1] Therefore, The key idea of solving dynamic portfolio optimization problems is to use the dynamic programming principle(DPP) to the optimal value function, see [2]. This method will produce a second-order nonlinear partial differential equation(PDE) called Hamiltonian-Jacobian-Berman (HJB) equation. However, when solving this equation, we also obtain the optimal control in the form of "feedback." This means that if we find an exact formula of the HJB, then we can use the martingale principle of optimality to verify that it is the value function and find the optimal feedback control simultaneously. Otherwise, it can be proved that the value function is a unique solution of the HJB equation, see [3]. For a complete market with some special utilities, it is easy to find a closed-form solution to the nonlinear HJB equation using the dual control method. Bian and Zheng[4] show that for a complete market with closed convex cone constraints for controls and some particular continuous concave utilities, there exists a closed-form solution to the HJB equation.

However, for an incomplete market is driven by some Markovian processes such as the Ornstein-Uhlenbeck(OU) process, it isn't easy to solve the fully nonlinear HJB PDE directly. The reason is that the HJB equation has two state variables, wealth X_t and drift μ_t. The OU process is not a geometric Brownian motion which leads to the dual HJB equation is an equally difficult nonlinear PDE with two state variables. In this paper, we discuss the method of finding an approximate solution to the HJB equation under the OU process. Since the OU process is an affine stochastic process, for a power utility, one may decompose the solution to reduce the dimensionality of state variables by one and get a simplified nonlinear PDE with one state variable (drift). A clever transformation proposed by Zariphopoulou simplifies the nonlinear equation further into an equivalent linear PDE. Then we can derive the classical solution by Feynman-Kac Theorem; see more details from [5].

However, for other general utilities, we cannot find a closed-form solution of the HJB equation. Although the dual control method cannot solve general utility maximization problems for our model, it provides valuable information for the optimal value function. Because of the weak duality relation, we can construct the upper bound from the dual value function for the primal value function and then construct a feasible control to find a good lower bound for the primal value function under the OU process. If the gap between the upper and lower bounds is small, we can approximate the primal value function. This idea has been applied successfully to find the approximate optimal value function for the Heston stochastic volatility model and regime-switching asset price model with general utility functions, see [6] and [7].

This paper derives the dual control problem and recovers the optimal solution for power utility in [5] and [8]. We use the dual control Monte Carlo method to compute the upper and lower bounds for general utilities' primal value function. For power and non-HARA utilities with a particular dual control, the upper bounds can be computed efficiently with the closed-form formula. Numerical tests for power and non-HARA utilities show that these bounds are tight, which provides a good approximation to the primal value function.

The rest of the paper is arranged as follows. Chapter 1 introduces the classical solution to the

HJB equation for some special continuous concave utility functions, including power, non-HARA, Yarri utilities, using the dual control method, and representing the solution to the HJB equation in terms of that of the dual HJB equation. Chapter 2 derives the optimal controls under the two-dimensional cases in constraint problems using the Kuhn-Tucker condition and gives some simple numerical examples. Chapter 3 discusses the dual control method and derives the closed-form solution for power utility under the OU process and use the solution of the Riccati equation and numerical methods to test and verify primal and dual control methods produce the same primal value function and feedback control. Chapter 4 presents the dual control Monte Carlo method for computing tight upper and lower bounds of the value function and derives the closed-form upper bound for power and non-HARA utilities with a specific form of the dual control. Then performs numerical tests to see the efficiency, accuracy, and robustness of the method. Chapter 5 concludes. Appendix A gives the closed-form solution of the Riccati equation.

Chapter 1

HJB equation and dual control method

1.1 Utility maximization with dual control method

Assume that a financial market has two assets, risk-free savings account B and risky stock S, satisfying the following stochastic differential equations(SDEs):

$$dB_t = rB_t dt, \qquad dS_t = S_t(\mu dt + \sigma dW_t), \qquad 0 \leq t \leq T$$

with initial prices $B_0 = 1$ and $S_0 = s > 0$, where interest rate $r > 0$ and W is a standard Brownian motion on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t\geq 0}, P)$.
Assume that π is an $\mathcal{F}_t$-adapted process, $\int_0^T \pi_t^2 dt < \infty$, a.s. and π is a progressively measurable control process satisfying $\pi_t \in \mathcal{K}$, a closed convex cone in R, a.s. for $t \in [0, T]$ a.e. π is proportional portfolio process. Assume X_t is wealth at time t. Then $\pi_t X_t$ is amount of money invested in S and $(1 - \pi_t)X_t$ is amount of money in bank account B. The wealth process X_t satisfies the stochastic differential equation(SDE):

$$\begin{aligned} dX_t &= \pi_t X_t(\mu dt + \sigma dW_t) + (1 - \pi_t)X_t r dt \\ &= rX_t dt + \pi_t X_t((\mu - r)dt + \sigma dW_t) \end{aligned} \tag{1.1.1}$$

with the initial wealth $X_0 = x$.
The utility maximization problem is given by

$$\sup_{\pi} E[U(X_T)] \quad \text{subject to (1.1.1)}, \tag{1.1.2}$$

where U is a continuous, increasing and concave (but not necessarily strictly increasing and strictly concave) utility function on $[0, \infty)$, satisfying $U(0) = 0$ and $U(x) \leq C(1 + x^p)$, $x \leq 0$ for some constants $C > 0$ and $0 < p < 1$.

Remark 1.1.1. U increasing ($U' > 0$) means investors prefer having more wealth to having less. U concave ($U'' < 0$) means investors are risk averse and having diminishing marginal utility, that is, prefer a payoff $E(X)$ to a random payoff X, the first one gives utility $U(E(X))$, and the second one gives expected utility $E(U(X))$.

1.1.1 Dynamic programming and the primal HJB equation

To solve (1.1.2) with the stochastic control method, the value function $V : [0, T] \times R \to R$ is defined by

$$V(t, x) = \sup_{\pi \in \mathcal{K}} E_{t,x}[U(X_T)|X_t = x], \quad t \geq 0 \text{ and } x \in R. \tag{1.1.3}$$

Theorem 1.1.2. *(Dynamic Programming Principle) Assume on $[t, t + h]$, the control process π and X evolves from (t, x) to $(t + h, X_{t+h})$, then the value function $V(t, x)$ is given by*

$$V(t, x) = \sup_{\pi \in \mathcal{K}} E[V(t + h, X_{t+h})|X_t = x] \tag{1.1.4}$$

Assume $V(\cdot,x)\in C^1$, $V(t,\cdot)\in C^2$. By Ito's formula,

$$\begin{aligned}V(t+h,X_{t+h}) &= V(t,x)+\int_t^{t+h}(\frac{\partial V(s,X_s)}{\partial s}ds+\frac{\partial V(s,X_s)}{\partial x}dX_s+\frac{1}{2}\frac{\partial^2 V(s,X_s)}{\partial x^2}d[X,X]_s)\\ &= V(t,x)+\int_t^{t+h}(\frac{\partial V}{\partial s}+\frac{\partial V}{\partial x}(rX_s+\pi_s X_s(\mu-r))+\frac{1}{2}\frac{\partial^2 V}{\partial x^2}\pi_s^2X_s^2\sigma^2)ds\\ &+\int_t^{t+h}\frac{\partial V}{\partial x}\pi_s X_s\sigma dW_s\end{aligned} \tag{1.1.5}$$

Substitute equation (1.1.5) to equation (1.1.4) to obtain

$$\sup_{\pi\in[t,t+h]}E[\int_t^{t+h}(\frac{\partial V}{\partial s}+\frac{\partial V}{\partial x}(rX_s+\pi_s X_s(\mu-r))+\frac{1}{2}\frac{\partial^2 V}{\partial x^2}\pi_s^2X_s^2\sigma^2)ds|X_t=x]=0$$

When $h\to 0$, $\pi_s\to\pi_t=\pi$ and by Mean Value Theorem, $s\to t$, $X_s\to X_t=x$

$$\sup_{\pi\in\mathcal{K}}(\frac{\partial V}{\partial t}+\frac{\partial V}{\partial x}(rx+\pi x(\mu-r))+\frac{1}{2}\frac{\partial^2 V}{\partial x^2}\pi^2x^2\sigma^2)=0$$

As a result, the primal Hamilton-Jacobi-Bellman(HJB) equation is given by

$$\frac{\partial V}{\partial t}+rx\frac{\partial V}{\partial x}+\sup_{\pi\in\mathcal{K}}(\pi x(\mu-r)\frac{\partial V}{\partial x}+\frac{1}{2}\pi^2x^2\sigma^2\frac{\partial^2 V}{\partial x^2})=0 \tag{1.1.6}$$

with the terminal condition

$$V(T,x)=\sup_{\pi}E[U(X_T)|X_T=x]=U(x)$$

Value function can be obtained by solving PDE (1.1.6). If V is a strictly concave function in x, that is, $V_{xx}<0$. The maximum point is obtained by solving

$$x(\mu-r)V_x+\pi x^2\sigma^2V_{xx}=0$$

which gives

$$\pi^*_{t,x}=-\frac{\mu-r}{\sigma^2}\frac{V_x}{xV_{xx}} \tag{1.1.7}$$

Inserting π^* into HJB (1.1.6) and simplifying the expression, we obtain a nonlinear PDE which is hard to solve

$$\frac{\partial V}{\partial t}+rx\frac{\partial V}{\partial x}-\frac{1}{2}(\frac{\mu-r}{\sigma})^2\frac{V_x^2}{V_{xx}}=0 \tag{1.1.8}$$

with the terminal condition $V(T,x)=U(x)$. If the utility function is power utility, that is $U(x)=x^p/p$, we know

$$V(t,x)=U(x)g(t)$$

and solve ODE for g to obtain $V(t,x)$. However, if U is not power utility, it is impossible to solve the nonlinear HJB PDE.

1.1.2 Dual control method

We can use the dual control method to find the solution of the nonlinear primal HJB equation. Dual utility function of U is defined by

$$\widetilde{U}(y)=\sup_{x\geq 0}(U(x)-xy),\quad y\geq 0, \tag{1.1.9}$$

The function $\widetilde{U}$ is continuous convex and decreasing on $[0,\infty)$ with $\widetilde{U}(0)=U(\infty)$ and $\widetilde{U}(\infty)=0$, satisfying

$$\widetilde{U}(y)\leq C(1+y^q),\qquad y>0$$

for some constant $C > 0$ and $q = p/(p-1) < 0$.[4]
Define a dual process Y satisfies the SDE

$$dY_t = Y_t(-rdt - (\sigma^{-1}\widetilde{\pi}_t + (\mu - r)\sigma^{-1})dW_t), \qquad Y_0 = y, \tag{1.1.10}$$

where $\widetilde{\pi}$ is a progressively measurable dual control process satisfying $\widetilde{\pi}_t \in \widetilde{\mathcal{K}}$, the positive polar cone of $\mathcal{K}$ in R, a.s. for $t \in [0, T]$ a.e..
The dual minimization problem is defined by

$$\inf_{\widetilde{\pi} \in \widetilde{\mathcal{K}}} E[\widetilde{U}(Y_T)]. \tag{1.1.11}$$

For $0 \leq t \leq T$ and $y \geq 0$, the dual value function is defined by

$$\widetilde{V}(t, y) = \inf_{\widetilde{\pi} \in \widetilde{\mathcal{K}}} E[\widetilde{U}(Y_T)|Y_t = y]. \tag{1.1.12}$$

The primal value function and dual value function have the following relationship[6]

$$V(t, x) = \inf_{y \geq 0}(\widetilde{V}(t, y) + xy) \tag{1.1.13}$$

Since $\widetilde{V}(t, \cdot)$ is strictly convex, $\widetilde{V}_y(t, \cdot)$ is strictly increasing. Thus, the minimum in (1.1.13) is achieved at

$$\frac{\partial \widetilde{V}(t, x)}{\partial y} + x = 0. \tag{1.1.14}$$

For every $x > 0$, let the unique solution be $\hat{y}(t, x)$, satisfying

$$V(t, x) = \widetilde{V}(t, \hat{y}(t, x)) + x\hat{y}(t, x). \tag{1.1.15}$$

The partial derivatives of the equation(1.1.15) are given by

$$\begin{aligned} \frac{\partial V}{\partial t} &= \frac{\partial \widetilde{V}}{\partial t} + \frac{\partial \widetilde{V}}{\partial y}\frac{\partial y}{\partial t} + x\frac{\partial y}{\partial t} = \frac{\partial \widetilde{V}}{\partial t}, \\ \frac{\partial V}{\partial x} &= \frac{\partial \widetilde{V}}{\partial y}\frac{\partial y}{\partial x} + y + x\frac{\partial y}{\partial x} = y, \\ \frac{\partial^2 V}{\partial x^2} &= \frac{\partial y}{\partial x} = -\frac{1}{\widetilde{V}_{yy}} \end{aligned} \tag{1.1.16}$$

Substituting (1.1.14) and (1.1.16) into (1.1.8), $\widetilde{V}$ satisfies a linear PDE

$$\frac{\partial \widetilde{V}}{\partial t} - ry\frac{\partial \widetilde{V}}{\partial y} + \frac{1}{2}(\frac{\mu - r}{\sigma})^2 y^2 \frac{\partial^2 \widetilde{V}}{\partial y^2} = 0 \tag{1.1.17}$$

with initial condition $\widetilde{V}(T, y) = \widetilde{U}(y)$. This PDE is called the dual HJB equation. We can find a representation of the classical solution to the primal HJB equation by two convex dual operations and solution of a linear PDE, all are relatively easy to perform. By the Feynman-Kac Theorem, the solution to dual HJB equation(1.1.16) has the following representation

$$\widetilde{V}(t, y) = E[\widetilde{U}(Y_T)|Y_t = y], \tag{1.1.18}$$

where

$$\begin{aligned} &dY = -rYdt - \theta YdW \\ &\Rightarrow Y = y_0 \exp(\int_t^T (-r - \frac{1}{2}\theta^2)ds - \int_t^T \theta dW_s), \quad \theta = \frac{\mu - r}{\sigma}, \end{aligned}$$

with the initial value $Y_0 = y_0$. Since r, μ and σ are constant, then

$$Y = y_0 \exp[(-r - \frac{1}{2}\theta^2)(T - t) - \theta(W_T - W_t)] \tag{1.1.19}$$

with the initial value $Y_0 = y_0$. Therefore, using the dual control method, we can find the optimal control π^*, value function V and optimal wealth process X^* in two ways:
1. Find the value function V by equation (1.1.15), the corresponding optimal control π^* by equation (1.1.7) and optimal wealth X^* by solving SDE(1.1.1).
2. Find the value function V by equation 1.1.15, then solve equation (1.1.14) with time $t = 0$ to get y_0, where $X_0 = x$ is the initial wealth at time $t = 0$, then find the dual process Y by equation (1.1.19) and the optimal wealth process X^* at time t is given by

$$X_t^* = -\widetilde{V}_y(t, Y_t).$$

Find the SDE satisfied by X_t^* and the corresponding optimal control π^*.

1.2 Dual control method for three different utility functions

1.2.1 Power utility function

Power utility function is defined by

$$U(x) = \frac{1}{p}x^p$$

where $p \in (0, 1)$ is a constant. The dual power utility function is given by

$$\widetilde{U}(y) = \sup_{x>0}(U(x) - xy) = \sup_{x>0}(\frac{1}{p}x^p - xy), \quad y > 0.$$

Since the optimal point x satisfies the equation

$$U'(x) - y = x^{p-1} - y = 0.$$

Hence,

$$\widetilde{U}(y) = -\frac{1}{q}y^q, \quad y > 0 \tag{1.2.1}$$

where $q = p/(p-1) < 0$. By equation(1.1.18) and (1.1.19), the dual value function is given by

$$\widetilde{V}(t, y) = -\frac{1}{q}y^q \exp[(\frac{1}{2}q(q-1)\theta^2 - qr)(T-t)], \quad \theta = \frac{\mu - r}{\sigma}. \tag{1.2.2}$$

The minimum point is obtained at point $\hat{y} = \hat{y}(t, x)$ which is solves the equation(1.1.14)

$$\hat{y}(t, x) = x^{\frac{1}{q-1}} \exp[(-\frac{1}{2}q\theta^2 + \frac{qr}{q-1})(T-t)], \quad \theta = \frac{\mu - r}{\sigma}.$$

Then by using dual relation to find primal value function from dual value function

$$\begin{aligned} V(t, x) =& \widetilde{V}(t, \hat{y}(t, x)) + x\hat{y}(t, x) \\ =& \frac{q-1}{q}x^{\frac{q}{q-1}} \exp[(-\frac{1}{2}q\theta^2 + \frac{qr}{q-1})(T-t)] \\ =& \frac{1}{p}x^p \exp[(\frac{1}{2}\frac{p}{1-p}\theta^2 + pr)(T-t)] \end{aligned}$$

Therefore, the primal optimal control is achieved at

$$\begin{aligned} \pi^*(t, x) &= -\frac{\mu - r}{\sigma^2}\frac{V_x}{xV_{xx}} \\ &= \frac{\mu - r}{(1-p)\sigma^2}. \end{aligned}$$

Therefore, the wealth process X_t satisfies a linear SDE

$$dX_t = X_t \left((\frac{(\mu - r)^2}{(1-p)\sigma^2} + r)dt + \frac{\mu - r}{\sigma(1-p)}dW_t \right)$$

Theorem 1.2.1. *Consider $dY_t = m_t Y_t dt + \nu_t Y_t dW_t$ (Geometric Brownian Motion), with m_t and ν_t deterministic function of time. Set $Z_t = \ln(Y_t) =: \phi(Y_t)$. By Ito formula the SDE of Z_t is given by*

$$dZ_t = (m_t - \frac{1}{2}\nu_t^2)dt + \nu_t dW_t \qquad \textit{(Arithmetic BM)}$$

By Theorem1.2.1, the optimal wealth process is given by

$$X_t = x\exp\left[\left(\frac{(\mu - r)^2}{(1-p)\sigma^2} + r - \frac{1}{2}\frac{(\mu - r)^2}{\sigma^2(1-p)^2}\right)t + \frac{\mu - r}{\sigma(1-p)}dW_t\right].$$

1.2.2 Non-Hara utility function

Non-Hara utility function is defined by

$$U(x) = \frac{1}{3}H(x)^{-3} + H(x)^{-1} + xH(x)$$

for $x > 0$, where

$$H(x) = \left(\frac{2}{\sqrt{1+4x} - 1}\right)^{1/2}$$

Theorem 1.2.2. *Non-Hara utility function U is strictly increasing, strictly concave, $U(0) = 0$, $U(\infty) = \infty$, $U'(0) = \infty$ and $U'(\infty) = 0$.*

Proof. The first derivative function of $H(x)$ is

$$H'(x) = -\frac{1}{2\sqrt{1+4x}}H(x)^3$$

The first derivative function of $U(x)$ is

$$\begin{aligned}
U'(x) &= -H(x)^{-4}H'(x) - H(x)^{-2}H'(x) + H(x) + xH'(x) \\
&= \frac{1}{2\sqrt{1+4x}}\left(H(x)^{-1} + H(x) - xH(x)^3\right) + H(x) \\
&= \frac{1}{2\sqrt{1+4x}} \times 0 + H(x) \\
&= H(x) > 0
\end{aligned}$$

which shows U is strictly increasing. The second derivative function of $U(x)$ is given by

$$U''(x) = H'(x) = -\frac{1}{2\sqrt{1+4x}}H(x)^3 < 0$$

which shows U is strictly concave. Since $H(0) = \lim_{x\to 0} H(x) = \infty$, then

$$\begin{aligned}
&U(0) = \lim_{x\to 0} H(x) = \frac{1}{3}H(0)^{-3} + H(0)^{-1} + 0 \times H(0) = 0, \\
&U'(0) = H(0) = \infty.
\end{aligned}$$

Since $H(\infty) = 0$, then $U(\infty) = \infty$ and $U'(\infty) = H(\infty) = 0$. At the meanwhile, U is a utility function is also proved.

□

The dual non-Hara utility function is defined by

$$\widetilde{U}(y) = \sup_{x>0}(U(x) - xy)$$

The maximum is achieved at

$$U'(x) - y = H(x) - y = 0.$$

Solving this equation, the solution is given by

$$x = \frac{1}{y^2} + \frac{1}{y^4}.$$

Substituting x into $U(x) - xy$, the dual utility function is given by

$$\widetilde{U}(y) = \frac{1}{y} + \frac{1}{3y^3}.$$

By equation(1.2.1), the dual non-Hara utility function and the dual non-Hara utility function have the following relationship

$$\widetilde{U}(y) = -\frac{1}{q_1}y^{q_1} - \frac{1}{q_2}y^{q_2}$$

where $q_1 = -1$ and $q_2 = -3$. By equation(1.2.2), the dual value function is given by

$$\begin{aligned}\widetilde{V}(t,y) &= -\frac{1}{q_1}y^{q_1}\exp[(\frac{1}{2}q_1(q_1-1)\theta^2 - q_1 r)(T-t)] - \frac{1}{q_2}y^{q_2}\exp[(\frac{1}{2}q_2(q_2-1)\theta^2 - q_2 r)(T-t)] \\ &= \mathrm{e}^{(\theta^2+r)(T-t)}\frac{1}{y} + \mathrm{e}^{(6\theta^2+3r)(T-t)}\frac{1}{3y^3}, \quad \theta = \frac{\mu - r}{\sigma}.\end{aligned}$$

Furthermore, since $\hat{y}$ is the solution to $\widetilde{V}_y(t,y) + x = 0$, by solving the equation

$$-\mathrm{e}^{(\theta^2+r)(T-t)}\frac{1}{y^2} - \mathrm{e}^{(6\theta^2+3r)(T-t)}\frac{1}{y^4} + x = 0, \tag{1.2.3}$$

the minimum point is obtained at point

$$\hat{y}(t,x) = \left(\frac{1}{2x}\left(\mathrm{e}^{(r+\theta^2)(T-t)} + \sqrt{\mathrm{e}^{2(r+\theta^2)(T-t)} + 4x\mathrm{e}^{3(r+2\theta^2)(T-t)}}\right)\right)^{1/2}. \tag{1.2.4}$$

Furthermore, we can derive an explicit solution to the primal HJB equation by the dual HJB equation

$$\begin{aligned}V(t,x) &= \widetilde{V}(t,\hat{y}(t,x)) + x\hat{y}(t,x) \\ &= \mathrm{e}^{(\theta^2+r)(T-t)}\frac{1}{\hat{y}} + \mathrm{e}^{(6\theta^2+3r)(T-t)}\frac{1}{3\hat{y}^3} + x\hat{y} \\ &= \frac{\hat{y}}{3}\left(3\mathrm{e}^{(\theta^2+r)(T-t)}y^{-2} + \mathrm{e}^{(6\theta^2+3r)(T-t)}\hat{y}^{-4} + 3x\right)\end{aligned}$$

By equation(1.2.3)

$$\mathrm{e}^{(6\theta^2+3r)(T-t)}\hat{y}^{-4} = -\mathrm{e}^{(\theta^2+r)(T-t)}\frac{1}{y^2} + x.$$

Then,

$$\begin{aligned}V(t,x) =& \frac{\hat{y}}{3}\left(3\mathrm{e}^{(\theta^2+r)(T-t)}y^{-2} - \mathrm{e}^{(\theta^2+r)(T-t)}\frac{1}{y^2} + x + 3x\right) \\ =& \frac{2}{3}\left(\mathrm{e}^{(r+\theta^2)(T-t)}\hat{y}^{-1} + 2x\hat{y}\right),\end{aligned} \tag{1.2.5}$$

where $\hat{y} > 0$ is given by (1.2.4). The optimal control π^* can be computed by equation(1.1.7).

1.2.3 Yarri utility function

Yarri utility function is defined by

$$U(x) = x \wedge m = \min(x, m),$$

where m is a positive constant. The dual Yarri function of U is given by

$$\begin{aligned}
\widetilde{U}(y) = \sup_{x>0}(\min(x,m) - xy) &= \begin{cases} \sup\limits_{x>0}(x - xy), \text{ if } x \leq m,\ y > 0 \\ \sup\limits_{x>0}(m - xy), \text{ if } x > m,\ y > 0 \end{cases} \\
&= \begin{cases} m(1-y), \text{if } 1 - y > 0 \\ 0, \qquad\quad \text{if } 1 - y \leq 0 \end{cases} \\
&= m(1-y)^+, \quad y > 0.
\end{aligned}$$

By equation(1.1.17), the dual value function is given by

$$\widetilde{V}(t,y) = E[\widetilde{U}(Y_T)|Y_t = y] = mE[(1 - Y_T)^+|Y_t = y].$$

By the equation(1.1.18),

$$Y_T = y_0 \exp[-(r + \frac{1}{2}\theta^2)(T-t) - \theta\sqrt{T-t}Z],$$

with the initial value $Y_0 = y_0$, where Z is the standard normal distribution, i.e. $Z \sim N(0,1)$. Hence, the dual value function is given by

$$\widetilde{V}(t,y) = mE\left[\left(1 - y\exp\left(-(r + \frac{1}{2}\theta^2)(T-t) - \theta\sqrt{T-t}Z\right)\right)^+\right].$$

Noting that

$$1 - y\exp\left(-(r + \frac{1}{2}\theta^2)(T-t) - \theta\sqrt{T-t}Z\right) \geq 0 \Leftrightarrow Z \geq \frac{\log y - (r + \frac{1}{2}\theta^2)(T-t)}{\theta\sqrt{T-t}} := k.$$

As a result,

$$\begin{aligned}
\widetilde{V}(t,y) &= m\int_k^\infty \left(1 - y\exp\left(-(r + \frac{1}{2}\theta^2)(T-t) - \theta\sqrt{T-t}z\right)\right)\frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}dz \\
&= -mye^{-r(T-t)}\Phi(-k - \theta\sqrt{T-t}) + m\Phi(-k) \\
&= -mye^{-r(T-t)}\Phi(-\frac{1}{\theta\sqrt{T-t}}\log y + \frac{r}{\theta}\sqrt{T-t} - \frac{1}{2}\theta\sqrt{T-t}) \\
&+ m\Phi(-\frac{1}{\theta\sqrt{T-t}}\log y + \frac{r}{\theta}\sqrt{T-t} + \frac{1}{2}\theta\sqrt{T-t}), \quad \theta = \frac{(\mu - r)}{\sigma}
\end{aligned}$$

where Φ is the cumulative distribution function of a standard normal variable. The partial derivative of $\widetilde{V}(t,y)$ is given by

$$\begin{aligned}
\frac{\partial\widetilde{V}}{\partial y} &= -me^{-r(T-t)}\Phi(-k - \theta\sqrt{T-t}) + m(\phi(-k) - ye^{-r(T-t)}\phi(k - \theta\sqrt{T-t}))(-\frac{\partial k}{\partial y}) \\
&= -me^{-r(T-t)}\Phi(-k - \theta\sqrt{T-t}) \\
&= -me^{-r(T-t)}\Phi(-\frac{1}{\theta\sqrt{T-t}}\log y + \frac{r}{\theta}\sqrt{T-t} - \frac{1}{2}\theta\sqrt{T-t}).
\end{aligned}$$

By solving the equation(1.1.13), the unique solution $\hat{y} = \hat{y}(t,x)$ is given by

$$\hat{y}(t,x) = \exp\left(-\theta\sqrt{T-t}\Phi^{-1}(\frac{x}{m}e^{r(T-t)}) + (r - \frac{1}{2}\theta^2)(T-t)\right). \tag{1.2.6}$$

If the initial wealth $x \geq m\mathrm{e}^{-r(T-t)}$ at time t, investors can invest $X_t = x$ in a riskless asset at time t, and then the wealth is $X_T = x\mathrm{e}^{r(T-t)}$, which is greater than or equal to m. Investors are guaranteed to achieve the maximum possible wealth m at time T with a trading strategy $\pi_t = 0$. Thus, the optimal trading strategy is to put all money in the bank account and do not invest in risky asset.
Hence, by using dual relation to find primal value function from dual value function

$$V(t,x) = \begin{cases} \widetilde{V}(t,\hat{y}(t,x)) + x\hat{y}(t,x), & \text{if } 0 \leq x \leq m\mathrm{e}^{-r(T-t)}, \\ m, & \text{if } x \geq m\mathrm{e}^{-r(T-t)}, \end{cases}$$
$$= \begin{cases} m\Phi\left(\Phi^{-1}(\frac{x}{m}\mathrm{e}^{r(T-t)}) + \theta\sqrt{T-t}\right), & \text{if } 0 \leq x \leq m\mathrm{e}^{-r(T-t)} \\ m, & \text{if } x \geq, m\mathrm{e}^{-r(T-t)}. \end{cases}$$

In the region of $\{(x,t) : x \geq m\mathrm{e}^{-r(T-t)}, 0 < t < T\}$, the primal optimal control is achieved at $\pi^*(t,x) = 0$. In the region of $\{(x,t) : 0 < x < m\mathrm{e}^{-r(T-t)}, 0 < t < T\}$, the primal optimal control is achieved at

$$\pi^*(t,x) = -\frac{\mu - r}{\sigma^2}\frac{V_x}{xV_{xx}} = \frac{m\mathrm{e}^{-r(T-t)}}{x\sigma\sqrt{T-t}}\phi\left(\Phi^{-1}(\frac{x}{m}\mathrm{e}^{r(T-t)})\right).$$

Chapter 2

Dual control method under the multidimensional model

Assume that W is an n-dimensional standard Brownian motion on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t\geq 0}, P)$ satisfying the usual conditions. The financial market consists of one savings account and n risky stocks. The discounted bank account and stocks price processes B and $S = (S^1, S^2, ..., S^n)^T$ satisfies the following stochastic differential equations(SDEs)

$$dB_t = rB_t dt, \qquad dS_t = \mathrm{diag}(S_t)(\mu dt + \sigma dW_t), \qquad 0 \leq t \leq T \tag{2.0.1}$$

with the initial price $S_0 = s$, where the positive constant r is the riskless interest rate, $\mathrm{diag}(S_t)$ is a diagonal $n \times n$ matrix with diagonal elements S_t^i and all other elements zero, $(S^1, S^2, ..., S^n)^T$ is the transpose of $S = (S^1, S^2, ..., S^n)$, $\mu = (\mu_1, \mu_2, ..., \mu_n)^T$ is a vector of stock returns and σ is a $n \times n$ nonsingular matrix of volatilities.
Assume that π is an $\mathcal{F}_t$-adapted process, $\int_0^T \pi_t^2 dt < \infty$, a.s. and $\pi_t = (\pi_1, \pi_2, ..., \pi_n)^T$ is a progressively measurable control process satisfying $\pi_t \in \mathcal{K}$, a closed convex cone in R^n, a.s. for $t \in [0, T]$ a.e. π is proportional portfolio process. Assume X_t is wealth at time t. Then $\pi_t X_t$ is amount of money invested in S and $(1 - \pi_t)X_t$ is amount of money in bank account B. The discounted wealth process X satisfies the SDE:

$$\begin{aligned} dX_t &= (1 - \pi_t^T)X_t r dt + \pi_t^T X_t(\mu dt + \sigma dW_t) \\ &= rX_t dt + \pi_t^T X_t((\mu - r\mathbf{1})dt + \sigma dW_t) \\ &= X_t((\pi_t^T(\mu - r\mathbf{1}) + r)dt + \pi_t^T \sigma dW_t), \end{aligned} \tag{2.0.2}$$

with initial wealth $X_0 = x$, where $\mathbf{1}$ is a vector with all components 1.
Similarly, the utility maximization problem is defined by

$$\sup_{\pi\in\mathcal{K}} E[U(X_T)] \quad \text{subject to (2.0.2).} \tag{2.0.3}$$

The value function is given by

$$V(t, x) = \sup_{\pi\in\mathcal{K}} E[U(X_T)|X_t = x].$$

The primal HJB equation is given by

$$\frac{\partial V}{\partial t} + rx\frac{\partial V}{\partial x} + \sup_{\pi\in\mathcal{K}} \left(\pi^T(\mu - r\mathbf{1})x\frac{\partial V}{\partial x} + \frac{1}{2}|\sigma^T\pi|^2 x^2 \frac{\partial^2 V}{\partial x^2} \right) = 0 \tag{2.0.4}$$

for $x > 0$ and $0 < t < T$ with the terminal condition $V(T, x) = U(x)$. The dual utility function of U is given by

$$\widetilde{U}(y) = \sup_{x\geq 0}(U(x) - xy)$$

for $y \geq 0$. The dual process Y satisfies the SDE

$$dY_t = Y_t(-rdt - (\sigma^{-1}\widetilde{\pi}_t + \sigma^{-1}(\mu - r\mathbf{1}))^T dW_t), \quad Y_0 = y,$$

where $\widetilde{\pi}$ is a progressively measurable dual control process satisfying $\widetilde{\pi}_t \in \widetilde{\mathcal{K}}$, the positive polar cone of $\mathcal{K}$ in R^n, a.s. for $t \in [0,T]$ a.e..
Furthermore, the dual minimization problem is given by

$$\inf_{\widetilde{\pi}\in\widetilde{\mathcal{K}}} E[\widetilde{U}(Y_T)]. \tag{2.0.5}$$

The dual value function is given by

$$\widetilde{V}(t,y) = \inf_{\widetilde{\pi}\in\widetilde{\mathcal{K}}} E[\widetilde{U}(Y_T)|Y_t = y] \tag{2.0.6}$$

for $0 \le t \le T$ and $y \ge 0$. The dual HJB equation is given by

$$\frac{\partial \widetilde{V}}{\partial t} - ry\frac{\partial \widetilde{V}}{\partial y} + \frac{1}{2}|\hat{\theta}|^2 y^2 \frac{\partial^2 \widetilde{V}}{\partial y^2} = 0 \tag{2.0.7}$$

for $y > 0$ and $0 < t < T$ with the terminal condition $\widetilde{V}(T,y) = \widetilde{U}(y)$, where $\hat{\theta} = \sigma^{-1}(\mu - r\mathbf{1}) + \sigma^{-1}\hat{\pi}$ and $\hat{\pi}$ is the unique minimizer of $f(\widetilde{\pi}) = |\sigma^{-1}(\mu - r\mathbf{1}) + \sigma^{-1}\hat{\pi}|^2$ over $\widetilde{\pi} \in \widetilde{\mathcal{K}}$. Assume that $\hat{\theta}$ is continuous on $[0,T]$ and there is a positive constant θ_0 such that $|\hat{\theta}| \ge \theta_0$ for all $t \in [0,T]$.
Now we consider three different cases of $\mathcal{K}$ under two-dimensional model.

2.1 Unconstrained problem

$\mathcal{K}$ is the whole space R^n, that is, there is no trading constraints, then the positive polar cone $\widetilde{\mathcal{K}}$ is $\{0\}$, and the dual optimal control solution $\hat{\pi}_t = 0$ for all t. Thus, $\hat{\theta}_t = \theta = \sigma(t)^{-1}(\mu - r\mathbf{1})$ is a nonzero continuous vector-valued function on $[0,T]$ and by the assumption, θ_0 is the minimum value of $|\theta|$ on $[0,T]$.
Therefore, the dual HJB equation is given by

$$\frac{\partial \widetilde{V}}{\partial t} - ry\frac{\partial \widetilde{V}}{\partial y} + \frac{1}{2}|\sigma^{-1}(\mu - r\mathbf{1})|^2 y^2 \frac{\partial^2 \widetilde{V}}{\partial y^2} = 0$$

The corresponding primal optimal control is given by

$$\pi^*(t,x) = -(\sigma^T)^{-1}\theta \frac{V_x(t,x)}{xV_{xx}(t,x)} \in R^2, \quad \theta = \sigma^{-1}(\mu - r\mathbf{1}).$$

2.2 Constraint problem

If $f : \mathbf{dom}(f) \to \mathbb{R}$ is twice differentiable(meaning that the function $\bigtriangledown f$ is differentiable), convexity can be characterized as follows.

Lemma 2.2.1. *Suppose that* $\mathbf{dom}(f)$ *is open and that* f *is twice differentiable; in particular, the Hessian(matrix of second partial derivatives)*

$$\bigtriangledown^2 f(\mathbf{x}) = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1 \partial x_1}(\mathbf{x}) & \frac{\partial^2 f}{\partial x_1 \partial x_2}(\mathbf{x}) & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_d}(\mathbf{x}) \\ \frac{\partial^2 f}{\partial x_2 \partial x_1}(\mathbf{x}) & \frac{\partial^2 f}{\partial x_2 \partial x_2}(\mathbf{x}) & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_d}(\mathbf{x}) \\ \vdots & \vdots & \cdots & \vdots \\ \frac{\partial^2 f}{\partial x_d \partial x_1}(\mathbf{x}) & \frac{\partial^2 f}{\partial x_d \partial x_2}(\mathbf{x}) & \cdots & \frac{\partial^2 f}{\partial x_d \partial x_d}(\mathbf{x}) \end{bmatrix} \tag{2.2.1}$$

exists at every point $\mathbf{x} \in \mathbf{dom}(f)$ *and is symmetric. Then* f *is convex if and only if* $\mathbf{dom}(f)$ *is convex, and for all* $\mathbf{x} \in \mathbf{dom}(f)$*, we have*

$$\bigtriangledown^2 f(\mathbf{x}) \ge 0 \tag{2.2.2}$$

i.e. $\bigtriangledown^2 f(\mathbf{x})$ *is positive semidefinite.*

Lemma 2.2.2. *If* $f : \mathbf{dom}(f) \to \mathbb{R}$ *is convex and differentiable over an open domain* $\mathbf{dom}(f) \subseteq \mathbb{R}^d$*. Let* $\mathbf{x} \in \mathbf{dom}(f)$*. If* $\bigtriangledown f(\mathbf{x}) = 0$*, then* $\mathbf{x}$ *is a global minimum.*

2.2.1 Kuhn-Tucker condition

In mathematical optimization, the Kuhn-Tucker conditions, are first-order derivative tests (sometimes called first-order necessary conditions), and the solution used in nonlinear programming is optimal, as long as it satisfies some regularity conditions. Kuhn-Tucker condition is a necessary condition to judge the feasible point of constrained nonlinear programming problem as the minimum point.

For convex programming, it is a necessary and sufficient condition for judging the minimum point. For a constrained nonlinear programming problem (NP), let $f(\mathbf{x})$, $g_i(\mathbf{x})(i = 1, 2, ...p)$ and $h_j(\mathbf{x})(j = 1, 2, ..., q)$ have the first order continuously differentiable on an open set of R.
Consider the optimization problem

$$\min f(\mathbf{x})$$

subject to

$$g_i(\mathbf{x}) \leq 0,$$
$$h_j(\mathbf{x}) = 0.$$

where $\mathbf{x} \in \mathbf{X}$ is the optimization variable chosen from a convex subset of R^n, f is the objective or utility function, g_i are the inequality constraint functions and h_j are the equality constraint functions. The number of inequalities and equalities are denoted by p and q respectively.
A Lagrange function is defined by

$$L(\mathbf{x}, \lambda, \mu) = f(\mathbf{x}) + \lambda^T g(\mathbf{x}) + \mu^T h(\mathbf{x})$$

Theorem 2.2.3. *x^* is the minimum point of the problem and the regular point of the constraint condition, then there are vectors $\lambda^* = (\lambda_1^*, \lambda_2^*, ..., \lambda_p^*)^T$, $\mu^* = (\mu_1^*, \mu_2^*, ..., \mu_q^*)^T$, so that*

$$\begin{cases} \bigtriangledown f(x^*) + \sum_{i=1}^{p} \lambda_i^* \bigtriangledown g_i(x^*) + \sum_{j=1}^{q} \mu_j^* \bigtriangledown h_j(x^*) = 0, \\ \lambda_i^* g_i(x^*) = 0, \ i = 1, 2, ..., p \\ \mu_j^* \geq 0, \ j = 1, 2, ..., q \end{cases} \tag{2.2.3}$$

This is the Kuhn-Tucker condition of the constrained nonlinear programming problem (NP), which is also called the first-order necessary condition. $\lambda_i^*(i = 1, 2, ..., p)$ and $\mu_j^*(j = 1, 2, ..., q)$ is called Kuhn-Tucker multiplier. From the Kuhn-Tucker condition mentioned above, it can be seen that only when $g_i(x)$ is a functional constraint at x^* point, there may be $\lambda_i^* \neq 0$; Otherwise, $\lambda_i^* = 0$.

Theorem 2.2.4. *If f is a convex function and $g_1, ..., g_m$ are linear functions, then x^* is minimum point if and only if x^*, λ^* and μ^* satisfies KT condition.*

2.2.2 $\mathcal{K} = R_+^2$

$\mathcal{K}$ is the nonnegative part of the whole space R_+^2, that is, there is short selling contraints, then the positive polar cone $\widetilde{\mathcal{K}}$ is also R_+^2. The above assumption is automatically satisfied if all components of $\mu - r\mathbf{1}$ are positive, the dual optimal control is given by $\hat{\pi}_t = 0$ for all t. However, if not all components of $\mu - r\mathbf{1}$ are positive numbers. The situation becomes complicated.

Let $\sigma^{-1} = \begin{bmatrix} \sigma_1 & \sigma_2 \\ \sigma_3 & \sigma_4 \end{bmatrix}$, $b = (\mu - r\mathbf{1}) = \begin{bmatrix} \mu_1 - r \\ \mu_2 - r \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$, $\widetilde{\pi} = \begin{bmatrix} \widetilde{\pi}_1 \\ \widetilde{\pi}_2 \end{bmatrix} \geq \mathbf{0}$, then

$$\theta + \sigma^{-1}\widetilde{\pi} = \begin{bmatrix} \sigma_1(b_1 + \widetilde{\pi}_1) + \sigma_2(b_2 + \widetilde{\pi}_2) \\ \sigma_3(b_1 + \widetilde{\pi}_1) + \sigma_4(b_2 + \widetilde{\pi}_2) \end{bmatrix}$$

$$\bigtriangledown^2 f(\widetilde{\pi}) = \begin{bmatrix} 2(\sigma_1^2 + \sigma_3^2) & 2(\sigma_1\sigma_2 + \sigma_3\sigma_4) \\ 2(\sigma_1\sigma_2 + \sigma_3\sigma_4) & 2(\sigma_2^2 + \sigma_4^2) \end{bmatrix} \tag{2.2.4}$$

Therefore, the above problem is transformed into solving quadratic programming with K-T condition. The standardized model is

$$\begin{aligned} \min f(\widetilde{\pi}) =& (\sigma_1(b_1 + \widetilde{\pi}_1) + \sigma_2(b_2 + \widetilde{\pi}_2))^2 + (\sigma_3(b_1 + \widetilde{\pi}_1) + \sigma_4(b_2 + \widetilde{\pi}_2))^2 \\ =& (\sigma_1^2 + \sigma_3^2)(b_1 + \widetilde{\pi}_1)^2 + (\sigma_2^2 + \sigma_4^2)(b_2 + \widetilde{\pi}_2)^2 + 2(\sigma_1\sigma_2 + \sigma_3\sigma_4)(b_1 + \widetilde{\pi}_1)(b_2 + \widetilde{\pi}_2) \end{aligned} \tag{2.2.5}$$

subject to

$$\begin{cases} g_1(\widetilde{\pi}) = \widetilde{\pi}_1 \geq 0 \\ g_2(\widetilde{\pi}) = \widetilde{\pi}_2 \geq 0 \end{cases} \tag{2.2.6}$$

The gradient of each function is

$$\bigtriangledown f(\widetilde{\pi}) = \begin{bmatrix} \frac{\partial f}{\partial \widetilde{\pi}_1} \\ \frac{\partial f}{\partial \widetilde{\pi}_2} \end{bmatrix} = \begin{bmatrix} 2(\sigma_1^2 + \sigma_3^2)(b_1 + \widetilde{\pi}_1) + 2(b_2 + \widetilde{\pi}_2)(\sigma_1\sigma_2 + \sigma_3\sigma_4) \\ 2(\sigma_2^2 + \sigma_4^2)(b_2 + \widetilde{\pi}_2) + 2(b_1 + \widetilde{\pi}_1)(\sigma_1\sigma_2 + \sigma_3\sigma_4) \end{bmatrix} \tag{2.2.7}$$

$g_1(\widetilde{\pi}) = (1,0)^T$, $g_2(\widetilde{\pi}) = (0,1)^T$. Lagrange multipliers γ_1^*, γ_2^* are introduced for the two constraints respectively. The following K-T condition is obtained

$$\begin{aligned} &\bigtriangledown f(\widetilde{\pi}) - \gamma_1^* \bigtriangledown g_1(\widetilde{\pi}) - \gamma_2^* \bigtriangledown g_2(\widetilde{\pi}) = 0 \\ &\Rightarrow \begin{bmatrix} 2(\sigma_1^2 + \sigma_3^2)(b_1 + \widetilde{\pi}_1^*) + 2(b_2 + \widetilde{\pi}_2^*)(\sigma_1\sigma_2 + \sigma_3\sigma_4) \\ 2(\sigma_2^2 + \sigma_4^2)(b_2 + \widetilde{\pi}_2^*) + 2(b_1 + \widetilde{\pi}_1^*)(\sigma_1\sigma_2 + \sigma_3\sigma_4) \end{bmatrix} - \gamma_1^* \begin{bmatrix} 1 \\ 0 \end{bmatrix} - \gamma_2^* \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 0 \end{aligned} \tag{2.2.8}$$

It can be decomposed into

$$\begin{cases} 2(\sigma_1^2 + \sigma_3^2)(b_1 + \widetilde{\pi}_1^*) + 2(b_2 + \widetilde{\pi}_2^*)(\sigma_1\sigma_2 + \sigma_3\sigma_4) - \gamma_1^* = 0 \\ 2(\sigma_2^2 + \sigma_4^2)(b_2 + \widetilde{\pi}_2^*) + 2(b_1 + \widetilde{\pi}_1^*)(\sigma_1\sigma_2 + \sigma_3\sigma_4) - \gamma_2^* = 0 \\ \gamma_1^*\widetilde{\pi}_1^* = 0 \\ \gamma_2^*\widetilde{\pi}_2^* = 0 \\ \gamma_1^*, \gamma_2^* \geq 0 \end{cases}$$

As a result, by identifying and classifying model coefficients, there are four cases come up.
(1) For $\gamma_1^* > 0$, $\gamma_2^* > 0$, that is, $\widetilde{\pi}_1^* = \widetilde{\pi}_2^* = 0$ and

$$\begin{cases} b_1(\sigma_1^2 + \sigma_3^2) + b_2(\sigma_1\sigma_2 + \sigma_3\sigma_4) > 0 \\ b_2(\sigma_2^2 + \sigma_4^2) + b_1(\sigma_1\sigma_2 + \sigma_3\sigma_4) > 0 \end{cases}$$

The dual optimal control is given by

$$\widetilde{\pi}^* = [0,0]^T. \tag{2.2.9}$$

Since $\gamma_1^* > 0$, $\gamma_2^* > 0$ i.e. $\bigtriangledown f(\widetilde{\pi}) > 0$, $f(\widetilde{\pi})$ is strictly increasing, then $\widetilde{\pi}^* = \mathbf{0}$ is the unique minimizer of quadratic function f. Therefore, $\hat{\theta} = \theta$.
The primal optimal control is given by

$$\pi^*(t,x) = -(\sigma^T)^{-1}\theta \frac{V_x(t,x)}{xV_{xx}(t,x)} \in R^2, \quad \theta = \sigma^{-1}(\mu - r\mathbf{1}).$$

(2) $\gamma_1^* > 0$, $\gamma_2^* = 0$: $\widetilde{\pi}_1^* = 0$
By K-T condition,

$$\begin{cases} 2b_1(\sigma_1^2 + \sigma_3^2) + 2(b_2 + \widetilde{\pi}_2^*)(\sigma_1\sigma_2 + \sigma_3\sigma_4) = \gamma_1^* \\ 2(\sigma_2^2 + \sigma_4^2)(b_2 + \widetilde{\pi}_2^*) + 2b_1(\sigma_1\sigma_2 + \sigma_3\sigma_4) = 0 \end{cases} \tag{2.2.10}$$

By solving the equation, we can obtain

$$\widetilde{\pi}_2^* = -\frac{b_1(\sigma_1\sigma_2 + \sigma_3\sigma_4)}{\sigma_2^2 + \sigma_4^2} - b_2 > 0 \tag{2.2.11}$$

$$\gamma_1^* = \frac{2b_1(\sigma_2\sigma_3 - \sigma_1\sigma_4)^2}{\sigma_2^2 + \sigma_4^2} > 0 \tag{2.2.12}$$

Hence, for $b_1 > 0$, $b_2(\sigma_2^2 + \sigma_4^2) + b_1(\sigma_1\sigma_2 + \sigma_3\sigma_4) > 0$, the dual optimal control is given by

$$\widetilde{\pi}^* = [0, -\frac{b_1(\sigma_1\sigma_2 + \sigma_3\sigma_4)}{\sigma_2^2 + \sigma_4^2} - b_2]^T.$$

The primal optimal control is given by

$$\pi^*(t,x) = -(\sigma^T)^{-1}\hat{\theta}\frac{V_x(t,x)}{xV_{xx}(t,x)} \in R_+^2, \quad \hat{\theta} = \sigma^{-1}(\mu - r\mathbf{1}) + \sigma^{-1}\widetilde{\pi}^*.$$

(3) $\gamma_1^* = 0$, $\gamma_2^* > 0$: $\pi_2^* = 0$
By K-T condition,

$$\begin{cases} 2(b_1 + \widetilde{\pi}_1^*)(\sigma_1^2 + \sigma_3^2) + 2b_2(\sigma_1\sigma_2 + \sigma_3\sigma_4) = 0 \\ 2b_2(\sigma_2^2 + \sigma_4^2) + 2(b_1 + \widetilde{\pi}_1^*)(\sigma_1\sigma_2 + \sigma_3\sigma_4) = \gamma_2^* \end{cases} \tag{2.2.13}$$

By solving the equation, we can obtain

$$\widetilde{\pi}_1^* = -\frac{b_2(\sigma_1\sigma_2 + \sigma_3\sigma_4)}{\sigma_1^2 + \sigma_3^2} - b_1 > 0 \tag{2.2.14}$$

$$\gamma_2^* = \frac{2b_2(\sigma_2\sigma_3 - \sigma_1\sigma_4)^2}{\sigma_1^2 + \sigma_3^2} > 0 \tag{2.2.15}$$

Thus, for $b_2 > 0$ and $b_1(\sigma_1^2 + \sigma_3^2) + b_2(\sigma_1\sigma_2 + \sigma_3\sigma_4) > 0$, the dual optimal control is given by

$$\widetilde{\pi}^* = [-\frac{b_2(\sigma_1\sigma_2 + \sigma_3\sigma_4)}{\sigma_1^2 + \sigma_3^2} - b_1, 0]^T.$$

The primal optimal control is given by

$$\pi^*(t,x) = -(\sigma^T)^{-1}\hat{\theta}\frac{V_x(t,x)}{xV_{xx}(t,x)} \in R_+^2, \quad \hat{\theta} = \sigma^{-1}(\mu - r\mathbf{1}) + \sigma^{-1}\widetilde{\pi}^*.$$

(4) $\gamma_1^* = 0$, $\gamma_2^* = 0$: i.e. $\bigtriangledown f(\widetilde{\pi}) = 0$

$$\begin{cases} 2(b_1 + \widetilde{\pi}_1^*)(\sigma_1^2 + \sigma_3^2) + 2(b_2 + \widetilde{\pi}_2^*)(\sigma_1\sigma_2 + \sigma_3\sigma_4) = 0 \\ 2(b_2 + \widetilde{\pi}_2^*)(\sigma_2^2 + \sigma_4^2) + 2(b_1 + \widetilde{\pi}_1^*)(\sigma_1\sigma_2 + \sigma_3\sigma_4) = 0 \end{cases} \tag{2.2.16}$$

Since matrix σ^{-1} is inverse, $\sigma_1\sigma_4 \neq \sigma_2\sigma_3$, i.e. $\sigma_1^2 + \sigma_3^2 > 0$ and $\sigma_2^2 + \sigma_4^2 > 0$.
Solving the above system of linear equations in two unknowns. The solution is given by

$$\begin{cases} (b_1 + \widetilde{\pi}_1^*)(\sigma_1^2\sigma_4^2 + \sigma_2^2\sigma_3^2 - 2\sigma_1\sigma_2\sigma_3\sigma_4) = 0 \\ (b_2 + \widetilde{\pi}_2^*)(\sigma_1^2\sigma_4^2 + \sigma_2^2\sigma_3^2 - 2\sigma_1\sigma_2\sigma_3\sigma_4) = 0, \end{cases} \tag{2.2.17}$$

To solve dual control problem, we need to find optimal dual control $\widetilde{\pi}^*$. We need to assume the minimum value is continuous on $[0,T]$ and non-zero, i.e., $\hat{\theta} = \sigma^{-1}\widetilde{\pi}^* + \theta \neq 0$, that is, there is a positive constant θ_0 such that $|\hat{\theta}(t)| \geq \theta_0$ for all $t \in [0,T]$. Therefore, the solution of equations(2.2.17) is given by

$$\begin{aligned} &\sigma_1^2\sigma_4^2 + \sigma_2^2\sigma_3^2 - 2\sigma_1\sigma_2\sigma_3\sigma_4 = 0 \\ &\Rightarrow (\sigma_1\sigma_4 - \sigma_2\sigma_3)^2 = 0 \\ &\Rightarrow \sigma_1\sigma_4 = \sigma_2\sigma_3 \end{aligned}$$

However, σ^{-1} is nonsingular matrix, that is, $|\sigma^{-1}| = \sigma_1\sigma_4 - \sigma_2\sigma_3 \neq 0$. Thus, for $b_1 = -\pi_1^*$, $b_2 = -\pi_2^*$, there not exists the dual optimal control.

Example 2.2.1. *Let the inverse of volatility matrix* $\sigma^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, *the stock excess returns* $b = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $\widetilde{\pi} = \begin{bmatrix} \widetilde{\pi}_1 \\ \widetilde{\pi}_2 \end{bmatrix} \geq \mathbf{0}$, *then*

$$\bigtriangledown f(\widetilde{\pi}) = \begin{bmatrix} 2(\widetilde{\pi}_1 + 1) \\ 2(\widetilde{\pi}_2 - 1) \end{bmatrix}, \quad \bigtriangledown^2 f(\widetilde{\pi}) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} > 0$$

We can obtain the following K-T condition

$$\begin{cases} 2(\widetilde{\pi}_1+1)-\gamma_1^*=0 \\ 2(\widetilde{\pi}_2-1)-\gamma_2^*=0 \\ \gamma_1^*\widetilde{\pi}_1^*=0 \\ \gamma_2^*\widetilde{\pi}_2^*=0 \\ \gamma_1^*,\gamma_2^*\geq 0 \end{cases}$$

(1) For $\gamma_1^*>0$, $\gamma_2^*>0$, we can obtain the dual optimal control $\widetilde{\pi}^*=[0,0]^T$ and the Lagrange multiplier $\gamma^*=[2,-2]^T$ which does not meet the K-T condition. Thus, there is not exist an optimal control.
(2)For $\gamma_1^*>0$, $\gamma_2^*=0$, the dual optimal control is given by $\widetilde{\pi}^*=[0,1]^T$ and the Lagrange multiplier is given by $\gamma^*=[2,0]^T$.
(3)For $\gamma_1^*=0$, $\gamma_2^*>0$, we can obtain the dual optimal control $\widetilde{\pi}^*=[-1,0]^T$ and the Lagrange multiplier $\gamma^*=[0,-2]^T$ which does not meet the K-T condition. Thus, there is not exist a optimal control.
To sum up, the dual optimal control is given by $\widetilde{\pi}^*=[0,1]^T$ and then the primal is given by

$$\pi^*(t,x)=-(\sigma^T)^{-1}\hat{\theta}\frac{V_x(t,x)}{xV_{xx}(t,x)}\in R_+^2,\quad \hat{\theta}=[1,0]^T.$$

Example 2.2.2. *Let the inverse of volatility matrix $\sigma^{-1}=\begin{bmatrix}1 & -1\\ -1 & 2\end{bmatrix}$, the stock excess returns $b=\begin{bmatrix}-1\\ 1\end{bmatrix}$, $\widetilde{\pi}=\begin{bmatrix}\widetilde{\pi}_1\\ \widetilde{\pi}_2\end{bmatrix}\geq \mathbf{0}$, then*

$$\triangledown f(\widetilde{\pi})=\begin{bmatrix}4(\widetilde{\pi}_1-1)-6(\widetilde{\pi}_2+1)\\ 10(\widetilde{\pi}_2+1)-6(\widetilde{\pi}_1-1)\end{bmatrix},\quad \triangledown^2 f(\widetilde{\pi})=\begin{bmatrix}4 & -6\\ -6 & 10\end{bmatrix}$$

We can obtain the following K-T condition

$$\begin{cases} 4\widetilde{\pi}_1-6\widetilde{\pi}_2-10-\gamma_1^*=0 \\ -6\widetilde{\pi}_1+10\widetilde{\pi}_2+16-\gamma_2^*=0 \\ \gamma_1^*\widetilde{\pi}_1^*=0 \\ \gamma_2^*\widetilde{\pi}_2^*=0 \\ \gamma_1^*,\gamma_2^*\geq 0 \end{cases}$$

(1) For $\gamma_1^*>0$, $\gamma_2^*>0$, we can obtain the dual optimal control $\widetilde{\pi}^*=[0,0]^T$ and the Lagrange multiplier $\gamma^*=[-10,16]^T$ which does not meet the K-T condition. Thus, there is not exist an optimal control.
(2)For $\gamma_1^*>0$, $\gamma_2^*=0$, we can obtain the dual optimal control $\widetilde{\pi}^*=[0,-8/5]^T$ which does not meet the K-T condition. Thus, there is not exist an optimal control.
(3)For $\gamma_1^*=0$, $\gamma_2^*>0$, the dual optimal control is given by $\widetilde{\pi}^*=[5/2,0]^T$ and the Lagrange multiplier is given by $\gamma^*=[0,1]^T$.
To sum up, the dual optimal control is given by $\widetilde{\pi}^*=[0,1]^T$ and then the primal is given by

$$\pi^*(t,x)=-(\sigma^T)^{-1}\hat{\theta}\frac{V_x(t,x)}{xV_{xx}(t,x)}\in R_+^2,\quad \hat{\theta}=[\frac{1}{2},\frac{1}{2}]^T.$$

2.2.3 $\mathcal{K}=R\times 0$

$\mathcal{K}$ is the nonnegative part of the whole space $R\times 0$, then the positive polar cone $\widetilde{\mathcal{K}}$ is $0\times R$. Thus, the $\widetilde{\pi}_1^*=0$ and the quadratic function is given by

$$\begin{aligned} f(\widetilde{\pi})=&(\sigma_1b_1+\sigma_2(b_2+\widetilde{\pi}_2))^2+(\sigma_3b_1+\sigma_4(b_2+\widetilde{\pi}_2))^2\\ =&(\sigma_1^2+\sigma_3^2)b_1^2+(\sigma_2^2+\sigma_4^2)(b_2+\widetilde{\pi}_2)^2+2b_1(b_2+\widetilde{\pi}_2)(\sigma_1\sigma_2+\sigma_3\sigma_4)\end{aligned}$$

The first derivative is given by

$$\frac{\partial f}{\partial \widetilde{\pi}_2}=2(\sigma_2^2+\sigma_4^2)(b_2+\widetilde{\pi}_2)+2b_1(\sigma_1\sigma_2+\sigma_3\sigma_4)$$

The second derivative is given by

$$\frac{\partial^2 f}{\partial \widetilde{\pi}_2^2} = 2(\sigma_2^2 + \sigma_4^2) > 0$$

Therefore, $f(\widetilde{\pi})$ is convex. By lemma 2.2.2, if $\bigtriangledown f = 0$, then $\widetilde{\pi}^*$ is global minimum.

$$\widetilde{\pi}_2^* = -\frac{b_1(\sigma_1\sigma_2 + \sigma_3\sigma_4)}{\sigma_2^2 + \sigma_4^2} - b_2$$

Therefore, the dual optimal control is given by

$$\widetilde{\pi}^* = [0, -\frac{b_1(\sigma_1\sigma_2 + \sigma_3\sigma_4)}{\sigma_2^2 + \sigma_4^2} - b_2]^T.$$

The primal optimal control is given by

$$\pi^*(t,x) = -(\sigma^T)^{-1}\hat{\theta}\frac{V_x(t,x)}{xV_{xx}(t,x)} \in R \times 0, \quad \hat{\theta} = \sigma^{-1}(\mu - r\mathbf{1}) + \sigma^{-1}\widetilde{\pi}^*.$$

Chapter 3

The Ornstein–Uhlenbeck process and dual control method

Assume that $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ is a given probability space with filtration $\mathcal{F}_t$ generated by standard Brownian motions W^s and W^μ with coefficient ρ. The market consists of two traded assets, one savings account B with risk-free interest rate r and one risky asset S satisfying a stochastic differential equaiton(SDE):

$$dS_t = S_t(\mu_t dt + \sigma dW_t^s). \tag{3.0.1}$$

The drift μ_t satisfies an Ornstein-Uhlenbeck(OU) process

$$d\mu_t = \kappa(\theta - \mu_t)d_t + \sigma dW_t^\mu, \qquad \alpha = (\kappa, \theta, \sigma) \tag{3.0.2}$$

where $\kappa > 0$, $\sigma > 0$, θ are parameters. In financial mathematics, this is also called the Vasicek model. the Vasicek model has some advantages. This equation is linear and can be solved explicitly. In addition, the model is mean-reverting, and the expected value of drift tends to a constant value θ. As time goes on to infinity, the velocity depends on κ, and its variance will not explode at infinity. Then the wealth process X satisfies the SDE:

$$dX_t = X_t\,[\ (r + \pi_t(\mu_t - r))dt + \pi_t \sigma dW_t^s] \tag{3.0.3}$$

where π is a progressively measurable control process.
The utility maximization problem is defined by

$$\sup_{\pi \in \mathcal{K}} E[U(X_T)] \quad \text{subject to (3.0.3)}, \tag{3.0.4}$$

Stochastic control method can be use to solve equation (3.0.4). Then we define the value function

$$V(t, x, \mu) = \underset{\pi \in \mathcal{K}}{sup} E_{t,x,\mu}[U(X_T)], \tag{3.0.5}$$

where $E_{t,x,\mu}$ is the conditional expectation operator given by $X_t = x$ and $\mu_t = \mu$, and $\pi_t \in \mathcal{K} = R$.

By Dynamic Programming Principle (**??**) and Ito formula, we obtain

$$\begin{aligned} V(t+h, X_{t+h}, \mu_{t+h}) &= V(t, x, \mu) + \int_t^{t+h} (\frac{\partial V}{\partial s} ds + \frac{\partial V}{\partial x} dX_s + \frac{1}{2}\frac{\partial^2 V}{\partial x^2} d[X, X]_s + \frac{\partial V}{\partial \mu} d\mu_s \\ &+ \frac{1}{2}\frac{\partial^2 V}{\partial \mu^2} d[\mu, \mu]_s + \frac{\partial^2 V}{\partial x \partial \mu} dX_s d\mu_s) \\ &= V(t, x, \mu) + \int_t^{t+h} (\frac{\partial V}{\partial s} + \frac{\partial V}{\partial x}(rX_s + \pi_s X_s(\mu_s - r)) + \frac{1}{2}\frac{\partial^2 V}{\partial x^2}\pi_s^2 X_s^2 \sigma^2 \\ &+ \frac{\partial V}{\partial \mu}\kappa(\theta - \mu_s) + \frac{1}{2}\frac{\partial^2 V}{\partial \mu^2}\sigma^2 + \frac{\partial^2 V}{\partial x \partial \mu} X_s \pi_s \sigma^2 \rho) ds \\ &+ \int_t^{t+h} (\frac{\partial V}{\partial x}\pi_s X_s \sigma + \frac{\partial V}{\partial \mu}\sigma) dW_s \end{aligned} \tag{3.0.6}$$

As a result, the primal HJB equation is given by

$$\frac{\partial V}{\partial t}+rxV_x+\kappa(\theta-\mu)V_\mu+\frac{1}{2}\sigma^2V_{\mu\mu}+\sup_{\pi\in\mathcal{K}}\left\{\pi x(\mu-r)V_x+\frac{1}{2}\pi^2x^2\sigma^2V_{xx}+\rho x\pi\sigma^2V_{x\mu}\right\}=0 \quad (3.0.7)$$

with the terminal condition $V(T,x,\mu)=U(x)$. The maximum in (3.0.7) is achieved at

$$\pi^*=-\frac{(\mu-r)V_x}{x\sigma^2V_{xx}}-\frac{\rho V_{x\mu}}{xV_{xx}} \quad (3.0.8)$$

Substituting (3.0.8) into (3.0.7) gives a nonlinear PDE

$$\frac{\partial V}{\partial t}+rxV_x+\kappa(\theta-\mu)V_\mu+\frac{1}{2}\sigma^2V_{\mu\mu}-\frac{1}{2\sigma^2V_{xx}}[(\mu-r)V_x+\sigma^2\rho V_{x\mu}]^2=0 \quad (3.0.9)$$

3.1 Power utility and primal HJB equation

For a power utility $U(x)=1/px^p$, $0<p<1$, the solution of equation (3.0.9) can be expressed in a separable form

$$V(t,x,\mu)=U(x)g(t,\mu) \quad (3.1.1)$$

The component $g(t,\mu)$ is unknown except for some special cases. In fact, $g(t,\mu)$ solves a nonlinear equation for which no closed-form solutions are available.
The partial derivative of V is given by

$$\begin{aligned}&\frac{\partial V}{\partial t}=\frac{1}{p}x^p\frac{\partial g}{\partial t},\quad \frac{\partial V}{\partial x}=x^{p-1}g,\quad \frac{\partial V}{\partial \mu}=\frac{1}{p}x^pg_\mu,\\ &\frac{\partial^2V}{\partial x^2}=(p-1)x^{p-2}g,\quad \frac{\partial^2V}{\partial\mu^2}=\frac{1}{p}x^pg_{\mu\mu},\quad \frac{\partial^2V}{\partial x\partial\mu}=x^{p-1}g_\mu\end{aligned} \quad (3.1.2)$$

for some function g which satisfies a nonlinear PDE

$$\frac{\partial g}{\partial t}+prg+\kappa(\theta-\mu)g_\mu+\frac{1}{2}\sigma^2g_{\mu\mu}-\frac{p}{2(p-1)\sigma^2g}[(\mu-r)g+\sigma^2\rho g_\mu]^2=0 \quad (3.1.3)$$

with the terminal condition $g(T,\mu)=1$. The primal optimal control is given by

$$\pi^*=-\frac{\mu-r}{\sigma^2(p-1)}-\frac{\rho g_\mu}{(p-1)g}. \quad (3.1.4)$$

To remove certain non-linearities that arise due to the stochastic factor, we now make the following transformation proposed by T.Zariphopoulou[5] to obtain a new equation of value function

$$V(t,x,\mu)=U(x)g(t,\mu)=\frac{x^p}{p}\hat{g}(t,\mu)^\lambda,\quad \lambda=\frac{1-p}{1-p+\rho^2p} \quad (3.1.5)$$

Partial derivatives of $\hat{g}(t,\mu)$ are given by

$$\frac{\partial g}{\partial t}=\lambda\frac{\partial\hat{g}}{\partial t}\hat{g}^{\lambda-1},\quad \frac{\partial g}{\partial\mu}=\lambda\frac{\partial\hat{g}}{\partial\mu}\hat{g}^{\lambda-1},\quad \frac{\partial^2g}{\partial\mu^2}=\lambda\frac{\partial^2\hat{g}}{\partial\mu^2}\hat{g}^{\lambda-1}+\lambda(\lambda-1)(\frac{\partial\hat{g}}{\partial\mu})^2\hat{g}^{\lambda-2}. \quad (3.1.6)$$

Inserting the above derivatives in PDE (3.1.3) gives

$$\frac{\partial\hat{g}}{\partial t}+\frac{pr}{\lambda}\hat{g}+\kappa(\theta-\mu)\hat{g}_\mu+\frac{1}{2}\sigma^2\hat{g}_{\mu\mu}-\frac{p}{2(p-1)\sigma^2\lambda}((\mu-r)^2\hat{g}+2(\mu-r)\sigma^2\lambda\rho\hat{g}_\mu)=0 \quad (3.1.7)$$

with the terminal condition $\hat{g}(T,\mu)=1$. The PDE (3.1.7) can be solved by Feynman-Kac Theorem and the exact solution can be obtained.
Suppose that $\hat{g}(t,\mu)$ is the solution to the PDE

$$\begin{cases}\frac{\partial\hat{g}}{\partial t}+b(t,\mu)\frac{\partial\hat{g}}{\partial\mu}+\frac{1}{2}\sigma^2(t,\mu)\frac{\partial^2\hat{g}}{\partial\mu^2}=r(t,\mu)\hat{g}, & t<T;\\ \hat{g}(T,\mu)=1 & t=T.\end{cases} \quad (3.1.8)$$

where

$$b(t,\mu) = \kappa(\theta - \mu) - \frac{p\rho(\mu - r)}{p-1}, \quad \sigma(t,\mu) = \sigma, \quad r(t,\mu) = \frac{p(\mu - r)^2}{2(p-1)\sigma^2\lambda} - \frac{pr}{\lambda} \tag{3.1.9}$$

Then the $\hat{g}$ is given by

$$\begin{aligned}\hat{g}(t,\mu) &= \mathbb{E}^{(t,\mu)}[e^{-\int_t^T r(u,\mu_u)du}] \\ &= \mathbb{E}^{(t,\mu)}[\exp\left(-\int_t^T (\frac{p(\mu_u - r)^2}{2(p-1)\sigma^2\lambda} - \frac{pr}{\lambda})du\right)]\end{aligned}$$

where $\mathbb{E}^{t,\mu}[\cdot]$ denotes the conditional expectation operator given by the information up to time t with $\mu_t = \mu$, and $\mu = (\mu_s)_{s\in[t,T]}$ is the solution to the SDE

$$\begin{aligned}d\mu_s &= b(s,\mu_s)ds + \sigma(t,\mu_s)dW_s \\ &= \left(\kappa(\theta - \mu_s) - \frac{p\rho(\mu_s - r)}{p-1}\right)ds + \sigma dW_s.\end{aligned}$$

The optimal control is given by

$$\pi^* = -\frac{\mu - r}{\sigma^2(p-1)} - \frac{\rho\lambda\hat{g}_\mu}{(p-1)\hat{g}}. \tag{3.1.10}$$

Since the Vasicek model has an affine term structure, we use the idea in [8, Proposition 5.1, page 307]. The function g can be expressed in an analytical form

$$g(t,\mu) = \exp(A(t) + B(t)\mu + C(t)\mu^2) \tag{3.1.11}$$

where $A(\cdot)$, $B(\cdot)$ and $C(\cdot)$ are continuous differentiable functions of time t with the terminal condition $A(T) = 0$, $B(T) = 0$ and $C(T) = 0$. Simply calculus shows that

$$\begin{aligned}&\frac{\partial g}{\partial t} = g(A'(t) + B'(t)\mu + C'(t)\mu^2), \quad g_\mu = g(B(t) + 2C(t)\mu) \\ &g_{\mu\mu} = g(B(t) + 2C(t)\mu)^2 + 2gC(t)\end{aligned}$$

Substituting these derivatives into PDE (3.1.3), the functions $A(\cdot)$, $B(\cdot)$ and $C(\cdot)$ are solutions of the ordinary differential equations (ODEs):

$$\begin{aligned}A'(t) &= -rp + \frac{pr^2}{2(p-1)\sigma^2} - (\kappa\theta + \frac{p\rho r}{p-1})B(t) + \frac{1}{2}\sigma^2(\frac{p\rho^2}{p-1} - 1)B^2(t) - \sigma^2 C(t) \\ B'(t) &= (\kappa + \frac{p\rho}{p-1})B(t) + 2\sigma^2(\frac{p\rho^2}{p-1} - 1)B(t)C(t) - 2(\kappa\theta + \frac{r\rho p}{p-1})C(t) - \frac{pr}{(p-1)\sigma^2} \\ C'(t) &= 2(\kappa + \frac{p\rho}{p-1})C(t) + 2\sigma^2(\frac{p\rho^2}{p-1} - 1)C^2(t) + \frac{p}{2\sigma^2(p-1)}\end{aligned} \tag{3.1.12}$$

with terminal conditions $A(T) = 0$, $B(T) = 0$ and $C(T) = 0$. We can solve the Riccati equation to get a closed-form solution $C(t)$ and then find $B(t)$ and $A(t)$ once $C(t)$ is known, see Appendix A. The primal optimal control is given by

$$\pi^*(t) = -\frac{\mu - r}{\sigma^2(p-1)} - \frac{\rho}{p-1}(B(t) + 2C(t)\mu).$$

3.2 Dual HJB equation

Since, we assume that the utility function is a power utility, the nonlinear PDE (3.0.9) can be simplified to a solvable nonlinear PDE (3.1.4) so that obtaining analytical solution. However, for general utility functions, like non-HARA utility and Yarri utility, it is impossible to find the exact solutions. We need to use dual control method to find the exact solutions. The dual utility function is defined by

$$\widetilde{U}(y) = \sup_{x\geq 0}(U(x) - xy)$$

for $y \geq 0$. Assume that a dual process Y has the following form

$$dY_t = Y_t[\alpha_t dt + \beta_t dW_t^s + \gamma_t dW_t^\mu]$$

with the initial condition $Y_0 = y$. If XY is super-martingale for any control process π, then

$$E[X_{T,\pi}Y_T] \leq xy = X_0 Y_0.$$

Since $\widetilde{U}$ is dual value function of U, by equation (1.1.9)

$$\begin{aligned} &\widetilde{U}(y) \leq U(x) + xy \\ \Rightarrow &U(x) \leq \widetilde{U}(y) + xy \end{aligned}$$

for $x \geq 0$ and $y \geq 0$. Thus,

$$E[U(X_{T,\pi})] \leq E[\widetilde{U}(Y_T)] + E[X_{T,\pi}Y_T] \leq E[\widetilde{U}(Y_T)] + xy,$$

where $X_0 = x$ and $Y_0 = y$ are the initial conditions, which leads to a weak duality relation

$$\sup_{\pi} E[U(X_{T,\pi})] \leq \inf_{y \geq 0} (E[\widetilde{U}(Y_T)] + xy). \tag{3.2.1}$$

To make sure XY is super-martingale, we apply Ito's formula to obtain

$$\begin{aligned} dX_tY_t &= Y_t dX_t + X_t dY_t + d[X,Y]_t \\ &= Y_t X_t[(r + \pi_t(\mu_t - r))dt + \pi_t \sigma dW_t^s] + X_t Y_t[\alpha_t dt + \beta_t dW_t^s + \gamma_t dW_t^\mu] + X_t Y_t[\pi_t \sigma \beta_t dt + \pi_t \sigma \gamma_t \rho dt] \\ &= X_t Y_t[(r + \pi_t \mu_t - \pi_t r + \alpha_t + \pi_t \sigma \beta_t + \pi_t \sigma \gamma_t \rho)dt + (\pi_t \sigma + \beta_t)dW_t^s + \gamma_t dW_t^\mu] \end{aligned}$$

Since XY is super-martingale,

$$\begin{aligned} &r + \pi_t \mu_t - \pi_t r + \alpha_t + \pi_t \sigma \beta_t + \pi_t \sigma \gamma_t \rho \leq 0 \\ \Rightarrow &r + \alpha_t + \pi_t(\mu_t + \sigma \beta_t + \sigma \gamma_t \rho - r) \leq 0 \\ \Rightarrow &\mu_t + \sigma \beta_t + \sigma \gamma_t \rho - r = 0, \quad r + \alpha_t \leq 0 \\ \Rightarrow &\alpha_t \leq -r, \quad \beta_t = -\frac{\mu_t - r}{\sigma} - \rho \gamma_t. \end{aligned}$$

Since U is increasing concave and $\widetilde{U}$ is decreasing convex. We shall choose Y_T as large as possible, that is, we shall choose the drift α as large as possible. Therefore, the largest $\alpha_t = -r$ and the dual process is given by

$$dY_t = Y_t[-rdt - (\frac{\mu_t - r}{\sigma} + \rho \gamma_t)dW_t^s + \gamma_t dW_t^\mu] \tag{3.2.2}$$

with the initial condition $Y_0 = y$. The solution to the PDE of dual process at time T is given by

$$Y_T = y\exp\left(-\int_t^T (r + \frac{1}{2}(1-\rho^2)\gamma_u^2 + \frac{(\mu_u - r)^2}{2\sigma^2})du - \int_t^T (\frac{\mu_u - r}{\sigma} + \rho \gamma_u)dW_u^s + \int_t^T \gamma_u dW_u^\mu\right).$$

The dual value function is defined by

$$\widetilde{V}(t, y, \mu) := \inf_{\gamma} E[\widetilde{U}(Y_T)|Y_t = y].$$

By dynamic programming principle,

$$\widetilde{V}(t, y, \mu) = \inf_{\gamma} E[\widetilde{V}(t + h, X_{t+h}, \mu_{t+h}) =)|X_t = x, \mu_t = \mu].$$

By Ito's formula,

$$
\begin{aligned}
\widetilde{V}(t+h, Y_{t+h}, \mu_{t+h}) &= \widetilde{V}(t,y,\mu) + + \int_t^{t+h} (\frac{\partial \widetilde{V}}{\partial s} ds + \frac{\partial \widetilde{V}}{\partial y} dY_s + \frac{1}{2}\frac{\partial^2 \widetilde{V}}{\partial y^2} d[Y,Y]_s + \frac{\partial \widetilde{V}}{\partial \mu} d\mu_s \\
&+ \frac{1}{2}\frac{\partial^2 \widetilde{V}}{\partial \mu^2} d[\mu,\mu]_s + \frac{\partial^2 \widetilde{V}}{\partial y \partial \mu} dY_s d\mu_s) \\
&= \widetilde{V}(t,y,\mu) + \int_t^{t+h} (\frac{\partial \widetilde{V}}{\partial s} - \frac{\partial \widetilde{V}}{\partial y} rY_s + \frac{1}{2}\frac{\partial^2 \widetilde{V}}{\partial y^2} Y_s^2 [\frac{(\mu_s - r)^2}{\sigma^2} + (1-\rho^2)\gamma_s^2] \\
&+ \frac{\partial \widetilde{V}}{\partial \mu} \kappa(\theta - \mu_s) + \frac{1}{2}\frac{\partial^2 \widetilde{V}}{\partial \mu^2}\sigma^2 + \frac{\partial^2 \widetilde{V}}{\partial y \partial \mu} Y_s[\sigma\gamma_s(1-\rho^2) - \rho(\mu_s - r)])ds \\
&+ \int_t^{t+h} \left(\frac{\partial \widetilde{V}}{\partial y} Y_s[-(\frac{\mu_s - r}{\sigma} + \rho\gamma_s)dW_s^s + \gamma_s dW_s^\mu] + \frac{\partial \widetilde{V}}{\partial \mu}\sigma dW_s^\mu \right)
\end{aligned}
$$

As a result, $\widetilde{V}$ satisfies the following dual HJB equation

$$
\begin{aligned}
&\frac{\partial \widetilde{V}}{\partial t} - ry\widetilde{V}_y + \kappa(\theta-\mu)\widetilde{V}_\mu + \frac{1}{2}y^2\frac{(\mu-r)^2}{\sigma^2}\widetilde{V}_{yy} + \frac{1}{2}\sigma^2\widetilde{V}_{\mu\mu} - \rho(\mu - r)y\widetilde{V}_{y\mu} \\
&+ \inf_\gamma \left\{ \frac{1}{2}y^2\gamma^2(1-\rho^2)\widetilde{V}_{yy} + \sigma\gamma(1-\rho^2)y\widetilde{V}_{y\mu} \right\} = 0
\end{aligned}
$$

with the terminal condition $\widetilde{V}(T,y,\mu) = \widetilde{U}(y)$. The minimum of the dual control γ in the dual HJB equation is achieved at

$$\gamma^* = -\frac{\sigma \widetilde{V}_{y\mu}}{y\widetilde{V}_{yy}}. \tag{3.2.3}$$

Substituting (3.2.3) into the dual HJB equation gives

$$\frac{\partial \widetilde{V}}{\partial t} - ry\widetilde{V}_y + \kappa(\theta-\mu)\widetilde{V}_\mu + \frac{(\mu-r)^2}{2\sigma^2}y^2\widetilde{V}_{yy} - \frac{\sigma^2(1-\rho^2)}{2}\frac{\widetilde{V}_{y\mu}^2}{\widetilde{V}_{yy}} - \rho(\mu-r)y\widetilde{V}_{y\mu} + \frac{1}{2}\sigma^2\widetilde{V}_{\mu\mu} = 0. \tag{3.2.4}$$

The dual value function and primal value function have the following relationship

$$V(t,x,\mu) = \inf_{y\geq 0}(\widetilde{V}(t,y,\mu) + xy).$$

Then the primal value function is given by

$$V(t,x,\mu) = \widetilde{V}(t,\hat{y}(t,x,\mu),\mu) + x\hat{y}(t,x,\mu). \tag{3.2.5}$$

where $\hat{y}(t,x,\mu)$ is the solution of the equation

$$\frac{\partial \widetilde{V}(t,y,\mu)}{\partial y} + x = 0. \tag{3.2.6}$$

Simple calculus shows that

$$\frac{\partial V}{\partial t} = \frac{\partial \widetilde{V}}{\partial t} + \frac{\partial \widetilde{V}}{\partial y}\frac{\partial y}{\partial t} + x\frac{\partial y}{\partial t} = \frac{\partial \widetilde{V}}{\partial t}, \quad \frac{\partial V}{\partial x} = \frac{\partial \widetilde{V}}{\partial y}\frac{\partial y}{\partial x} + y + x\frac{\partial y}{\partial x} = y, \quad \frac{\partial V}{\partial \mu} = \frac{\partial \widetilde{V}}{\partial \mu} + \frac{\partial \widetilde{V}}{\partial y}\frac{\partial y}{\partial \mu} + x\frac{\partial y}{\partial \mu} = \frac{\partial \widetilde{V}}{\partial \mu},$$

and by (3.2.6)

$$\frac{\partial x}{\partial y} = -\widetilde{V}_{yy}, \qquad \frac{\partial x}{\partial \mu} = -\widetilde{V}_{y\mu},$$

Then

$$
\begin{aligned}
\frac{\partial^2 V}{\partial x^2} &= \frac{\partial y}{\partial x} = -\frac{1}{\widetilde{V}_{yy}}, \qquad \frac{\partial^2 V}{\partial x \partial \mu} = \frac{\partial y}{\partial \mu} = \frac{\partial y}{\partial x}\frac{\partial x}{\partial \mu} = \frac{\widetilde{V}_{y\mu}}{\widetilde{V}_{yy}}, \\
\frac{\partial^2 V}{\partial \mu^2} &= \frac{\partial^2 \widetilde{V}}{\partial x \partial \mu}\frac{\partial x}{\partial \mu} + \frac{\partial^2 \widetilde{V}}{\partial \mu^2} = -\frac{(\widetilde{V}_{y\mu})^2}{\widetilde{V}_{yy}} + \widetilde{V}_{\mu\mu}.
\end{aligned}
$$

Inserting these partial derivatives into dual HJB equation(3.2.4) gives that V satisfies the primal HJB equation(3.0.9) with the terminal condition $V(T,x,\mu) = U(x)$. Then the primal optimal control is derived form the above dual relations of the partial derivatives and equation(3.0.8)

$$\pi^*(t,x,\mu) = \frac{\mu - r}{\sigma^2 x}\hat{y}(t,x,\mu)\widetilde{V}_{yy}(t,\hat{y}(t,x,\mu),\mu) - \frac{\rho}{x}\widetilde{V}_{y\mu}(t,\hat{y}(t,x,\mu),\mu) \tag{3.2.7}$$

3.2.1 Power utility function

Power utility function is defined by

$$U(x) = \frac{1}{p}x^p$$

where $p \in (0,1)$ is a constant. The dual power utility function is given by

$$\widetilde{U}(y) = -\frac{1}{q}y^q$$

for $y > 0$, where $q = p/(p-1) < 0$. By equations(3.1.11) and (3.2.5), we may assume

$$\widetilde{V}(t,y,\mu) = \widetilde{U}(y)\widetilde{f}(t,\mu) = -\frac{1}{q}y^q\widetilde{f}(t,\mu)$$

Simple calculus shows that

$$\begin{aligned}
\frac{\partial \widetilde{V}}{\partial t} &= -\frac{1}{q}y^q\frac{\partial \widetilde{f}}{\partial t}, \quad \frac{\partial \widetilde{V}}{\partial y} = -y^{q-1}\widetilde{f}, \quad \frac{\partial \widetilde{V}}{\partial \mu} = -\frac{1}{q}y^q\widetilde{f}_\mu \\
\widetilde{V}_{yy} &= -(q-1)y^{q-2}\widetilde{f}, \quad \widetilde{V}_{y\mu} = -y^{q-1}\widetilde{f}_\mu, \quad \widetilde{V}_{\mu\mu} = -\frac{1}{q}y^q\widetilde{f}_{\mu\mu}
\end{aligned}$$

and substitute it into the equation (3.2.4) to obtain the PDE equation of $\widetilde{f}$

$$\frac{\partial \widetilde{f}}{\partial t} - rq\widetilde{f} + \kappa(\theta-\mu)\widetilde{f}_\mu + \frac{1}{2\sigma^2}q(q-1)(\mu-r)^2\widetilde{f} - \frac{q}{2(q-1)}\sigma^2(1-\rho^2)\frac{\widetilde{f}_\mu^2}{\widetilde{f}} - q\rho(\mu-r)\widetilde{f}_\mu + \frac{1}{2}\sigma^2\widetilde{f}_{\mu\mu} = 0 \tag{3.2.8}$$

with the terminal condition $\widetilde{f}(T,\mu) = 1$. Now suppose that

$$\widetilde{f}(t,\mu) = \exp(\widetilde{A}(t) + \widetilde{B}(t)\mu + \widetilde{C}(t)\mu^2)$$

and inserting $\widetilde{f}$ into PDE(3.2.8) to obtain ODEs for $A\widetilde{A}$, $\widetilde{B}$ and $\widetilde{C}$

$$\begin{aligned}
\widetilde{A}'(t) &= -(\kappa\theta + q\rho r)\widetilde{B}(t) + \frac{1}{2}\sigma^2\frac{1-q\rho^2}{q-1}\widetilde{B}^2(t) - \sigma^2\widetilde{C}(t) + rq - \frac{q(q-1)r^2}{2\sigma^2} \\
\widetilde{B}'(t) &= (\kappa + q\rho)\widetilde{B}(t) + 2\sigma^2\frac{1-q\rho^2}{q-1}\widetilde{B}(t)\widetilde{C}(t) - 2(\kappa\theta + q\rho r)\widetilde{C}(t) + \frac{q(q-1)r}{\sigma^2} \\
\widetilde{C}'(t) &= 2(\kappa + q\rho)\widetilde{C}(t) + 2\sigma^2\frac{1-q\rho^2}{q-1}\widetilde{C}^2(t) - \frac{q(q-1)}{2\sigma^2}
\end{aligned} \tag{3.2.9}$$

with the terminal conditions $\widetilde{A}(T) = 0$, $\widetilde{B}(T) = 0$ and $\widetilde{C}(T) = 0$. We can solve the Riccati-type equation to obtain a closed-form solution $\widetilde{C}(t)$ and then find $\widetilde{B}(t)$ and $\widetilde{A}(t)$ once $\widetilde{C}(t)$ is known, see Appendix. By solving the equation $\widetilde{V}_y + x = 0$ to obtain

$$\hat{y}(t,x,\mu) = [x\exp(-\widetilde{A}(t) - \widetilde{B}(t)\mu - \widetilde{C}(t)\mu^2)]^{p-1},$$

and the primal value function

$$V(t,x,\mu) = \widetilde{V}(t,\hat{y},\mu) + x\hat{y} = \frac{x^p}{p}\exp[(1-p)(\widetilde{A}(t) + \widetilde{B}(t)\mu + \widetilde{C}(t)\mu^2)]. \tag{3.2.10}$$

By equation (3.2.7), the primal optimal control is given by

$$\begin{aligned}
\pi^*(t,\mu) &= -\frac{\mu-r}{\sigma^2 x}(q-1)\hat{y}^{q-1}\widetilde{f} + \frac{\rho}{x}\hat{y}^{q-1}(\widetilde{B}(t) + 2\widetilde{C}(t)\mu)\widetilde{f} \\
&= \frac{\mu-r}{\sigma^2}(1-q) + \rho(\widetilde{B}(t) + 2\widetilde{C}(t)\mu)
\end{aligned}$$

3.3 Numerical test

Equations for $A(t)$ and $B(t)$ are too complicated to find closed-form solutions, even though they are first-order linear ODE. Hence, we need to use numerical method to solve. A first-order differential equation is an Initial value problem(IVP) of the form,[9]

$$\begin{aligned} &B'(t) = f(t, B(t)), \quad B(T) = B_T = 0, \\ &A'(t) = g(t), \quad A(T) = A_T = 0, \end{aligned} \tag{3.3.1}$$

where f is a function $f : [t_0, \infty) \times \mathbb{R}^d \to \mathbb{R}^d$ and the terminal condition $B_T \in \mathbb{R}^d$ is a given vector. In the paper, we use Euler method to solve the first-order IVPs. Starting with the differential equation (3.3.1), we replace the derivative B' by the backward difference approximation

$$B'(t) \approx \frac{B(t) - B(t-h)}{h},$$

which when re-arranged yields the following formula

$$B(t-h) \approx B(t) - hB'(t)$$

and then by equation (3.3.1)

$$B(t-h) \approx B(t) - hf(t, B(t)). \tag{3.3.2}$$

Thus, we choose a step size $h = T/N$ where N is a integer and construct the sequence t_0, $t_1 = t_0 + h$, $t_2 = t_0 + 2h$,...,$t_N = t_0 + Nh$ and then denote by B_n a numerical estimate of the exact solution $B(t_n)$. Motivated by (3.3.2), we compute these estimates by the following recursive explicit scheme

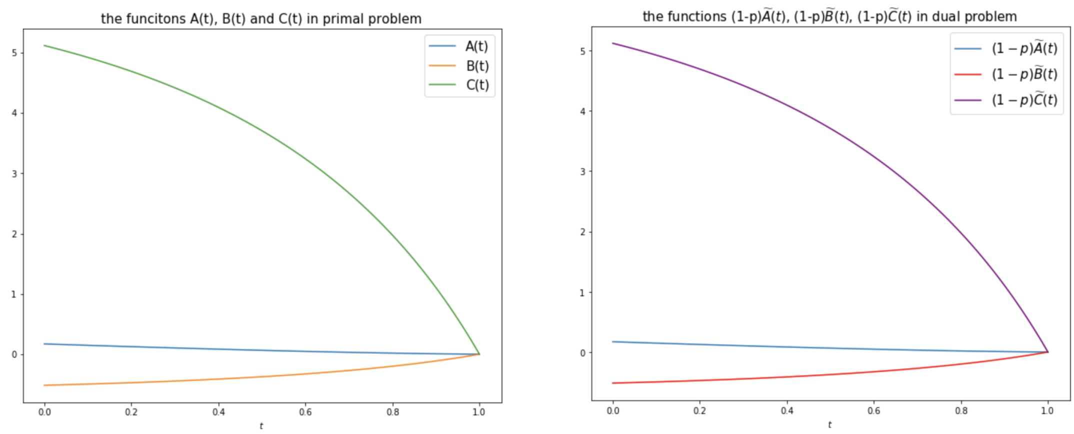
$$B_{n-1} = B_n - hf(t_n, B_n), \quad n = N, N-1, ..., 1$$

Similarly, we can obtain

$$A_{n-1} = A_n - hg(t_n), \quad n = N, N-1, ..., 1$$

We use Python to find numerical approximations to the solutions of ODEs and let parameters

$$r = 0.05, \quad \sigma = 0.2 \quad \rho = -0.5, \quad \kappa = 0.8, \quad \theta = 0.05, \quad T = 1, \quad p = 1. \tag{3.3.3}$$



(a) The functions A(t), B(t) and C(t) computed by primal HJB equation

(b) The functions $(1-p)\widetilde{A}(t)$, $(1-p)\widetilde{B}(t)$, and $(1-p)\widetilde{C}(t)$ computed by dual HJB equation

Figure 3.1: The functions A(t), B(t) and C(t) in primal problem and the functions $(1-p)\widetilde{A}(t)$, $(1-p)\widetilde{B}(t)$, and $(1-p)\widetilde{C}(t)$ in dual problem.

Figure3.1 illustrates one graph of $A(t)$, $B(t)$ and $C(t)$ using primal solution over $t \in [0, 1]$ and one graph of $(1-p)\widetilde{A}(t)$, $(1-p)\widetilde{B}(t)$, and $(1-p)\widetilde{C}(t)$ using dual solution over $t \in [0, 1]$

separately and then shows they essentially produce the same graph. By equations (3.1.1), (3.1.11) and (3.2.10), we can obtain

$$A(t) + B(t)\mu + C(t)\mu^2 = (1-p)(\widetilde{A}(t) + \widetilde{B}(t)\mu + C\widetilde{C}(t)\mu^2).$$

Hence, the figure proofs that dual control method and primal control method can produce the same optimal value function.
Furthermore, we use numerical method for SDEs to approximate drift μ. By equation (3.0.2) and Euler-Maruyama method, the drift μ_t can be discretized in the following form

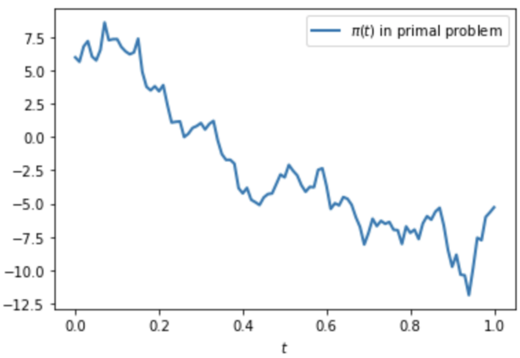
$$\mu_n = \mu_{n-1} + \kappa(\theta - \mu_{n-1})h + \sigma Z_1 \sqrt{h},$$

with the initial condition μ_0 where Z_1 is chosed from the standard normal distribution $N(0,1)$. Therefore, the primal optimal control using primal solution is given by

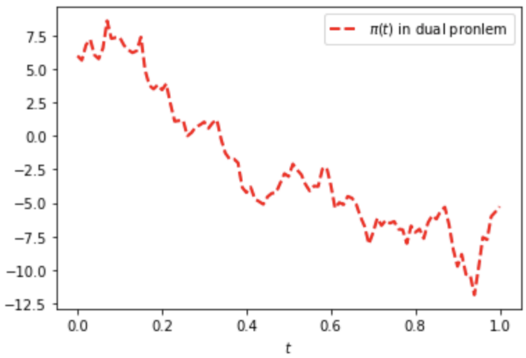
$$\pi^*(t_n) = -\frac{\mu_n - r}{\sigma^2(p-1)} - \frac{\rho}{p-1}(B(t_n) + 2C(t_n)\mu_n),$$

and the primal optimal control using dual solution is given by

$$\pi^*(t_n) = \frac{\mu_n - r}{\sigma^2}(1-q) + \rho(\widetilde{B}(t_n) + 2\widetilde{C}(t_n)\mu_n)$$



(a) The primal control π^* computed by primal control method over time t



(b) The primal control π^* computed by dual control method over time t

Figure 3.2: The primal control π^* computed by primal control method and dual control method over time t respectively

Figure3.2 shows the primal control computed by primal HJB equation and dual HJB equation over time $t \in [0,1]$, respectively. The figure shows that the primal control method and dual control method produce the same optimal control.

Chapter 4

Upper and lower bounds

For general utility functions, we cannot solve the primal problem by using the relationship

$$V(t,x,\mu) = \widetilde{V}(t,\hat{y},\mu) + x\hat{y},$$

since the dual problem is quiet difficult. we has already obtain a weak duality relation (3.2.1)

$$\sup_{\pi} E[U(X_{T,\pi})] \leq \inf_{y>0}(\inf_{\gamma} E[U(X_{T,\pi})] + xy) \leq \inf_{y>0}(E[\widetilde{U}(Y_T)] + xy),$$

for all dual controls γ.
For every fixed γ, the upper bound is defined by

$$\overline{V}(t,x,\mu) = \inf_{y>0}(E_{t,y,\mu}[\widetilde{U}(Y_T^{\gamma})] + xy). \tag{4.0.1}$$

Y_T can be computed for fixed γ and then we can compute the upper bound by Monte Carlo simulation.

Theorem 4.0.1. *Let Γ be a set of dual controls γ and the upper bound $\overline{V}(t,x,\mu)$ is given by* (4.0.1)*. Then the optimal value function $V(t,x,\mu)$ satisfies*

$$V(t,x,\mu) \leq \inf_{\gamma\in\Gamma} \overline{V}(t,x,\mu).$$

Assume that

$$\mathcal{M}(t,y,\mu) = E_{t,y,\mu}[\widetilde{U}(Y_T)]$$

is twice differentiable and strictly convex for $y > 0$ with fixed t and μ. Suppose that $y^* = \hat{y}(t,x,\mu;\gamma)$ is the solution of the equation

$$\frac{\partial \mathcal{M}}{\partial y}(t,y,\mu) + x = 0,$$

and the primal control is given by

$$\overline{\pi}^*(t,x,\mu) = \frac{\mu - r}{\sigma^2 x} y^* \mathcal{M}_{yy}(t,y^*,\mu) - \frac{\rho}{x}\mathcal{M}_{y\mu}(t,y^*,\mu). \tag{4.0.2}$$

Assume that $\overline{X}$ is the unique solution of SDE

$$dX_t = X_t\,[\,(r + \pi_t(\mu_t - r))dt + \pi_t \sigma dW_t^s]$$

with the primal control $\pi_t = \overline{\pi}(t,\overline{X}_t,\mu_t)$ for $t \in [0,T]$.

Theorem 4.0.2. *Define the lower bound*

$$\underline{V}(t,x,\mu) = E_{t,x,\mu}[U(\overline{X}_T)]$$

for the fixed primal control $\pi_t = \overline{\pi}(t,\overline{X}_t,\mu_t)$. Then the optimal value function satisfies

$$V(t,x,\mu) \geq \sup_{\gamma\in\Gamma} \underline{V}(t,x,\mu).$$

4.1 Monte Carlo lower and upper bounds

Let $\Gamma \subset \Gamma_1$, then

$$V(t,x,\mu) \le \inf_{\gamma\in\Gamma_1} \overline{V}(t,x,\mu) \le \inf_{\gamma\in\Gamma} \overline{V}(t,x,\mu).$$

Hence, Using Γ_1 instead of Γ gives tighter upper and lower bounds. For numerical tests in section 3, we apply the idea in [7] and refer the equation (3.2.5) to choose the set Γ to contain dual control: $m(t)$, $m(t)\mu_t$ and $m(t)+\eta(t)\mu_t$ where m and η are piecewise constant functions

$$m(t) = \sum_{i=1}^{n} m_i 1_{(t_{i-1},t_i]}(t),$$
$$\eta(t) = \sum_{i=1}^{n} \eta_i 1_{(t_{i-1},t_i]}(t),$$

with $0 = t_0 < t_1 < t_2 < ... < t_{n-1} < t_n = T$ for $n \ge 1$, m_i and η_i for $i = 1, ..., n$ being arbitrary constants. Then assume that the dual function $\widetilde{U}$ has the following form

$$\widetilde{U}(y) = \sum_{j=1}^{N} \widetilde{U}_j(y) = \sum_{j=1}^{N} \left(-\frac{1}{q_j} y^{q_j}\right) \tag{4.1.1}$$

for $q_j < 0$ and $j = 1, ..., N$, then

$$\mathcal{M}(t,y,\mu) = \sum_{j=1}^{N} \widetilde{U}_j(y) F_j(t,\mu) \tag{4.1.2}$$

By equation (4.2.3), the upper bound $\overline{V}$ is given by

$$\overline{V}(t,x,\mu) = \sum_{j=1}^{N} \widetilde{U}_j(y^*) F_j(t,\mu) + xy^*. \tag{4.1.3}$$

where $y^* = \hat{y}(t,x,\mu)$ is the unique solution of equation

$$\sum_{j=1}^{N} y^{q_j-1} F_j(t,\mu) = x.$$

By equation (4.0.2), the primal control $\overline{\pi}^*$ is given by

$$x\overline{\pi}^*(t,x,\mu) = \sum_{j=1}^{N} \left[\frac{\mu-r}{\sigma^2}(1-q_j)F_j(t,\mu) + \rho\frac{\partial}{\partial\mu}F_j(t,\mu)\right](y^*)^{q_j-1} \tag{4.1.4}$$

For fixed dual control γ_t, one can compute $F_j(t,\mu)$ using the Monte Carlo simulation and approximate $\frac{\partial}{\partial\mu}F_j(t,\mu)$ using finite difference method

$$\frac{\partial}{\partial\mu}F_j(t,\mu) = \frac{F_j(t,\mu+h) - F_j(t,\mu-h)}{2h}$$

for small h. If $N = 1$, there is a closed-form solution for y^*. When $N \ge 2$, the Newton Raphson method can be used to find y^*.
As a result, we can use the Monte-Carlo methods to find the tight upper and lower bounds. We describe the process of the Monte Carlo simulation for computing tight upper and lower bounds at time $t = 0$.
Step 1: Denote by Γ the set of vectors $\mathbf{M} := (m_1, m_2, ..., m_n)$ and $\eta := (\eta_1, \eta_2, ..., \eta_n)$ which form the coefficients of the function m and η. Fix the vector $\mathbf{M}, \eta \in \Gamma$ and a form of dual control γ_t.
Step 2: Generate M sample paths of W^s and W^μ

$$W^\mu = \sqrt{h}Z_1, \quad W^s = \rho\sqrt{h}Z_1 + \sqrt{1-\rho^2}\sqrt{h}Z_2.$$

Discretize the SDE of the processes Y_t (3.2.2)

$$Y_{t+h} = Y_t \exp\left[(-r - \frac{1}{2}(1-\rho^2)\gamma_t^2 - \frac{(\mu_t - r)^2}{2\sigma^2})h - (\frac{\mu_t - r}{\sigma} + \rho\gamma_t)(\rho\sqrt{h}Z_1 + \sqrt{1-\rho^2}\sqrt{h}Z_2) + \gamma_t\sqrt{h}Z_1\right]$$

where Z_1 and Z_2 are chosed from the standard normal distribution $N(0,1)$. Compute Y_T with the initial value $Y_0 = y$. Compute the average derivative

$$\frac{\partial M(0,y,\mu)}{\partial y} \approx \frac{1}{y}\frac{1}{M}\sum_{l=1}^{M} Y_T \widetilde{U}'(Y_T).$$

Step 3: Solve equation $M_y(t,y,\mu) + x = 0$ by the bisection method and then find the solution $y \approx y^*$.
Step 4: Compute the upper bound

$$\overline{V}(0,x,\mu) \approx M(0,y^*,\mu) + xy^* = \sum_{j=1}^{N} \widetilde{U}_j(y^*)F_j(0,\mu) + xy^*.$$

Step 5: Generate the drift μ_t by the Euler method

$$\mu_{t+h} = \mu_t + \kappa(\theta - \mu_t)h + \sigma Z_1\sqrt{h}$$

and compute the control process $\overline{\pi}$ in (4.0.2). Generate the wealth process $\overline{X}$ by the Euler method

$$\overline{X}_{t+h} = \overline{X}_t + \overline{X}_t(r + \overline{\pi}_t(\mu_t - r))h + \overline{\pi}_t\overline{X}_t\sigma(\rho\sqrt{h}Z_1 + \sqrt{1-\rho^2}\sqrt{h}Z_2)$$

where the initial value $\overline{X}_0 = x$, Z_1 and Z_2 are chosed from the standard normal distribution $N(0,1)$. Since wealth process $\overline{X}_t$ is driven by $\overline{\pi}_t$, it is possible for an investor to lose all his money before maturity T. Therefore, if $\overline{X}_t \le 0$, we set $\overline{X}_T = 0$ for this path.
Step 6: Compute the lower bound

$$\underline{V}(t,x,\mu) \approx \frac{1}{M}\sum_{l=1}^{M} U(\overline{X}_T).$$

Step 7: Repeat Steps 1 to 6 N_1 times with different $\mathbf{M}, \eta \in \Gamma$ to derive the tight upper bound $\inf_{\mathbf{M},\eta\in\Gamma} \overline{V}(0,x,\mu)$ and the tight lower bound $\sup_{\mathbf{M},\eta\in\Gamma} \underline{V}(0,x,\mu)$.

4.2 Closed-form upper bounds for power utility and non-Hara utility

For some special utility functions, it is easy to find the closed-form upper bounds. The dual process Y satisfies a linear SDE(3.2.2) and the dual value function $\widetilde{U}$ is decreasing and convex. Therefore, $\mathcal{M}(t,y,\mu)$ is decreasing and strictly convex for $y > 0$ with fixed t and μ. By Feynman-Kac theorem, $\mathcal{M}$ satisfies the following linear PDE

$$\frac{\partial \mathcal{M}}{\partial t} - ry\mathcal{M}_y + \kappa(\theta-\mu)\mathcal{M}_\mu + \frac{1}{2}y^2[\frac{(\mu-r)^2}{\sigma^2} + \gamma^2(1-\rho^2)]\mathcal{M}_{yy} + \frac{1}{2}\sigma^2\mathcal{M}_{\mu\mu} + y[\sigma\gamma(1-\rho^2) - \rho(\mu-r)]\mathcal{M}_{y\mu} = 0 \tag{4.2.1}$$

with the terminal condition $\mathcal{M}(T,y,\mu) = \widetilde{U}(y)$. If the dual control $\gamma_t = m(t) + \eta(t)\mu_t$ where m and η are piecewise constant functions

$$m(t) = \sum_{i=1}^{n} m_i 1_{(t_{i-1},t_i]}(t),$$
$$\eta(t) = \sum_{i=1}^{n} \eta_i 1_{(t_{i-1},t_i]}(t),$$

with $0 = t_0 < t_1 < t_2 < ... < t_{n-1} < t_n = T$ for $n \geq 1$ and m_i and η_i for $i = 1, ..., n$ being arbitrary constants. And the dual function $\widetilde{U}$ has the following form

$$\widetilde{U}(y) = \sum_{j=1}^{N} \widetilde{U}_j(y) = \sum_{j=1}^{N} \left(-\frac{1}{q_j} y^{q_j} \right) \tag{4.2.2}$$

for $q_j < 0$ and $j = 1, ..., N$. The solution of PDE (4.2.1) is given by

$$\mathcal{M}(t, y, \mu) = \sum_{j=1}^{N} \widetilde{U}_j(y) F_j(t, \mu) = \sum_{j=1}^{N} \widetilde{U}_j(y) \exp(\widetilde{A}_j(t) + \widetilde{B}_j(t)\mu + \widetilde{C}_j(t)\mu^2), \tag{4.2.3}$$

where $\widetilde{A}_j$, $\widetilde{B}_j$ and $\widetilde{C}_j$ satisfies the following ODEs

$$\begin{aligned}
\widetilde{A}'_j(t) =& -\left(\kappa\theta + q_j\rho r + \sigma(1-\rho^2)q_j m(t)\right)\widetilde{B}_j(t) - \frac{1}{2}\sigma^2 \widetilde{B}_j^2(t) - \sigma^2 \widetilde{C}_j(t) \\
& + r q_j - \frac{q_j(q_j-1)}{2}\left(\frac{r^2}{\sigma^2} + m^2(t)(1-\rho^2)\right) \\
\widetilde{B}'_j(t) =& (\kappa + q_j\rho - \sigma q_j(1-\rho^2)\eta(t))\widetilde{B}_j(t) - 2\sigma^2 \widetilde{B}_j(t)\widetilde{C}_j(t) - 2(\kappa\theta + q_j\rho r + \sigma(1-\rho^2)q_j\eta(t))\widetilde{C}_j(t) \\
& + \frac{q_j(q_j-1)r}{\sigma^2} - q_j(q_j-1)(1-\rho^2)m(t)\eta(t) \\
\widetilde{C}'_j(t) =& -2\sigma^2 \widetilde{C}_j^2(t) + 2(\kappa + q_j\rho - \sigma q(1-\rho^2)\eta(t))\widetilde{C}_j(t) - \frac{q_j(q_j-1)}{2}\left(\frac{1}{\sigma^2} + (1-\rho^2)\eta^2(t)\right)
\end{aligned}$$

with the terminal conditions $A_j(T) = 0$, $B_j(T) = 0$ and $C_j(T) = 0$. Moreover, $\widetilde{C}_j(t)$ is given by

$$\widetilde{C}_j(t) = \sum_{i=1}^{n} \widetilde{C}_{ij}(t) 1_{(t_{i-1}, t_i]}(t),$$

The $\widetilde{C}_{ij}$ are computed recursively as follows:
for $i = n$,

$$\widetilde{C}'_{nj}(t) = a_j \widetilde{C}_{nj}^2(t) + b_j(t_n)\widetilde{C}_{nj}(t) + d_j(t_n), \quad t \in [t_{n-1}, t_n]$$

with terminal condition $\widetilde{C}_{nj}(t_n) = 0$.
For $i = n-1, ..., 1$,

$$\widetilde{C}'_{ij}(t) = a_j \widetilde{C}_{ij}^2(t) + b_j(t_i)\widetilde{C}_{ij}(t) + d_j(t_i), \quad t \in [t_{i-1}, t_i]$$

with the terminal condition $\widetilde{C}_{ij}(t_i) = \widetilde{C}_{i+1,j}(t_i)$ where

$$a_j = -2\sigma^2, \quad b_j(t) = 2(\kappa + q_j\rho - \sigma q(1-\rho^2)\eta(t)), \quad d_j(t) = -\frac{q_j(q_j-1)}{2}\left(\frac{1}{\sigma^2} + (1-\rho^2)\eta^2(t)\right).$$

The closed-form solutions of $\widetilde{C}_{ij}(t)$ are given by Appendix A. Equations for $\widetilde{A}_{ij}(t)$ and $\widetilde{B}_{ij}(t)$ are too complicated to find closed-form solutions, even though they are first order linear ODEs. Hence we need to use numerical method to solve which is given by section 3.3. By equation (4.2.3), the upper bound $\overline{V}$ is given by

$$\overline{V}(t, x, \mu) = \sum_{j=1}^{N} \widetilde{U}_j(y^*) F_j(t, \mu) + xy^*. \tag{4.2.4}$$

By equation (4.0.2), the primal control $\overline{\pi}^*$ is given by

$$x\overline{\pi}^*(t, x, \mu) = \sum_{j=1}^{N} \left[\frac{\mu - r}{\sigma^2}(1-q) + \rho(\widetilde{B}_j(t) + 2\widetilde{C}_j(t)\mu) \right] (y^*)^{q_j - 1} F_j(t, \mu) \tag{4.2.5}$$

where $y^* = \hat{y}(t, x, \mu)$ is the unique solution of equation $\sum_{j=1}^{N} y^{q_j - 1} F_j(t, \mu) = x$ and $F_j(t, \mu) = \exp(\widetilde{A}_j(t) + \widetilde{B}_j(t)\mu + \widetilde{C}_j(t)\mu^2)$. We can find an exact solution of the linear PDE (4.2.1). Hence, there is no need to use Monte Carlo method to compute the upper bound. The closed-form solution makes the computation of the upper bound fast, such that power utility and non-Hara utility.

4.2.1 Power utility

Let $\gamma_t = m(t) + \eta(t)\mu_t$. This is a special case of equation (4.2.2) with $N = 1$ and $q_1 = q$. The dual value function is given by equation (4.2.3). Therefore by equation (3.2.10) and (4.0.1), the upper bound $\overline{V}$ is given by

$$\overline{V}(t, x, \mu) = \widetilde{V}(t, \hat{y}, \mu) + x\hat{y} = \frac{x^p}{p}\exp[(1-p)(\widetilde{A}(t) + \widetilde{B}(t)\mu + \widetilde{C}(t)\mu^2)].$$

The primal control $\overline{\pi}$ is given by

$$\overline{\pi}(t, \mu) = \frac{\mu - r}{\sigma^2}(1 - q) + \rho(\widetilde{B}(t) + 2\widetilde{C}(t)\mu)$$

Furthermore, we can use Monte Carlo method to generate sample paths of the wealth process $\overline{X}$ in (3.0.3) to compute the lower bound

$$\underline{V}(t, x, \mu) \approx \frac{1}{M}\sum_{l=1}^{M} U(\overline{X}_T).$$

4.2.2 Non-Hara utility

The dual function of non-HARA utility is given by

$$\widetilde{U}(y) = -\frac{1}{q_1}y^{q_1} - \frac{1}{q_2}y^{q_2}$$

where $q_1 = -1$ and $q_2 = -3$. Let $\gamma_t = m(t) + \eta(t)\mu_t$. Hence, this is a special case of (4.2.2) with $N = 2$ and $q_1 = -1$ and $q_2 = -3$. The dual value function is given by equation (4.2.3), the upper bound is given by equation (4.2.4) and the primal control is given by equation (4.2.5). Since y^* is the unique solution of equation

$$y^{-2}F_1(t, \mu) + y^{-4}F_2(t, \mu) = x,$$

y^* can be written as

$$y^* = \left(\frac{F_1(t, \mu) + \sqrt{F_1(t, \mu)^2 + 4xF_2(t, \mu)}}{2x}\right)^{\frac{1}{2}},$$

Furthermore, we can use Monte Carlo simulation to generate sample paths of the wealth process $\overline{X}$ in (3.0.3) to compute the lower bound.

4.3 Numerical tests

In the section, we solve the optimal control problem with power and non-Hara utilities by using the dual control Monte Carlo method. We use the closed-form solution to compute upper bounds for power and non-Hara utilities when $\gamma = m(t) + \eta(t)\mu_t$ and $\gamma_t = m(t)\mu_t$, and use the Monte Carlo method to compute the lower bounds and the upper bound for $m(t)$. Let path number 100,000 and time step 100 for discretizing SDEs.

4.3.1 Power utility

Our goal is solving the optimal control problem by lower and upper bounds method with power utility function. Let parameters

$$r = 0.05, \quad \sigma = 0.2 \quad \rho = -0.5, \quad \kappa = 3, \quad \theta = 0.05, \quad T = 1, \quad p = 0.5, \quad x_0 = 1, \quad \mu_0 = 0.5. \tag{4.3.1}$$

We compare performances of three different choices of dual control γ_t with $m(t)$ and $\eta(t)$ both being constants(n=1) which are independent uniform variables in $[-0.5, 0.5]$. We repeat the Monte-Carlo method step 1 to step 6 1, 5, 10, 20, 50, 100 and 200 times and then compute the tight upper bound $\inf_{\mathbf{M},\eta\in\Gamma} \overline{V}(0, x, \mu)$ and lower bound $\sup_{\mathbf{M},\eta\in\Gamma} \underline{V}(0, x, \mu)$.

By equations (3.1.11) and (3.1.12), We can compute the closed-form solution of $C(t)$ and approximate $A(t)$ and $B(t)$ once $C(t)$ is known. Using Python, we can obtain

$$A(0) = 0.092030, \qquad B(0) = -0.183145, \qquad C(0) = 1.831404.$$

Then the benchmark value is the primal value explicitly given by

$$V = \frac{x_0^p}{p}\exp((A(0) + B(0)\mu_0 + C(0)\mu_0^2)) = 3.162794$$

The numerical results are listed in Table 4.1.

Times	Benchmark	UB	LB	abs-diff	rel-diff
1	3.162794	3.163576	3.162234	0.001342	0.000425
5	3.162794	3.163576	3.162234	0.001342	0.000425
10	3.162794	3.163576	3.162234	0.001342	0.000425
20	3.162794	3.163351	3.162234	0.001117	0.000354
50	3.162794	3.163351	3.162234	0.001117	0.000354
100	3.162794	3.162842	3.162234	0.000608	0.000193
200	3.162794	3.162796	3.162234	0.000562	0.000178

Table 4.1: Upper bound (UB) lower bound (LB) for power utility($\gamma_t = m(t) + \eta(t)\mu_t$)

Times	Benchmark	UB	LB	abs-diff	rel-diff
1	3.162794	3.163754	3.162195	0.001559	0.000493
5	3.162794	3.163705	3.162195	0.001510	0.000478
10	3.162794	3.163668	3.162195	0.001473	0.000466
20	3.162794	3.163317	3.162195	0.001122	0.000355
50	3.162794	3.163256	3.162195	0.001061	0.000335
100	3.162794	3.162894	3.162195	0.000699	0.000221
200	3.162794	3.162843	3.162195	0.000648	0.000205

Table 4.2: Upper bound (UB) and lower bound (LB) for power utility($\gamma_t = m(t)\mu_t$)

Times	Benchmark	UB	LB	abs-diff	rel-diff
1	3.162794	3.163953	3.161932	0.002021	0.000639
5	3.162794	3.163729	3.161932	0.001797	0.000568
10	3.162794	3.163361	3.161932	0.001429	0.000452
20	3.162794	3.163361	3.161932	0.001429	0.000452
50	3.162794	3.163274	3.161932	0.001342	0.000424
100	3.162794	3.163029	3.161932	0.001097	0.000347
200	3.162794	3.163029	3.161932	0.001097	0.000347

Table 4.3: Upper bound (UB) and lower bound (LB) for power utility($\gamma_t = m(t)$)

The tables illustrate that the benchmark value is between the upper and lower bounds, and the absolute difference is proportional to 10^{-3} or 10^{-4} and the relative difference is proportional to 10^{-4}. As a result, the upper and lower bounds method is accurate and reliable. From tables 4.1, 4.2 and 4.3, the gap between the tight upper and lower bounds is very small. We also can obtain $\gamma_t = m(t) + \eta(t)\mu_t$ outperforms other choices of γ_t. In addition, we examine the robustness of the method for $\gamma_t = m(t) + \eta(t)\mu_t$. We choose the dual control $\gamma_t = m(t) + \eta(t)\mu_t$ with $m(t)$ and $\eta(t)$ both being constants(n=1) which are independent uniform variables in $[-0.5, 0.5]$. Then we repeat the Monte-Carlo method step 1 to step 6 1, 5, 10, 20, 50, 100 and 200 times and then compute the tight upper bound $\inf_{\mathbf{M},\eta\in\Gamma} \overline{V}(0, x, \mu)$ and lower bound $\sup_{\mathbf{M},\eta\in\Gamma} \underline{V}(0, x, \mu)$.

We randomly take 10 samples of interest rate r from the uniform distribution on $[0.01, 0.08]$, σ on $[0.1, 0.5]$, ρ on $[-1, 1]$, κ on $[1, 5]$, θ on $[0.01, 0.1]$. Let parameters $T = 1$, $x_0 = 1$ and $\mu_0 = 0.5$. We compute the mean and standard deviation of the absolute and relative difference between the upper and lower bounds and show the numerical results in the Table 4.4.

Times	mean abs-diff	std abs-diff	mean rel-diff	std rel-diff
1	0.001821	0.002639	0.00034434	0.0114325
5	0.001797	0.002568	0.00028595	0.0109843
10	0.001538	0.002452	0.00029425	0.0104357
20	0.001429	0.002241	0.00028532	0.0094641
50	0.001342	0.001924	0.00025320	0.0092543
100	0.000978	0.001847	0.00021249	0.0090568
200	0.000897	0.001832	0.00019951	0.0088093

Table 4.4: Mean and standard deviation of the absolute and relative difference between the upper and lower bounds for power utility with randomly sampled parameters-sets

The table illustrates that the algorithm is accurate and reliable. Furthermore, we compare performances of $\gamma_t = m(t) + \eta(t)\mu_t$ with $m(t)$ and $\eta(t)$ both being constants, that is $m(t) = m$ and $\eta(t) = \eta$ (the number of pieces $n1 = 1$) and being a two-piecewise constant function, that is $m(t) = m_1 1_{[0,1/2]}(t) + m_2 1_{(1/2,1]}(t)$ and $\eta(t) = \eta_1 1_{[0,1/2]}(t) + \eta_2 1_{(1/2,1]}(t)$ (the number of pieces $n1 = 2$. In the process of calculating the lower bound, we use a piece constant control

$$\overline{\pi}^*(t) = \sum_{k=1}^{n_2} \overline{\pi}(\bar{t}_k) 1_{(\bar{t}_{k-1},\bar{t}_k]}(t), \quad \bar{t}_k = k\frac{T}{n_2}, \quad n_2 = 100$$

to replace the feasible control $\overline{\pi}$. Moreover, we make the number of samples for each m_i and η_i, $i = 1, 2$, the same as that of m and η for $n_1 = 1$, so as to ensure that the piecewise functions with $n_1 = 2$ include all functions with $n_1 = 1$. The numerical results are listed in Table 4.5 and Table 4.6.

times	Benchmark	UB	LB	abs-diff	rel-diff
1	3.16279372	3.16357634	3.16223398	0.00134236	0.00042442
60	3.16279372	3.16297127	3.16223398	0.00073729	0.00023311
600	3.16279372	3.16279465	3.16223398	0.00056067	0.00017727
6000	3.16279372	3.16279399	3.16223398	0.00056001	0.00017706

Table 4.5: Upper bound(UB) and lower bound(LB) for power utility with piecewise constant control ($n_1 = 1$)

Times	Benchmark	UB	LB	abs-diff	rel-diff
1	3.16279372	3.16301876	3.16223398	0.00078478	0.00024813
60^2	3.16279372	3.16281122	3.16223398	0.00057724	0.00018251
600^2	3.16279372	3.16279404	3.16223398	0.00056006	0.00017708
6000^2	3.16279372	3.16279384	3.16223398	0.00055986	0.00017701

Table 4.6: Upper bound(UB) and lower bound(LB) for power utility with piecewise constant control ($n_1 = 2$)

It is clear that the upper and lower bounds are tight. Our numerical results confirm the upper bound is decreased as the number of samples for m and η increased, that is the performance of $n_1 = 2$ is better than that of $n_1 = 1$, even if the rate of improvement is small. One possible reason is that the bounds are already very tight. Hence, we can increase the number of samples to reduce the gap of the upper and lower bounds. However, the time of computation will exponentially increase as increasing the number of samples. One needs to strike a balance of accuracy and cost of computation. According to the numerical results in Table 4.5 and Table 4.6, $n_1 = 1$ has already given good estimation of the bounds. we will use it for non-HARA utility.

4.3.2 Non-HARA utility

In this section, we check the correctness of the upper and lower bounds when process μ_t always constant over the time t. By subsection 1.2.2, there is a classical solution to the primal value equation. Let all parameters be the same as (4.3.1). By equation (1.2.4), we can obtain

$$y^* = \left(\frac{1}{2x_0} \left(e^{(r+\theta^2)T} + \sqrt{e^{2(r+\theta^2)T} + 4x_0 e^{3(r+2\theta^2)T)}} \right) \right)^{1/2} = 1.330956.$$

By equation (1.2.5), the benchmark value at time t is given by

$$V = \frac{2}{3} \left(\frac{e^{(r+\theta^2)T}}{y^*} + 2x_0 y^* \right) = 2.307806.$$

We compute the upper and lower bounds by Monte-Carlo method. Table 4.7 lists the numerical results.

The table illustrates that the benchmark is between upper and lower bounds, and the absolute

Benchmark	UB	LB	abs-diff	rel-diff
2.307806	2.307859	2.307702	0.000157	6.80e-05

Table 4.7: Upper bound(UB) and lower bound(LB) for Non-HARA utility

difference is proportional to 10^{-4} and the relative difference is proportional to 10^{-5}. As a result, the upper and lower bounds method is accurate and reliable.

In addition, we use the upper and lower bounds method to the non-HARA utility when drift μ_t following the OU process.We compare performances of three different choices of dual control γ_t with $m(t)$ and $\eta(t)$ both being constants(n=1) which are independent uniform variables in $[-0.5, 0.5]$. We repeat the Monte-Carlo method step 1 to step 6 1, 5, 10, 20, 50, 100 and 200 times and then compute the tight upper bound $\inf_{\mathbf{M},\eta\in\Gamma} \overline{V}(0, x, \mu)$ and lower bound $\sup_{\mathbf{M},\eta\in\Gamma} \underline{V}(0, x, \mu)$. Let $\mu_0 = 0.05$, $p_1 = 0.5$ and $p_2 = 0.75$. The other parameters are the same as in 4.3.1. The numerical results are listed in Table 4.8, Table 4.9 and Table 4.10.

Num m and η	UB	LB	abs-diff	rel-diff
1	3.55639620	3.55552074	0.00087546	0.00024623
5	3.55639620	3.55552074	0.00087546	0.00024623
10	3.55639620	3.55552074	0.00087546	0.00024623
20	3.55630068	3.55552074	0.00077994	0.00021936
50	3.55630068	3.55552074	0.00077994	0.00021936
100	3.55569335	3.55552074	0.00017261	4.855e-05
200	3.55557267	3.55552074	5.193e-05	1.461e-05

Table 4.8: Upper bound (UB) lower bound (LB) and the absolute and relative difference between upper and lower bounds for Non-HARA utility($\gamma_t = m(t) + \eta(t)\mu_t$)

Num m	UB	LB	abs-diff	rel-diff
1	3.55659998	3.55520307	0.00139691	0.00039288
5	3.55659998	3.55520307	0.00139691	0.00039288
10	3.55645730	3.55520307	0.00125423	0.00035276
20	3.55631066	3.55520307	0.00110759	0.00031151
50	3.55631066	3.55520307	0.00110759	0.00031151
100	3.55613581	3.55520307	0.00093274	0.00026234
200	3.55562488	3.55520307	0.00042181	0.00011864

Table 4.9: Upper bound (UB) and lower bound (LB) and the absolute and relative difference between upper and lower bounds for Non-HARA utility($\gamma_t = m(t)\mu_t$)

Num m	UB	LB	abs-diff	rel-diff
1	3.55667566	3.55515873	0.00151693	0.00042664
5	3.55667566	3.55515873	0.00151693	0.00042664
10	3.55642792	3.55515873	0.00126919	0.00035696
20	3.55642792	3.55515873	0.00126919	0.00035696
50	3.55642792	3.55515873	0.00126919	0.00035696
100	3.55621365	3.55515873	0.00105492	0.00029670
200	3.55565726	3.55515873	0.00049853	0.00014021

Table 4.10: Upper bound (UB) and lower bound (LB) and the absolute and relative difference between upper and lower bounds for Non-HARA utility($\gamma_t = m(t)$)

The numerical results also confirm that the choice $\gamma_t = m(t) + \eta(t)\mu_t$ outperforms the other choices of γ_t. We further check and test the robustness of the dual control Monte-Carlo methods for non-HARA utility with $\gamma_t = m(t)+\eta(t)\mu_t$ with $m(t)$ and $\eta(t)$ both being constants(n=1) which are independent uniform variables in $[-0.5, 0.5]$. Then we repeat the Monte-Carlo method step 1 to step 6 1, 5, 10, 20, 50, 100 and 200 times and then compute the tight upper bound $\inf_{\mathbf{M},\eta\in\Gamma} \overline{V}(0, x, \mu)$ and lower bound $\sup_{\mathbf{M},\eta\in\Gamma} \underline{V}(0, x, \mu)$.

We randomly take 10 samples of interest rate r from the uniform distribution on $[0.01, 0.08]$, σ on $[0.1, 0.5]$, ρ on $[-1, 1]$, κ on $[1, 5]$, θ on $[0.01, 0.1]$. Let parameters $T = 1$, $x_0 = 1$ and $\mu_0 = 0.05$. We compute the mean and standard deviation of the absolute and relative difference between the upper and lower bounds for non-HARA utility and show the numerical results in the Table 4.11.

Num m and η	mean abs-diff	std abs-diff	mean rel-diff	std rel-diff
1	0.008868	0.012592	0.219486	0.343412
5	0.008747	0.011095	0.194536	0.309677
10	0.008523	0.010178	0.185413	0.287890
20	0.008311	0.009639	0.181249	0.262324
50	0.007969	0.009414	0.169873	0.249088
100	0.007243	0.009135	0.163256	0.237766
200	0.007004	0.008871	0.159987	0.211909

Table 4.11: Mean and standard deviation of the absolute and relative difference between upper and lower bounds for non-HARA utility with randomly sampled parameters-sets

It is clear that the gap between the upper bound and lower bound is very small. Therefore, the algorithm is accurate and reliable.

Conclusion

In this paper, we use Lagrange multiplier and Kuhn-Tucker condition to solve constrained quadratic minimization problem and find the primal control. We find an exact solution for power utility using both primal HJB and dual HJB when the drift μ_t following the OU process. We further plot some graphs using the primal solution and dual solution separately and confirm they essentially produce the same results. Besides, we use the weak duality relation to construct the upper bound from the dual problem and then construct a feasible control to find the lower bound under the OU process with power and non-HARA utilities and estimate the gap of UB and LB. We apply the dual control Monte-Carlo method to compute the bounds and suggest some simple forms of the dual control γ_t. For power and non-HARA utilities, if γ is taken as $\gamma_t = m(t) + \eta(t)\mu_t$ with m and η being piecewise constant functions, we can find a closed-form formula of upper bound. For power utility, we use the benchmark value to check bounds tight. Numerical tests show that the choice $\gamma_t = m(t)+\eta(t)\mu_t$ outperforms the other choices of γ_t, and the difference between the upper and lower bounds can be decreased if the number of sampling increases. Moreover, the upper and lower bounds method is accurate, reliable and robust.

Appendix A

Technical Proofs

In this paper we need to solve a number of times the following equation to obtain a closed-form solution of $C(t)$.

$$C'(t) = aC^2(t) + bC(t) + d, \quad \underline{t} \le t \le \bar{t}$$

with the terminal conditions $C(\bar{t}) = c$, where all coefficients are constants. Assume that $b^2 - 4ad > 0$ and $\frac{m_1}{m_2} \notin [\mathrm{e}^{-k_1(\bar{t}-\underline{t}}, 1]$, where

$$k_1 = \sqrt{b^2 - 4ad}, \quad m_1 = \frac{-b - k_1}{2a}, \quad m_2 = \frac{-b + k_1}{2a}.$$

Then we can obtain

$$\left(\frac{1}{D - m_1} - \frac{1}{D - m_2} \right) dD = a(m_1 - m_2)dt$$

Solving the above equation, the closed-form solution for $C(t)$ on interval $[\underline{t}, \bar{t}]$ is given by

$$C(t) = \frac{m_1 - m_2}{1 - k_2 \exp(k_1(\bar{t} - t))} + m_2, \quad k_2 = \frac{c - m_1}{c - m_2}$$

In this paper, $c = 0$, $\underline{t} = 0$ and $\bar{t} = T$.

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