Application of Stochastic Control in Optimal Execution Algorithms

by

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Declaration

The work contained in this thesis is my own work unless otherwise stated.

Signature and date:

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To my mother, Leonor.

Abstract

In this thesis I study optimal execution models under different scenarios, using Stochastic Control as the main mathematical tool. Many features of electronic markets are also discussed, such as volume, volatility, and liquidity. Regarding the execution problems, I first solve a model with terminal and running penalty, I then move to a more realistic setup, with stochastic volatility and liquidity, which is solved and calibrated with real data, I also study a double liquidation problem, this involves modelling a joint liquidation in the equity market and in the foreign exchange market. Finally, I solve the double liquidation problem assuming ambiguity aversion in the mid-price drift.

Keywords: High Frequency Trading, Round-Trip Cost, Stochastic Control, Optimal Execution Models, Double Liquidation Problem, Ambiguity Aversion.

Contents

1 Introduction

The world of finance and how to trade a security have changed dramatically in the last ten years, new developments such as machine learning and on-line platforms have revolutionised the way in which banks and brokerage houses execute orders. A leading example is algorithmic trading, this new trading technique allows machines to trade any financial security establishing pre-defined rules, optimising profits, minimising price impact, or reading an alpha-signal.

Almgren and Chriss [2] introduced the design of optimal execution problems, in recent years this approach has been developed in a more general setup. In this thesis I study various models for the optimal execution problem, using stochastic control as the primary mathematical tool. In chapter 2, I discuss how the electronic market works, market participants and some financial variables such as volume, volatility, and liquidity.

In chapter 3 and 4, I develop the theory behind of stochastic control using as motivation the optimisation portfolio problem introduced by Merton (1971) in his work [20]. I also analyse the execution models in the basic setup, with a particular emphasis in inventory penalties that can be considered in the optimisation model.

All models presented previously assume that volatility and liquidity are fixed during the execution process. In chapter 5, I discuss the liquidation problem with stochastic volatility and liquidity, this problem was introduced by Almgren in his work [1]. I also calibrate this model using data from Starbucks stock.

In chapter 6, I explain and solve double liquidation problem which involves an execution process in the equity market and simultaneously another execution process in the foreign exchange market. My contribution for this model is to introduce a closed-form solution; As far as I know, this problem had not been solved in closed-form before.

Finally, in chapter 7 I extend the double liquidation problem, making it robust to misspecification. This technique is the so called ambiguity aversion, and for this specific model, I am able to find a closed-form solution.

2 Electronic Markets

Electronic markets are defined by the U.S Security and Exchange Commission (SEC) as "professional traders acting in a proprietary capacity that generate a large number of trades on a daily basis"; these markets have some characteristics such as:

1. Use programs to execute orders in high speed.
2. Submission of numerous orders which can be cancelled after submission.
3. The market participants seek to close their positions at the end of the day.

These markets have grown dramatically in the last ten years[1], due to this fact the market micro-structure has also changed, for instance, it can be observed a higher trading volume, the bid-ask spread for large-cap stocks have been tightened, and an increment in the large-cap stocks' volatility at the end of the trading day.

2.1 Assets traded in electronic markets

According to Cartea et al. [10], shares are the most common asset traded in electronic markets, also known as equity, shares are issued by companies to raise money through an Initial Public Offer (IPO) and are listed and traded in an exchange (for example, the New York Stock Exchange (NYSE), the Nasdaq, the London Stock Exchange (LSE) and the Tokyo Stock Exchange (TSE)). The investor receives one proportion of the corporation's profits as dividend and has the right to intervene in the corporate decisions, if and only if these shares are ordinary, which are the most common in the market[2].

In electronic markets also are traded financial contracts such as commodities, currencies, real state contracts, and derivatives. Usually, these assets are found in the form of mutual funds or exchange-traded funds (ETFs). A mutual fund is an investment vehicle that tracks an index, and collects money from different investors. On the other hand, when an investor buys an ETF, she delegates also her money to a portfolio manager. However, the main differences between a mutual fund and an ETF are that an exchange-traded fund generates the same return as a specific index (e.g., S&P500) and if the investor wants to close her participation in the fund, the issuer could give to the investor a basket of securities which has had the same performance as the ETF.

2.2 Market participants in an electronic market

The understanding of an electronic market is based on analysing its participants. Every market participant should has as a central purpose to generate profits, but the way that they produce

[1] It is estimated that electronic markets exceeded 50% of total volume in U.S equity.

[2] There exists another kind of shares called preferred stock. In this case, the holder cannot be part of company's decisions and receives a pre-arrange income, but is considered as equity from the legal point of view.

them is different. The main market participants are the following:

- Corporate issuer: As I mention previously, they are corporations which raise money via IPO. The reason for this transaction depends entirely on the origin or necessities of the corporation. Another feature of this market participants is that they can increase or reduce the supply of their shares using a secondary share offering (SSO), shares buybacks or converted bonds.
- Financial management companies: They are responsible for creating funds such as mutual funds or ETFs. These market participants can be divided into:
  - long-term investors: based on "the fundamental value".
  - short-term investors: the leading example is an ETF which seeks to replicate an index.
- Fundamental traders: These kinds of traders work using sources of information to make a market decision. That is to say, if news are released then they try to understand the implications in the stock's dynamics due to these news. They use sources of information such as economic reports, political factors, and rumours, among others.
- Technical traders: The primary source of information for technical traders are stock charts and trading information, they use tools such as momentum, stock patterns, support and resistance price points, or moving average. Technical traders assume that viewing the price history can predict the next price movement.
- High-frequency Traders: These traders seek to execute big orders prioritising the speed in their trades. For example:
  - Arbitrageurs: They use some technical indicators or price inefficiencies to execute their algorithms.
  - Execution trading: It is an algorithm programmed to execute an order within a limited time horizon, and maximising profits. This algorithm splits the order into smaller pieces minimising a possible market impact.
- Market maker (liquidity provider): It is a participant willing to buy and sell assets most of the time, the most common is a brokerage house. Their main roll is to create a smooth flow of transactions, making profits on the bid-ask spread.

2.3 Trading in electronic markets

In electronic markets there are mainly two kind of orders: markets orders (MOs) and limit orders (LOs), the main difference between them is the urgency of execution on each one.

- Market order: is considered the most aggressive order, since is used when an investor wants to execute immediately her order (buy or sell) at the best price available. The investor prioritises a full execution of her order over the price that she is going to receive for her transaction.

- Limit order: is used when a trader seeks to execute an order with a specific price, and up to specific quantities of shares. This order ensures control over the price, but could be or could not be executed; also a limit order does not execute immediately, the trader has to wait until her order is matched with a new order, or is cancelled.

Another important concept is the limit order book (LOB) which displays the current liquidity available in the market at time t. To match market orders with limit orders is employed "the matching algorithm", and has two main cases to consider:

- If a market order arrives at time t will be matched with a sell limit order, but
  - If the quantity demanded is less than what is offered at the best price available, the matching algorithm selects the earliest order, and match them until the market order is complete (price-time priority).
  - If the market order demands more quantity than what is offered at the best price available, then is matched all available quantity at the best price, and also is matched the market order with the second best price, then with the third best price and so on until the market order is complete, this procedure is known as "walking the book".

Figure 1 shows a limit order book and the cumulative volume for Starbucks Corporation stock (SBUX), on Feb 28, 2018 at 15:52:03. This stock is traded on Nasdaq stock market[3].

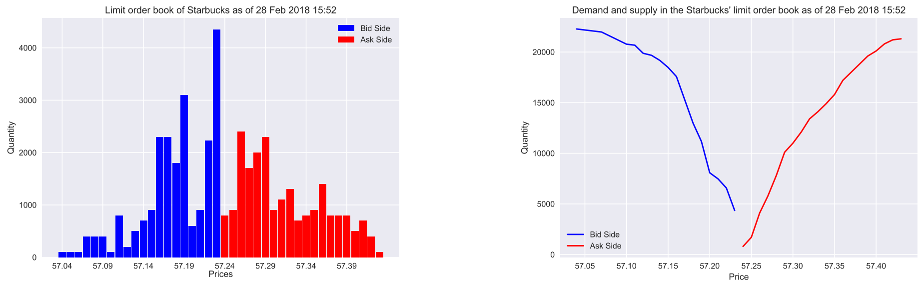

Figure 1: Limit order book and demand-supply of SBUX.

Different statistics can be extracted using a limit order book, for instance, the mid-price at time t is defined as

$$\text{Mid price}_t = \frac{1}{2}\left(P_t^{\text{best(ask)}} + P_t^{\text{best(ask)}}\right), \tag{2.1}$$

where $P_t^{\text{best(bid)}}$ is the best bid price, and $P_t^{\text{best(ask)}}$ is the best ask price in the limit order book at time t, is also known as “fair price” since this price does not include trading costs.

However, the mid-price does not consider the imbalance in the limit order book, that is to say, it is not taking into account the volume displays in both sides of the limit order book (bid side and ask side).

Bearing this in mind, we define the micro-price at time t as

$$\text{Micro Price}_t = \sum_{i=1}^{n}\left[\left(\frac{V_t^{b,i}}{V_t^{a,i}+V_t^{b,i}}\right)P_t^{b,i} + \left(\frac{V_t^{a,i}}{V_t^{a,i}+V_t^{b,i}}\right)P_t^{a,i}\right], \tag{2.2}$$

where $V_t^{b(a),i}$ is the volume in the level[4] i in the bid(ask) side and $P_t^{b(a),i}$ is the price displayed at level i in the bid(ask) side.

The micro-price can be also used as a measure of trend. For instance, if there were more buyers than sellers, the micro-price is pushed upwards meaning that is probably that the mid-price increases. Figure 2 shows the mid-price and the micro-price for SBUX, on Feb 28, 2018.

[3] American stock market, the second-largest exchange in the world.

[4] the first level in the limit order book is the best bid, the second level is the second-best bid and so on, analogously for the ask side.

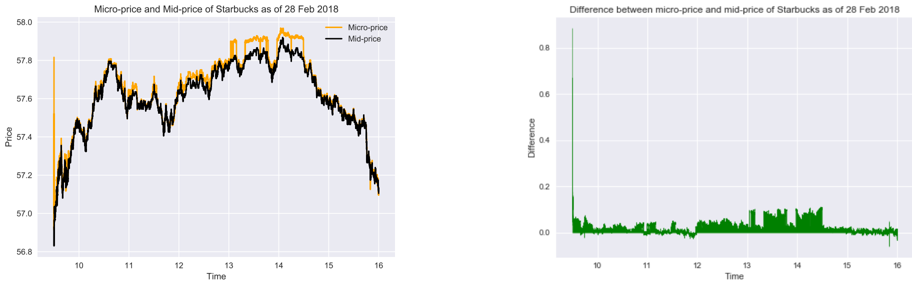

Figure 2: Micro-price and Mid-price of SBUX.

2.4 Intraday market patterns

Among the vast majority of assets, it is possible to observe some qualitative patterns in trading hours (intraday trading). These kind of structure are due to aspects such as re-balancing of portfolios, or closing positions. In this section, I discuss different market patterns for the following metrics:

- Volume.
- Volatility.
- Liquidity: bid-ask spread and round trip cost.
- Price impact.

To implement these qualitative patterns, I use SBUX data for 28th February 2018; it is essential to indicate that SBUX is the fourteenth most traded[5] stock in the Nasdaq. The data is provided by LOBSTER[6] in two files: "message.csv" and "order_book_file.csv".

Message file

Time (sec)	Event Type	Order ID	Size	Price	Direction
...	...	...	...	...	...
34713.685155243	1	206833312	100	118600	-1
34714.133632201	3	206833312	100	118600	-1
...	...	...	...	...	...

Time: Measured in seconds after midnight.

Event Type:

[5] I compute the daily trading volume during the last year.

[6] An on-line provider of limit order book data, is possible to have access to all Nasdaq traded stocks.

1 Submission of a new limit order.

2 Cancellation (partial deletion of a limit order).

3 Deletion (total deletion of a limit order).

4 Execution of a visible limit order.

5 Execution of a hidden limit order.

6 Indicates a cross trade, e.g. auction trade.

7 Trading halt indicator.

Order ID: A reference number.

Size: Quantity of shares.

Price: Dollar price multiplied by 10000.

Direction:

-1 Sell limit order

1 Buy limit order

Order book file

AskPrice 1	**AskSize 1**	**BidPrice 1**	**BidSize 1**	**...**
...	...	...	...	...
1186600	9484	118500	8800	...
1186600	9384	118500	8800	...
...	...	...	...	...

where ask (bid) price i corresponds the ask (bid) price in the level i, ask (bid) size i corresponds the ask (bid) size in the level i. During this work, I use twenty levels in each limit order book.

2.4.1 Trading volume patterns

Previous research have found an U-shape for intraday trading volume; these analysis indicate that the reasons for this shape is that at the beginning of the trading day can observed an increase in the volume for the incorporation of overnight information in trades, while at the end of the trading day some desks seek to close their open positions in assets, and also some traders re-balance their financial portfolios. Figure 3 shows a U-shape with a noticeable skewness at the end of the trading day.

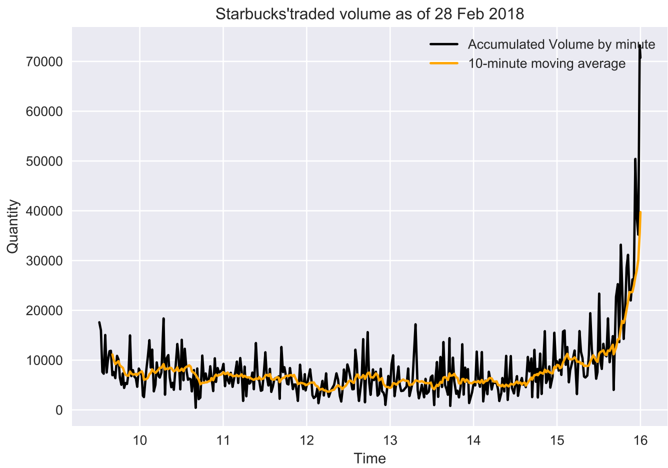

Figure 3: Intraday volume of SBUX.

I notice that the traded volume at the end of the day is considerably bigger than the rest of the trades, figure 4 shows that the last mass points of the distribution belong to the end of the trading day.

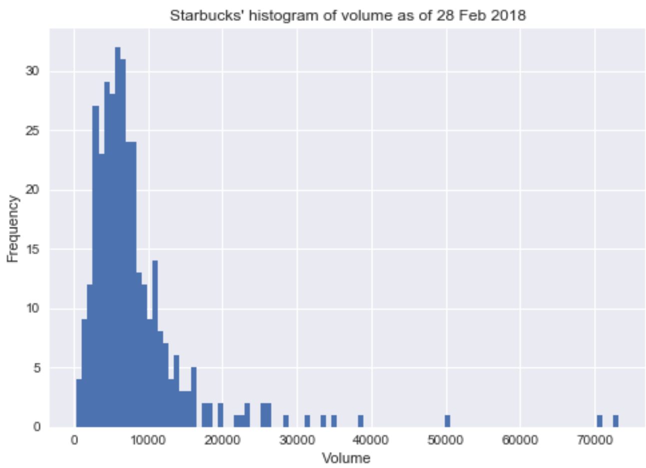

Figure 4: Histogram for traded volume of SBUX.

2.4.2 Intraday volatility pattern

Volatility measures price fluctuations, is considered as a measure of price-quality, and can be measured in different ways in the financial markets.In this thesis I employ realised volatility, which is calculated using the following equation:

$$\sigma_t = \sqrt{\frac{1}{\tau}\sum_{j=1}^{\tau} r_j^2}\,,$$

where r is the realised micro-price's return, τ is the number of different limit order books between the minute t and $t+1$. Returns are calculated using the micro-price, given that this measure is considered a good proxy of the limit order book's behaviour.

Figure 5 shows an J-shape in intraday volatility; this outline is standard among the vast majority of assets. This shape is given by the same features as trading volume pattern.

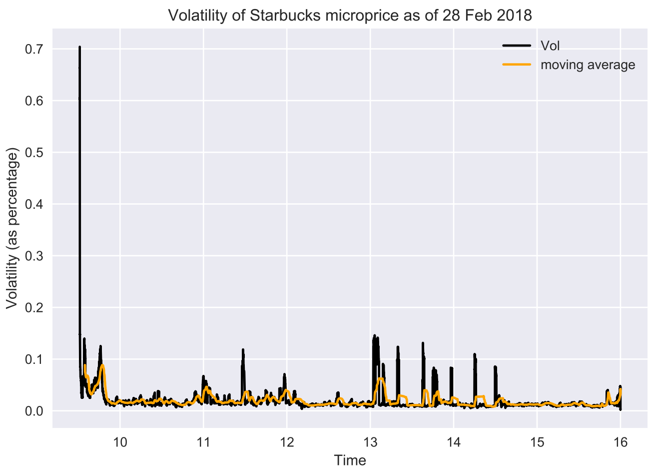

Figure 5: Intraday volatility of SBUX.

2.4.3 Liquidity: bid-ask spread and round trip cost

There are many definitions of liquidity in financial markets. A simple one is: "an asset is liquid if is easy to buy and sell it". Different theories mention that liquidity is incorporated into the market price as liquidity risk, for this reason, traders are concern about this metric; there are also a negative relationship between liquidity and market impact. Bearing this in mind, it is crucial to know how to measure liquidity for any asset.

Market participants and researches have not struck an agreement what it is the "best" or "correct" way to estimate the liquidity for any asset. The most popular among practitioners is bid-ask spread. However, this section is focused on round trip cost.

Bid-ask spread: is considered as a proxy of liquidity in the very short term since is computed as the distance between the best bid and the best ask

$$\text{bid-ask spread}_t = \left(P_t^{\text{best ask}} - P_t^{\text{best bid}}\right)\,, \tag{2.3}$$

where $P_t^{\text{best bid(ask)}}$ is the best bid(ask) price in the limit order book, and represents the potential cost of buying and selling the security immediately.

The simplicity of this measure is the main advantage; However, bid-ask spread does not consider the possible imbalance in the limit order book, also this metric has become useless in some exchanges due to the imposition of new regulations, for example in the United States, the minimum tick size[7] is one cent for stock prices at one dollar or more, inducing that the bid-ask spread will be equal or close to one cent for liquid stocks.

Figure 6 shows the classic U-shape, and is attributed to different reasons such as a positive relationship with trading volume, also because some traders adjust their bid and ask quotes to restore inventory imbalance.

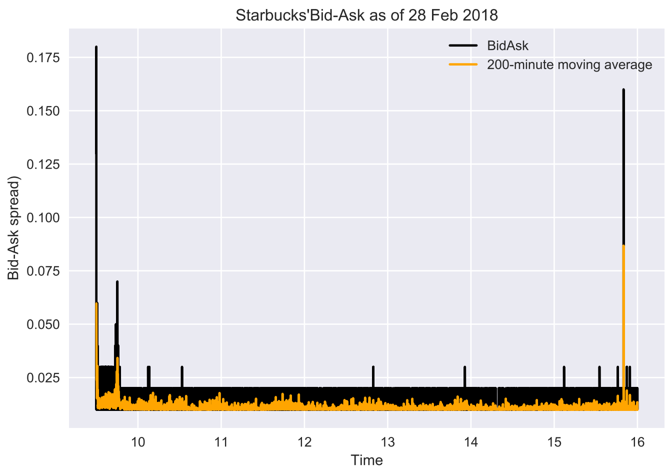

Figure 6: Bid-ask spread for SBUX.

Round trip cost: is considered as a general way to measure liquidity, since involves volume and liquidity, i.e., market depth. A stock market is deep when both sides of the limit order book have enough volume such that the market impact for a market order is negligible. It is important to indicate that high volume does not imply that the market is deep, since the price and volume posted in each level of the limit order book are considered as part of being a market depth.

Round trip cost represents a net loss of buying and immediately selling a given number of shares ("walking the book"), this metric induces a curve where x-axis represents quantity and in y-axis represents cost (see figure 7).

[7]it is the minimum difference between one price and the next one.

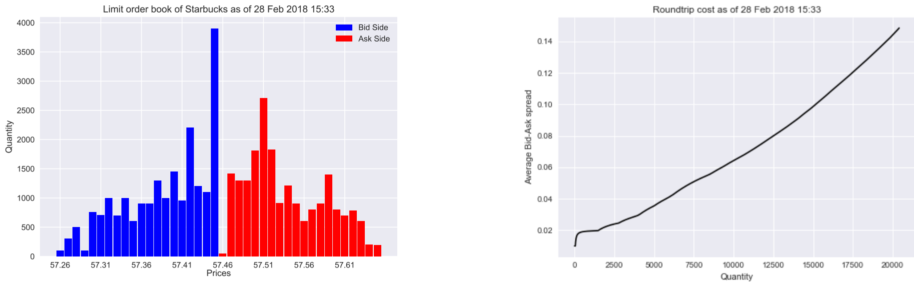

Figure 7: Limit order book and Round trip cost of SBUX.

To compare the level of market depth among different limit order books, I compute the round trip cost associated with 99-percentile volume in buying orders (see figure 8). The idea behind of choosing 99-percentile is to emphasise the concept of market depth.

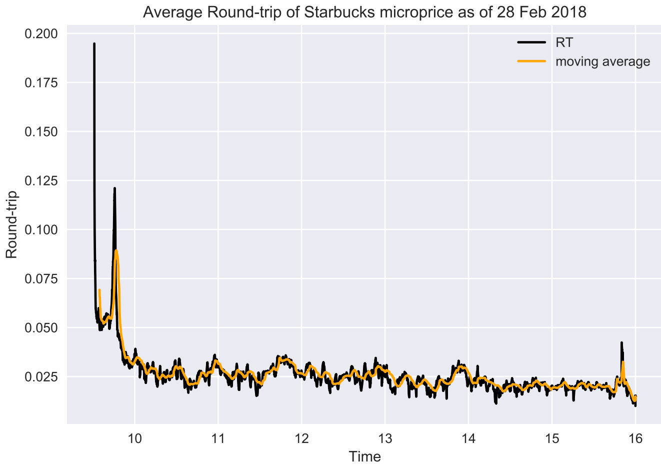

Figure 8: Round trip cost 99-percentile trading volume for SBUX.

Figure 9 shows different round trip cost curves by minute, the trading day is divided in quarters; a red curve indicates that the round trip cost curve is associated to the beginning of the quarter, and the colour converges into yellow when the limit order book is associated to close of the quarter. Between 11:30 a.m. and 15:00 p.m. the liquidity is stable, since we cannot differentiate between red curves and yellow curves.

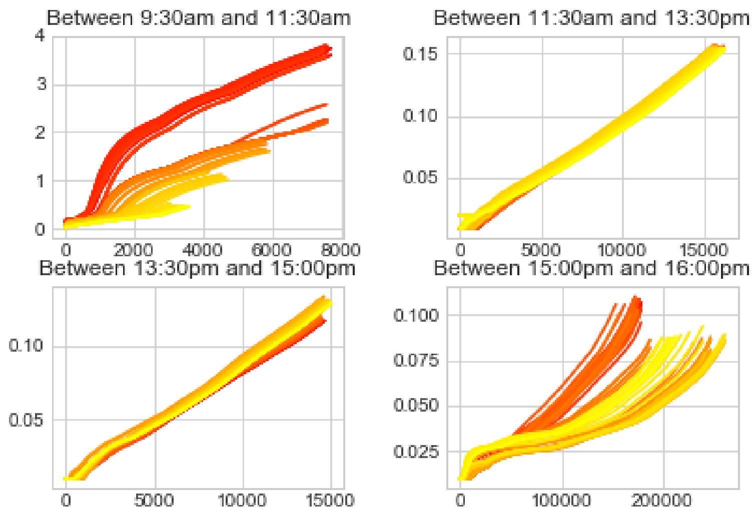

Figure 9: Round-Trip cost curve pattern for SBUX.

2.4.4 Market impact pattern

One important variable that high-frequency traders should have in mind when want to execute a large order is the potential adverse price impact. That is to say, a market order has zero price impact when is executed at best price available i.e., without walking the limit order book.

In the literature is consider two different market impacts: permanent and temporary. Both concepts are related to the price formation process, that is to say, every asset has a "fair price", so

- If the investor's view of the fair price moves in accordance with the new quoted price (from the mathematical point of view, this price impact changes the stochastic process' drift), then is a permanent price impact.
- If the impacted price is far from the investor's view of the fair price, then is a temporary price impact.

The temporary price impact is a concept which has a strong relationship with liquidity in the limit order book. For this reason, we can observe a J-shape in the temporary market impact (see figure 10).

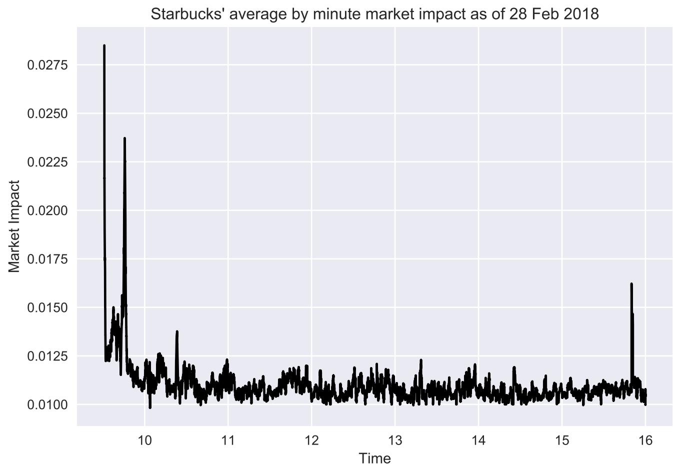

Figure 10: Temporary market impact for SBUX.

To estimate this metric is common to assume that temporary market impact is linear, then is taken a snapshot of the limit order book and perform the following linear regression:

$$S_t^{\text{exec,bid}}(Q_i) = S_t - k^{\text{bid}} Q_i + \epsilon_t^{\text{exec,bid}}(Q_i), \qquad S_t^{\text{exec,ask}}(Q_i) = S_t + k^{\text{ask}} Q_i + \epsilon_t^{\text{exec,ask}}(Q_i)\,,$$

where S_t is the mid-price at time t, Q_i is the volume at level i and $S_t^{\text{exec,bid(ask)}}(Q_i)$ is the liquidation price at level i in the bid(ask) side. The slope k is the theoretical temporary market impact. In this thesis, I estimate the temporary price impact through the slope of a lineal regression model which is calibrated using the short part of round-trip cost curve. Since the average traded volume in SBUX during 28th of February was 173 shares per trade, I consider the short term as the cost to trade one share to trade 173 shares.

2.5 Relationship between volatility and round trip cost.

The relationship between liquidity and volatility has been studied using different ways of computing the liquidity. For instance, in past studies such as [4] , indicates that if the traded volume is considered as a measure of liquidity, there exists a positive correlation between traded volume and volatility, while research such as [19], [21],[27] show that if it is considered the bid-ask spread as a measure of liquidity, this correlation is negative. However, in this thesis I analyse the correlation between volatility and liquidity, using round trip cost.

Figure 11 shows a positive correlation between volatility and round-trip cost, i.e., an increment in the volatility index means an increase in the round-trip cost. Concerning liquidity metric as round-trip cost, an increment in the volatility index implies a decrease in the market conditions (liquidity).

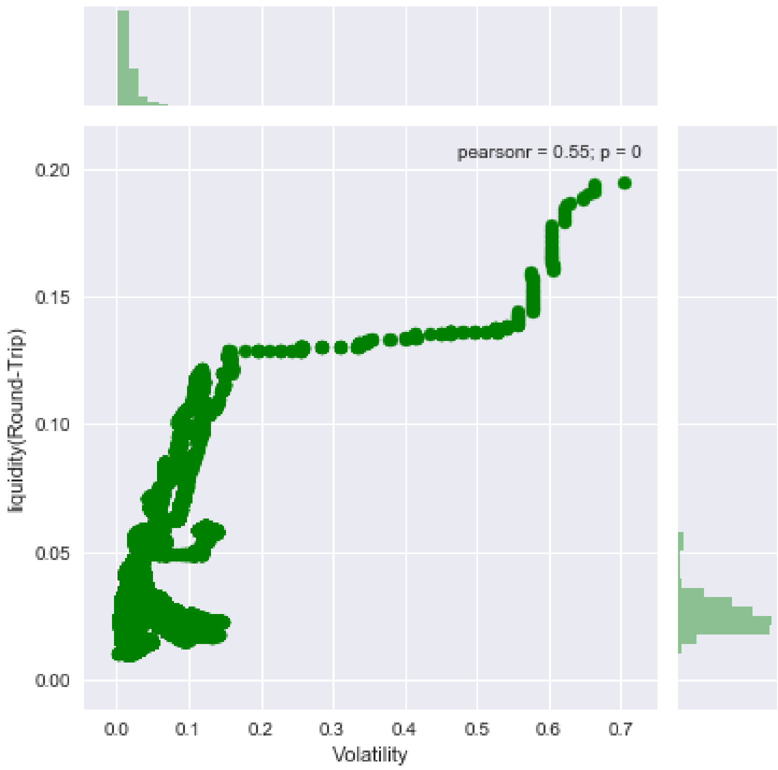

Figure 11: Volatility and Round trip cost of SBUX as of 28 Feb 2018 .

3 Stochastic Control

This sub-field of control theory studies dynamic systems subject to random perturbations, and has been applied in topics such as economics, management, and finance. Over recent years, the research over control theory has been developed, mainly by problems emerging from mathematical finance. In this section I discuss the theory behind stochastic control, using as motivation the portfolio optimisation problem, which was proposed by Merton (1971) in his work [20]. I use as main references for this section the following works: Cass [12], Cartea et al. [10], Björk [5], Merton [20],and Pham [23].

3.1 Portfolio optimisation problem

Consider an agent at time t who wishes to maximise her expected utility by allocating her wealth in a risk-free bank account or a risky asset. Let us define the following processes:

- $B = (B_t)_{0\leq t\leq T}$ is the risk-free bank account and satisfies

$$\mathrm{d}B_t = rB_t\mathrm{d}t\,.$$

- $W = (W_t)_{0\leq t\leq T}$ is a Brownian Motion.

- $S = (S_t)_{0\leq t\leq T}$ is the discounted risky price process and satisfies the following stochastic differential equation:

$$\mathrm{d}S_t = (\mu - r)S_t\mathrm{d}t + \sigma S_t\mathrm{d}W_t\,, S_0 = s.$$

- $\pi = (\pi_t)_{0\leq t\leq T}$ is a self-financing strategy, which indicates the amount of money allocated in the risky asset at time t.

- $X^\pi = (X_t^\pi)_{0\leq t\leq T}$ is the agent's discounted wealth given the strategy π, and satisfies the following stochastic differential equation:

$$\mathrm{d}X_t^\pi = (\pi_t(\mu - r) + rX_t^\pi)\mathrm{d}t + \pi_t\sigma\mathrm{d}W_t\,, X_0^\pi = x.$$

therefore the maximisation problem is formulated as follows:

$$H^{\pi,t}(s,x) = \sup_{\pi\in\mathcal{A}_{0,T}} \mathbb{E}_{s,x}\left[U(X_T^\pi)\right]\,, \tag{3.1}$$

where $U(x)$ is the agent's utility function, $\mathcal{A}_{t,T}$ is the set of all admissible strategies, corresponding to all $\mathcal{F}-$predictable self-financing strategies such that $\int_t^T \pi_s^2\mathrm{d}s < \infty$, and $\mathbb{E}_{s,x}[.]$ is the conditional expectation given $S_t = s$ and $X_t = x$.

This problem was introduced by Merton in 1971, and is a classical example of how to apply stochastic control. In the next subsections I prove and derive some mathematical results to solve problems such as (3.1).

3.2 Dynamic programming principle

We work with a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$, which satisfies the usual conditions[8], and with a controlled model $X = (X_t)_{t\geq 0}$ which is defined in $\mathbb{R}^n$ and given by

$$\mathrm{d}X_s = b(X_s, v_s)\mathrm{d}s + \sigma(X_s, v_s)\mathrm{d}W_s\,, \tag{3.2}$$

where $W = (W_t)_{t\geq 0}$ is a d-dimensional Brownian motion, and $v = (v_t)_{t\geq 0}$ is the control process, which satisfies the condition of being progressively measurable with respect to $\mathbb{F}$, and being defined in A, subset of $\mathbb{R}^n$.

The drift and dispersion processes of (3.2) are measurable functions such that

$$b : \mathbb{R}^n \times A \longmapsto \mathbb{R}^n\,,$$

$$\sigma : \mathbb{R}^n \times A \longmapsto \mathbb{R}^{n\times d}\,,$$

and satisfy the Lipschitz condition in the set A. We introduce also the notation of the set of stopping times valued in the interval $[t, T]$ as $\tau_{t,T}$. Moreover, we define $\mathcal{A}$ the set of control process v which holds

$$\mathbb{E}\left[\int_0^T (|b(0, v_t)|^2 + |\sigma(0, v_t)|^2)\mathrm{d}t\right] < \infty\,.$$

Let f and g be measurable functions such that

$$f : [0, T] \times \mathbb{R}^n \times A \longmapsto \mathbb{R}^n\,,$$

$$g : \mathbb{R}^n \longmapsto \mathbb{R}\,,$$

and we assume that g is lower-bounded or that g satisfies a quadratic growth condition, i.e. $|g(x)| \leq C(1 + |x|^2)$, for all x in $\mathbb{R}^n$, and for some constant C independent of x.

Consider for all (t, x) in $[0, T] \times \mathbb{R}^n$ a non-empty subset of controls v in $\mathcal{A}$, $\mathcal{A}(t, x)$ such that

$$\mathbb{E}\left[\int_t^T |f(s, X_s, v_s)|\mathrm{d}s\right] < \infty\,,$$

therefore the gain function is given by

$$H^v(t, x) = \mathbb{E}\left[\int_t^T f(s, X_s, v_s)\mathrm{d}s + g(X_T)\right]\,,$$

for all (t, x) in $[0, T] \times \mathbb{R}^n$ and v in $\mathcal{A}(t, x)$. The goal is to maximise the gain function over the control process, that is to say

$$H(t, x) = \sup_{v\in\mathcal{A}(t,x)} H^v(t, x)\,. \tag{3.3}$$

We say that v^* is an optimal control if $H(t, x) = H^{v^*}(t, x)$, and if the control has the form $v_s = a(s, X_s)$ for some measurable function $a : [0, T] \times \mathbb{R}^n \longmapsto A$, then is called Markovian control.

[8]This means that $\mathcal{F}$ is $\mathbb{P}$-complete, the filtrations are right-continuous, and $\mathcal{F}_0$ contains all $\mathbb{P}$-null subset of $\mathcal{F}$.

Theorem 3.1 (Dynamic programming principle). *Let us consider* (t, x) *in* $[0, T] \times \mathbb{R}^n$ *under the assumptions above mentioned, then the following equation holds*

$$\begin{aligned} H(t,x) &= \sup_{v \in \mathcal{A}(t,x)} \sup_{\theta \in \tau_{t,T}} \mathbb{E}\left[\int_t^\theta f(s, X_s^{t,x}, v_s) ds + H(\theta, X_\theta^{t,x})\right] \\ &= \sup_{v \in \mathcal{A}(t,x)} \inf_{\theta \in \tau_{t,T}} \mathbb{E}\left[\int_t^\theta f(s, X_s^{t,x}, v_s) ds + H(\theta, X_\theta^{t,x})\right]. \end{aligned}$$

Proof. First, by the Markovian property of X, we can assume that

$$X_s^{t,x} = X_s^{\theta, X_\theta^{t,x}}, \theta \leq s,$$

where $X_s^{t,x}$ denotes the process X at time s given $X_t = x$ with $t \leq s$, and θ is a stopping time defined in $[t, T]$. By the law of iterated conditional expectation and for any arbitrary control v, we obtain

$$H^v(t,x) = \mathbb{E}\left[\int_t^\theta f(s, X_s^{t,x}, v_s)\mathrm{d}s + H^v(\theta, X_\theta^{t,x})\right],$$

by construction $H^v(t,x) \leq H(t,x)$, this implies that

$$\begin{aligned} H^v(t,x) &\leq \inf_{\theta \in \tau_{t,T}} \mathbb{E}\left[\int_t^\theta f(s, X_s^{t,x}, v_s)\mathrm{d}s + H(\theta, X_\theta^{t,x})\right] \\ &\leq \sup_{v \in \mathcal{A}(t,x)} \inf_{\theta \in \tau_{t,T}} \mathbb{E}\left[\int_t^\theta f(s, X_s^{t,x}, v_s)\mathrm{d}s + H(\theta, X_\theta^{t,x})\right], \end{aligned}$$

taking supremum over all control v in the left-hand-side, we then get

$$H(t,x) \leq \sup_{v \in \mathcal{A}(t,x)} \inf_{\theta \in \tau_{t,T}} \mathbb{E}\left[\int_t^\theta f(s, X_s^{t,x}, v_s)\mathrm{d}s + H(\theta, X_\theta^{t,x})\right]. \tag{3.4}$$

Second, we fix an arbitrary control v in $\mathcal{A}(t,x)$ and a stopping time θ in $\tau_{t,T}$. By definition (3.3) and for any $\epsilon > 0$ and ω in Ω, there exist a control $v^{\epsilon,\omega}$ in $\mathcal{A}(\theta(\omega), X_{\theta(\omega)}^{t,x}(\omega))$ such that

$$H(\theta(\omega), X_{\theta(\omega)}^{t,x}(\omega)) - \epsilon \leq H^{v^{\epsilon,\omega}}(\theta(\omega), X_{\theta(\omega)}^{t,x}(\omega)). \tag{3.5}$$

Now, consider the control process

$$\hat{v}_0(\omega) = \begin{cases} v_s(\omega) & s \text{ in } [0, \theta(\omega)] \\ v_s^{\epsilon,\omega} & s \text{ in } (\theta(\omega), T] \end{cases},$$

using (3.5) and the law of iterated conditional expectation

$$\begin{aligned} H(t,x) \geqslant H^{\hat{v}}(t,x) &= \mathbb{E}\left[\int_t^\theta f(s, X_s^{t,x}, v_s)\mathrm{d}s + H^{v^{\epsilon,\omega}}(\theta, X_\theta^{t,x})\right] \\ &\geqslant \mathbb{E}\left[\int_t^\theta f(s, X_s^{t,x}, v_s)\mathrm{d}s + H(\theta, X_\theta^{t,x})\right] - \epsilon, \end{aligned}$$

due to the arbitrariness of v, θ and $\epsilon > 0$, we conclude that

$$H(t,x) \geqslant \sup_{v \in \mathcal{A}(t,x)} \sup_{\theta \in \tau_{t,T}} \mathbb{E}\left[\int_t^\theta f(s, X_s^{t,x}, v_s)\mathrm{d}s + H(\theta, X_\theta^{t,x})\right]. \tag{3.6}$$

By (3.4) and (3.6) we get the equations. □

3.3 Hamilton-Jacobi-Bellman equation

It describes the local behaviour of (3.3) when the stopping time θ tends to t, is also known as dynamic programming equation or infinitesimal version of the dynamic programming principle, and its formal derivation is the following:

Consider $\theta = t + h$, and a constant control $v = a$ then by 3.1

$$H(t,x) \geqslant \mathbb{E}\left[\int_t^{t+h} f(s, X_s^{t,x}, a)\mathrm{d}s + H(t+h, X_{t+h}^{t,x})\right], \tag{3.7}$$

by assuming that H is smooth enough such that we can apply Itô formula in the time interval $[t, t+h]$, thus

$$H(t+h, X_{t+h}^{t,x}) = H(t,x) + \int_t^{t+h} \left(\frac{\partial H}{\partial d} + \mathcal{L}^a H\right)(s, X_s^{t,x})\mathrm{d}s + \text{(local)martingale}\,,$$

where $\mathcal{L}_H^a$ is the infinitesimal operator associated to (3.2), defined by

$$\mathcal{L}^a H = b(x,a) D_x H + \frac{1}{2}\mathrm{tr}(\sigma(x,a)\sigma^T(x,a) D_{xx} H)\,, \tag{3.8}$$

then substituting (3.8) in (3.7), we obtain

$$0 \geqslant \mathbb{E}\left[\int_t^{t+h} \left(\frac{\partial H}{\partial t} + \mathcal{L}^a H\right)(s, X_s^{t,x}) + f(s, X_s^{t,x}, a)\mathrm{d}s\right], \tag{3.9}$$

divided (3.9) by h and consider the case when h goes to 0, we get

$$0 \geqslant \frac{\partial H}{\partial t}(t,x) + \mathcal{L}^a H(t,x) + f(t,x,a)\,,$$

since the last inequality is valid for any a in $\mathcal{A}$, then

$$-\frac{\partial H}{\partial t}(t,x) - \sup_{a\in\mathcal{A}} [\mathcal{L}^a H(t,x) + f(t,x,a)] \geqslant 0\,. \tag{3.10}$$

On the other hand, suppose that v^* is an optimal control, and by similar arguments we can conclude that

$$0 = -\frac{\partial H}{\partial t}(t,x) - \mathcal{L}^{v^*} H(t,x) - f(t,x,v^*)\,,$$

and

$$-\frac{\partial H}{\partial t}(t,x) - \sup_{a\in\mathcal{A}} [\mathcal{L}^a H(t,x) + f(t,x,a)] = 0\,, \text{for all } (t,x) \text{ in } (0,T] \times \mathbb{R}^n\,. \tag{3.11}$$

(3.11) is called dynamic programming equation or Hamilton-Jacobi-Bellman equation with terminal condition given by

$$H(T,x) = g(x)\,, \text{for all } x \text{ in } \mathbb{R}^n\,,$$

which results from (3.3).

3.4 Verification theorem

This theorem indicates that given a smooth solution to the Hamilton-Jacobi-Bellman equation, this candidate coincides to the solution of (3.3).

Theorem 3.2 (Verification theorem)**.** *Let w be a function in $\mathcal{C}^{1,2}([0,T]\times\mathbb{R}^n)\cap\mathcal{C}^0(([0,T]\times\mathbb{R}^n)$, and satisfies a quadratic growth condition, i.e. there exist a constant C independent of x such that*

$$|w(t,x)|\leq C(1+|x|^2)\,,\textit{for all } (t,x) \textit{ in } (0,T]\times\mathbb{R}^n\,.$$

i) Suppose that

$$-\frac{\partial w}{\partial t}(t,x)-\sup_{a\in\mathcal{A}}\left[\mathcal{L}^a w(t,x)+f(t,x,a)\right]\geq 0\,,\textit{for all } (t,x) \textit{ in } (0,T]\times\mathbb{R}^n\,,$$

$$\textit{and } w(T,x)\geq g(x)\,,\textit{for } x \textit{ in } \mathbb{R}^n\,,$$

then $w\geq H$ on $[0,T]\times\mathbb{R}^n$.

ii) Suppose that $w(T)=g$ and that exists a measurable function $\hat{v}(t,x)$ valued in A such that

$$-\frac{\partial w}{\partial t}(t,x)-\sup_{\hat{v}\in\mathcal{A}}\left[\mathcal{L}^{\hat{v}} w(t,x)+f(t,x,\hat{v})\right]=0\,,$$

If the stochastic differential equation

$$dX_s=b(X_s,\hat{v}(s,X_s))\,ds+\sigma(X_s,\hat{v}(s,X_s))\,dW_s\,,$$

has unique solution $(\hat{X}^{t,x}_s)$, and the process $\hat{v}(t,\hat{X}^{t,x}_s)$ is in $\mathcal{A}(t,x)$. We can conclude that

$$w=H\,,\textit{ on } [0,T]\times\mathbb{R}^n\,,$$

and $\hat{v}$ is an optimal Markovian control.

Proof. i) Since w in $\mathcal{C}^{1,2}([0,T]\times\mathbb{R}^n)$, for all controls v in $\mathcal{A}(t,x)$, and τ a stopping time, we can use Itô formula from t to $s\wedge\tau$, thus

$$w(s\wedge\tau,X^{t,x}_{s\wedge\tau})=w(t,x)+\int_t^{s\wedge\tau}\left(\frac{\partial w}{\partial t}(r,X^{t,x}_r)+\mathcal{L}^{u_r}w(r,X^{t,x}_r)\right)\mathrm{d}r+ \int_t^{s\wedge\tau}D_x w(r,X^{t,x}_r)^T\sigma(X^{t,x}_r,r)\mathrm{d}W_r.$$

we choose $\tau=\tau_n=\inf\{s\geq t:\int_t^s|D_x w(r,X^{t,x}_r)^T\sigma(X^{t,x}_r,r)|^2\mathrm{d}r\geq n\}$, then τ_n goes to infinity when n tends to infinity. Then the stopped process

$$\left(\int_t^{s\wedge\tau}D_x w(r,X^{t,x}_r)^T\sigma(X^{t,x}_r,r)\mathrm{d}W_r\right)_{t\leq s\leq T}\,,$$

is a martingale (See Cass [12]). Now by taking expectation, we get

$$\mathbb{E}[w(s\wedge\tau,X^{t,x}_{s\wedge\tau})]=w(t,x)+\mathbb{E}\left[\int_t^{s\wedge\tau}\left(\frac{\partial w}{\partial t}(r,X^{t,x}_r)+\mathcal{L}^{u_r}w(r,X^{t,x}_r)\right)\mathrm{d}r\right]\,,$$

using the assumptions of w, we obtain

$$\mathbb{E}[w(s\wedge\tau, X^{t,x}_{s\wedge\tau})] \leq w(t,x) + \mathbb{E}\left[\int_t^{s\wedge\tau} f(X^{t,x}_r, u_r)\mathrm{d}r\right] \text{ for all } v \text{ in } \mathcal{A}(t,x)\,,$$

therefore

$$\Big|\int_t^{s\wedge\tau} f(X^{t,x}_r, u_r)\mathrm{d}r\Big| \leq \int_t^T |f(X^{t,x}_r, u_r)|\mathrm{d}r\,,$$

since w satisfies a quadratic growth, and using dominated convergence theorem when n goes to infinity, we obtain

$$\mathbb{E}[g(X^{t,x}_T)] \leq w(t,x) + \mathbb{E}\left[\int_t^T f(X^{t,x}_r, u_r)\mathrm{d}r\right] \text{ for all } v \text{ in } \mathcal{A}(t,x)\,.$$

We conclude that $w(t,x) \leq H(t,x)$ for all (t,x) in $[0,T]\times\mathbb{R}^n$, since v is an arbitrary control in $\mathcal{A}(t,x)$.

ii) Using Itô formula in $w(r,\hat{X}^{t,x}_r)$ between t in $[0,T)$ and s in $[t,T)$, we then get

$$\mathbb{E}[w(s,\hat{X}^{t,x}_s)] = w(t,x) + \mathbb{E}\left[\int_t^s \left(\frac{\partial w}{\partial t}(r,\hat{X}^{t,x}_r) + \mathcal{L}^{\hat{v}(r,\hat{X}^{t,x}_r)} w(r,\hat{X}^{t,x}_r)\right)\mathrm{d}r\right]\,.$$

Now, by definition of the control $\hat{v}(t,x)$, we obtain

$$-\frac{\partial w}{\partial t}(t,x) - \sup_{\hat{v}\in\mathcal{A}}\left[\mathcal{L}^{\hat{v}} w(t,x) + f(t,x,\hat{v})\right] = 0\,,$$

and so

$$\mathbb{E}[w(s,\hat{X}^{t,x}_s)] = w(t,x) + \mathbb{E}\left[\int_t^s f(\hat{X}^{t,x}_r, \hat{v}(r,\hat{X}^{t,x}_r))\mathrm{d}r\right]\,,$$

if s tends to t, so

$$w(t,x) = \mathbb{E}\left[\int_t^T f(\hat{X}^{t,x}_r, \hat{v}(r,\hat{X}^{t,x}_r))\mathrm{d}r + g(\hat{X}^{t,x}_T)\right] = H^{\hat{v}}(t,x)\,,$$

that is to say, $H^{\hat{v}}(t,x) \geq H(t,x)$, and finally $w = H$ with $\hat{v}$ as an optimal Markovian control

□

The two previous theorems suggest the following strategy to solve a stochastic control problem. Let us consider a non-linear Hamilton-Jacobi-Bellman equation

$$-\frac{\partial w}{\partial t}(t,x) - \sup_{a\in\mathcal{A}}\left[\mathcal{L}^a w(t,x) + f(t,x,a)\right] = 0\,, \text{for all } (t,x) \text{ in } (0,T]\times\mathbb{R}^n\,,$$

with terminal condition $w(T,x) = g(x)$.

i) Let us fix (t,x) in $(0,T]\times\mathbb{R}^n$, and solve $\sup_{a\in\mathcal{A}}\left[\mathcal{L}^a w(t,x) + f(t,x,a)\right]$ as a maximisation problem in a.

ii) Denote $a^*(t,x)$ the value that reaches this maximum.

iii) If this non-linear Partial Differential Equation with terminal condition admits a smooth solution w, then w is the solution of the stochastic control problem (3.3), and $a^*(t,x)$ is the optimal Markovian control.

3.5 Portfolio optimisation solution

In this subsection, I incorporate the stochastic control theory discussed before to solve (3.1), then using (3.11), we get

$$0 = \left(\partial_t + rx\partial_x + \frac{1}{2}\sigma^2 s^2 \partial_{ss}\right) H + \sup_{\pi} \left[\pi((\mu - r)\partial_x + \sigma\partial_{xs})H + \frac{1}{2}\sigma^2\pi^2\partial_{xx}H\right] , \tag{3.12}$$

with terminal condition $H(T, x, s) = U(x)$. As long as $\partial_{xx}H(t, x, s) < 0$ the maximum exists and is attained at

$$\pi^* = -\frac{(\mu - r)\partial_x H + \sigma\partial_{xs}H}{\sigma^2\partial_{xx}H} , \tag{3.13}$$

therefore (3.12) in feedback form is

$$0 = \left(\partial_t + rx\partial_x + \frac{1}{2}\sigma^2 s^2 \partial_{ss}\right) H - \frac{((\mu - r)\partial_x H + \sigma\partial_{xs}H)^2}{2\sigma^2\partial_{xx}H} , \tag{3.14}$$

taking in consideration that the terminal condition ($H(T, x, s) = U(x)$), is independent of s, we propose the following ansatz $H(t, x, s) = h(t, x)$, then substituting it in (3.14), we obtain

$$0 = (\partial_t + rx\partial_x)h(t, x) - \frac{\lambda}{2\sigma} \frac{(\partial_x h(t, x))^2}{\partial_{xx}h(t, x)} ,$$

where $\lambda = \frac{\mu - r}{\sigma}$, so π^* is also reduced in

$$\pi^* = -\frac{\lambda}{\sigma} \left(\frac{\partial_x h}{\partial_{xx}h}\right) ,$$

that is to say, the form of π^* depends totally on the form of $U(x)$.

4 Optimal Executions in the Basic Model

During the last years, the interest to extend the understanding of the optimal execution models among practitioners and academics have increased considerably. The goal of trade scheduling is to buy or to sell a massive number of securities before a fixed time horizon, and maximising profits. In this chapter, I explain the theory of these models. The main references are Almgren et al. [2], Cartea et al. [7], and Graziano [14].

4.1 Basic model

Markets participants as pension funds, hedge funds, mutual funds or sovereign funds delegate their trades to a house of brokerage. These brokers find to slice the big order[9] into small ones or child orders; taking into consideration the possible price impact and the risk associated with asset's fluctuation (volatility). Therefore, the agent or broker should formulate a model to help her to decide how to split this order over the time.

To measure the performance during the execution process, are consider different benchmarks. The most common is implementation shortfall, which is defined as the difference between the execution price and the pre-trade price.

These models are front-loaded, that is to say they execute as much as possible early, and this is because the agent seeks to reduce the risk against the benchmark price (pre-trade price).

In the industry and the literature, there are others benchmarks, which involve averaging the market price during the trading interval, the most popular ones are:

- TWAP: Time-weighted average price.
- VWAP: Volume-weighted average price.

I analyse different versions of the execution model, the difference between them are the assumptions which are used.

4.1.1 Liquidation problem assuming only temporary impact

In this problem, the agent seeks to liquidate an order before T using market orders and cannot arrive at time T with a positive inventory, she starts her trading operation with M shares. The formulation problem is given by

- $v = (v_t)_{0 \le t \le T}$ is the trading rate at which the agent liquidates her stock.
- $Q^v = (Q^v_t)_{0 \le t \le T}$ is the agent's inventory, with $Q^v_0 = M$ and $Q^v_T = 0$.
- $S^v = (S^v_t)_{0 \le t \le T}$ is the mid-price process.

[9]when is higher in volume than the average traded volume or the average best bid/ask volume.

- $\hat{S}^v = (\hat{S}^v_t)_{0\leq t\leq T}$ is the execution price process.
- $X^v = (X^v_t)_{0\leq t\leq T}$ is the agent's cash process.

Remark 4.1. All processes are affected by the trading rate v, and satisfy the following equations:

- $\mathrm{d}Q^v_t = -v_t\mathrm{d}t$.
- $\mathrm{d}S^v_t = \sigma\mathrm{d}W^s_t$, where $W^s = (W^s_t)_{0\leq t\leq T}$ is a Brownian motion. We assume that the permanent price impact is zero.
- $\hat{S}^v_t = S_t - f(v_t)$, where $f : \mathbb{R}^+ \longmapsto \mathbb{R}^+$ such that $f(v_t) = kv_t$. In this model, I consider that temporary price impact is approximated by a linear model with slope k.
- $\mathrm{d}X^v_t = \hat{S}^v_t v_t\mathrm{d}t$, with $X^v_0 = 0$.

the agent wishes to maximise her the expected revenue, i.e.

$$H(t,s,q) = \sup_{v\in\mathcal{A}} \mathbb{E}_{t,s,q}\left[\int_t^T X^v_u \mathrm{d}u\right] = \sup_{v\in\mathcal{A}} \mathbb{E}_{t,s,q}\left[\int_t^T (S_u - kv_u)v_u\mathrm{d}u\right], \tag{4.1}$$

where $\mathbb{E}_{s,x}[.]$ is the conditional expectation given $S_t = S$,$Q_t = q$, $\mathcal{A}$ is the set of admissible strategies, then using (3.11) in (4.1), we obtain

$$\partial_t H + \frac{1}{2}\sigma^2\partial_{ss}H + \sup_v\{(S-kv)v - v\partial_q H\} = 0\,. \tag{4.2}$$

In this problem is not allowed to left inventory at time T, for this reason, I impose the following two terminal and initial condition: If t goes to T and q is not negative then $H(t,s,q)$ converges to $-\infty$, and if t converges to T and q is zero then $H(t,s,0)$ goes to zero.

By the first order condition applied to (4.2) then the maximum is attained at

$$v^* = \frac{1}{2k}(S - \partial_q H)\,,$$

(4.2) in the feedback form is:

$$\partial_t H + \frac{1}{2}\sigma^2\partial_{ss}H + \frac{1}{4}(S - \partial_q H)^2 = 0\,, \tag{4.3}$$

we propose the following ansatz $H(t,s,q) = qS + h(t,q)$, then (4.3) is reduced in

$$\partial_t h + \frac{1}{4k}(\partial_q h)^2 = 0\,, \tag{4.4}$$

if $h(t,q) = q^2 h_2(t)$ then (4.4) becomes into $\partial_t h_2 + \frac{1}{k}h_2^2 = 0$, therefore

$$h_2(t) = \left(\frac{1}{h_2(T)} - \frac{1}{k}(T-t)\right)^{-1},$$

to ensure that the terminal inventory is zero, we impose that if t goes to T, then $h_2(t)$ converges to $-\infty$.

On the other hand, the trading rate is also reduced to $v_t^* = -\frac{1}{k}h_2(t)Q_t^{v^*}$, and the inventory profile is given by:

$$\int_0^t \frac{\mathrm{d}Q_t^{v^*}}{Q_t^{V^*}} = \int_0^t \frac{h_2(s)}{k}\mathrm{d}s \text{ then } Q_t^{v^*} = \frac{(T-t) - \frac{k}{h_2(T)}}{T - \frac{k}{h_2(T)}} M\,,$$

To satisfy the condition $Q_T^{v^*} = 0$ and ensure that the term $h(t,q)$ is negative , I establish that if t goes to T, then $h_2(t)$ tends to $-\infty$. We conclude that

$$Q_t^{v^*} = \left(1 - \frac{t}{T}\right) M \text{ and } v_t^* = \frac{M}{T}\,, \tag{4.5}$$

that is to say, the shares must be liquidated at a constant rate, it is important to indicate that this strategy is the same as the time-weighted average price (TWAP).

4.1.2 Optimal acquisition with terminal penalty and only temporary impact

In this problem the agent seeks to acquire M shares before T using market orders, starting with zero inventory ($Q_0^v = 0$). However, the agent is allowed to arrive at time T with an inventory less than the acquisition target, i.e. $Q_T^v < M$, and in this case she must execute a buy market order for the remaining amount and will receive a penalty. The problem formulation is

- $v = (v_t)_{0\leq t\leq T}$ is the trading rate at which the agent acquires the stock S.
- $Q^v = (Q_t^v)_{0\leq t\leq T}$ is the agent's inventory, with $Q_0^v = 0$ and $Q_T^v = M$.
- $S^v = (S_t^v)_{0\leq t\leq T}$ is the mid price process.
- $\hat{S}^v = (\hat{S}_t^v)_{0\leq t\leq T}$ is the execution price process.
- $X^v = (X_t^v)_{0\leq t\leq T}$ is the agent's cash process.

Remark 4.2. All processes are affected by the trading rate v, and satisfy the following equations:

- $\mathrm{d}Q_t^v = v_t\mathrm{d}t$.
- $\mathrm{d}S_t^v = \sigma\mathrm{d}W_t^s$, where $(W_t^s)_{0\leq t\leq T}$ is a Brownian motion.
- $\hat{S}_t^v = S_t + f(v_t)$, where $f : \mathbb{R}^+ \longmapsto \mathbb{R}^+$ such that $f(v_t) = kv_t$.
- $\mathrm{d}X_t^v = \hat{S}_t^v v_t\mathrm{d}t$, with $X_0^v = 0$.

Then the expected cost is given by

$$EC^v = \mathbb{E}_{t,s,q}\left[\int_t^T \hat{S}^v v_u \mathrm{d}u + (M - Q_T^v)S_T + \alpha(M - Q_T^v)^2\right]\,, \tag{4.6}$$

where the first term is the cost associated to acquire the stock S during the execution process, the second term is the cost for the last market order execution at mid-price, while the last term is the terminal penalty with α as a penalty factor, and $\mathbb{E}_{s,x}[.]$ is the conditional expectation given $S_t = S$, $Q_t = q$.

Let us define $Y = (Y_t)_{0\le t\le T}$ such that $Y_t^v = M - Q_t^v$ then $\mathrm{d}Y_t^v = -v_t \mathrm{d}t$. then the agent wishes to minimise her cost, i.e.

$$H(t,s,q) = \inf_{v\in\mathcal{A}} \mathbb{E}_{t,s,q}\left[\int_t^T \hat{S}_u^v v_u du + Y_T^v S_T + \alpha (Y_T^v)^2\right] , \tag{4.7}$$

then using 3.11 in (4.7)

$$0 = \partial_t H + \frac{1}{2}\sigma^2 \partial_{ss} H + \inf_v \{(S + kv)v - v\partial_y H\} , \tag{4.8}$$

with terminal condition $H(T,s,y) = ys + \alpha y^2$. By the first order condition the minimum is attained at

$$v^* = \frac{1}{2K}(\partial_y H - S) ,$$

then the feedback form of (4.8) is:

$$\partial_t H + \frac{1}{2}\sigma^2 \partial_{ss} H - \frac{1}{4k}(\partial_y H - S)^2 = 0 , \tag{4.9}$$

Assuming that the solution of (4.9) has the form of $H(t,s,y) = ys + h_0(t) + h_1(t)y + h_2(t)y^2$, where $h_i(t)$ are deterministic functions, we can assume that the terminal conditions hold $h_2(T) = \alpha$ and $h_1(T) = h_0(T) = 0$; we get

$$\{\partial_t h_2 - \frac{1}{k}h_2^2\}y^2 + \{\partial_t h_1 - \frac{1}{2k}h_2 h_1\}y + \{\partial_t h_0 - \frac{1}{4k}h_1^2\} = 0 , \tag{4.10}$$

(4.10) should be valid for any y, thus each equation in the bracket should be zero, therefore $h_1(T) = 0$ and $h_2(T) = 0$ implies that $h_1(t) = 0$ and $h_2(t) = 0$. We obtain

$$H(t,S,y) = ys + h_2(t)y^2 ,$$

since $h_2(T) = \alpha$ then $h_2(t) = \left(\frac{1}{k}(T-t) + \frac{1}{\alpha}\right)^{-1}$.

Remark 4.3. If α tends to ∞ the acquisition problem converges to TWAP, then the agent avoids to arrive at time T with an inventory less than the objective acquisition volume. If α tends to 0 the optimal strategy is to purchase all shares at the end of execution problem.

Using $\mathrm{d}Y_t^{v^*} = -((T-t) + \frac{k}{\alpha})^{-1} Y_t^{v^*} \mathrm{d}t$ thus

$$Q_t^{v*} = \frac{t}{T + \frac{k}{\alpha}} M \text{ and } v_t^* = \frac{M}{T + \frac{k}{\alpha}} .$$

4.1.3 Optimal liquidation with permanent impact

In this problem, the agent uses only market orders to liquidate M shares, and we assume that the temporary and the permanent impact are different to zero. Also, the agent could arrive at time

T with a positive inventory, in this scenario should execute a market order to sell the remaining shares. Finally, I consider a new penalty named running inventory penalty, given by $\phi \int_t^T (Q_u^v)^2 \mathrm{d}u$. This penalty incorporates the urgency for executing trades, i.e., a higher value of ϕ implies a quicker liquidation process. The problem formulation is given by

- $v = (v_t)_{0 \leq t \leq T}$ is the trading rate at which the agent liquidates her stock.
- $Q^v = (Q_t^v)_{0 \leq t \leq T}$ is the agent's inventory, with $Q_0^v = M$.
- $S^v = (S_t^v)_{0 \leq t \leq T}$ is the mid price process.
- $\hat{S}^v = (\hat{S}_t^v)_{0 \leq t \leq T}$ is the execution price process.
- $X^v = (X_t^v)_{0 \leq t \leq T}$ is the agent's cash process.

Remark 4.4. All processes are affected by the trading rate v, and satisfy the following equations:

- $\mathrm{d}Q_t^v = -v_t \mathrm{d}t$.
- $\mathrm{d}S_t^v = -g(v_t)\mathrm{d}t + \sigma \mathrm{d}W_t^s$, where $W^s = (W_t^s)_{0 \leq t \leq T}$ is a Brownian motion. We assume that permanent price impact is given by $g : \mathbb{R}^+ \longmapsto \mathbb{R}^+$ such that $g(v_t) = bv_t$.
- $\hat{S}_t^v = S_t - f(v_t)$, where $f : \mathbb{R}^+ \longmapsto \mathbb{R}^+$ such that $f(v_t) = kv_t$. In this model, we assume that the temporary price impact is approximated by a linear model.
- $\mathrm{d}X_t^v = \hat{S}_t^v v_t \mathrm{d}t$, with $X_0^v = 0$.

the agent wishes to maximise her the expected revenue, i.e.

$$H(t,x,s,q) = \sup_{v \in \mathcal{A}} \mathbb{E}_{t,x,s,q} \left[X_T^v + Q_T^v(S_T^v - \alpha Q_T^v) - \phi \int_t^T (Q_u^v)^2 \mathrm{d}u \right], \tag{4.11}$$

where the first term is the terminal wealth, the second term corresponds to the terminal execution, and the last term is the inventory penalty, and $\mathbb{E}_{t,x,s,q}[.]$ is the conditional expectation given $X_t = x$,$S_t = S$,$Q_t = q$, $\mathcal{A}$ is the set of admissible strategies.

Using 3.11 in (4.7), we get

$$0 = \partial_t H + \frac{1}{2}\sigma^2 \partial_{ss} H - \phi q^2 + \sup_{v \in A} \{ v(s - kv)\partial_x H - bv\partial_s H - v\partial_q H \}, \tag{4.12}$$

with terminal condition $H(T,x,s,q) = x + sq - \alpha q^2$.
By the first order criteria the maximum is attained at

$$v^* = \frac{1}{2k} \frac{s\partial_x H - b\partial_s H - \partial_q H}{\partial_x H}, \tag{4.13}$$

(4.12) in the feedback form is

$$0 = \partial_t H + \frac{1}{2}\sigma^2 \partial_{ss} H - \phi q^2 + \frac{1}{4k} \frac{(s\partial_x H - b\partial_s H - \partial_q H)^2}{\partial_x H}. \tag{4.14}$$

Exploring the terminal condition, we propose the following ansatz:

$$H(t,x,s,q) = x + sq + h(t,s,q)\,,$$

with terminal condition $h(T,s,q) = -\alpha q^2$ then (4.14) is reduced to

$$0 = \partial_t h + \frac{1}{2}\sigma^2 \partial_{ss} h - \phi q^2 + \frac{1}{4k}(bq + b\partial_s h + \partial_q h)^2\,, \tag{4.15}$$

we observe that (4.15) is a new PDE, independent of (4.14), does not depend on s, and also the terminal condition of h does not depend on s, taking in consideration these facts we conclude that $\partial_s h(t,s,q) = 0$ and $h(t,s,q) = h(t,q)$, then (4.15) simplifies into

$$0 = \partial_t h(t,q) - \phi q^2 + \frac{1}{4k}(bq + \partial_q h(t,q))^2\,, \tag{4.16}$$

also, (4.13) is reduced in

$$v^* = \frac{-1}{2k}(\partial_q h(t,q) + bq)\,,$$

Analysing the last equation , I assume that the solution admits a separation of variables , i.e. $h(t,q) = g(t)q^2$ then (4.16) is reduces to

$$0 = \partial_t g - \phi + \frac{1}{k}(g + \frac{1}{2}b)^2 \tag{4.17}$$

with terminal condition $g(T) = -\alpha$. (4.17) is a ordinary differential equation, specifically a Riccati type. To solve it, consider $g(t) = -\frac{1}{2}b + g_1(t)$ then (4.17) is given by

$$\frac{\partial_t g_1}{k\phi - g_1^2} = \frac{1}{k}\,, \tag{4.18}$$

with terminal condition $g_1(T) = \frac{1}{2}b - \alpha$, thus the solution of (4.18) is

$$g_1(t) = \sqrt{k\phi}\frac{1 + \zeta e^{2\delta}}{1 - \zeta e^{2\delta}}\,,$$

where $\delta = \sqrt{\frac{\phi}{k}}$ and $\zeta = \frac{\alpha - 0.5b + \sqrt{k\phi}}{\alpha - 0.5b - \sqrt{k\phi}}$. Then the optimal trading rate is given by

$$v_t^* = \delta \frac{\zeta e^{\delta(T-t)} + e^{-\delta(T-t)}}{\zeta e^{\delta(T-t)} - e^{-\delta(T-t)}} Q_t^{v^*}\,,$$

Notice that the optimal trading rate is proportional in a non-linear way to the inventory level; therefore

$$Q_t^{v^*} = \frac{\zeta e^{\delta(T-t)} + e^{-\delta(T-t)}}{\zeta e^{\delta(T)} - e^{-\delta(T)}} M \text{ and } v^* = \delta \frac{\zeta e^{\delta(T-t)} + e^{-\delta(T-t)}}{\zeta e^{\delta(T)} - e^{-\delta(T)}} M\,.$$

5 Optimal Execution Strategy with Stochastic Volatility and Liquidity

All the models which I explore in the last chapter consider that volatility and liquidity are deterministic functions, and fixed during the execution time. Among large-capitalisation shares, these assumptions are a good approximation. For instance, the bid-ask spread (a measure of liquidity) has been converged to the ticket size and the volatility for these stocks have decreased.

To extend these models, Almgren in his work [1] proposed an execution model which considers these metrics as a stochastic processes. The main advantage of this model is that the agent could use it, in any stock. In this chapter, I discuss this model and calibrated it using a medium-capitalisation stock such as SBUX.

5.1 Model

Considering an acquisition problem where the agent seeks to buy M shares before time T, the formulation is the following:

- $v = (v_t)_{0\leq t\leq T}$ is the trading rate at which the agent acquires the stock.
- $Q^v = (Q^v_t)_{0\leq t\leq T}$ is the number of shares remaining to purchase, with $Q^v_0 = M$ and $Q^v_T = 0$.
- $S^v = (S^v_t)_{0\leq t\leq T}$ is the mid price process.
- $\hat{S}^v = (\hat{S}^v_t)_{0\leq t\leq T}$ is the execution price process.

Remark 5.1. All processes are affected by the trading rate v, and satisfy the following equations:

- $\mathrm{d}Q^v_t = -v_t \mathrm{d}t$.
- $\mathrm{d}S^v_t = \sigma_t \mathrm{d}W^s_t$, where $W^s = (W^s_t)_{0\leq t\leq T}$ is a Brownian motion. I assume that the permanent price impact is zero.
- $\hat{S}^v = S_t + \eta_t v_t$. In this model, I assume that the temporary price impact is given by the stochastic process η.
- $\zeta = (\zeta_t)_{0\leq t\leq T}$ is the market state process and satisfies

  $$\mathrm{d}\zeta_t = a(\zeta_t)\mathrm{d}t + b(\zeta_t)\mathrm{d}W^\zeta_t\,, \tag{5.1}$$

  where $W^\zeta = (W^\zeta_t)_{0\leq t\leq T}$ is a Brownian Motion such that $\mathrm{d}\langle W^\zeta, W^s\rangle_t = 0$, where $\mathrm{d}\langle . \rangle_t$ is defined as instantaneous quadratic variation.
- $\sigma = (\sigma_t)_{0\leq t\leq T}$ is the volatility process such that $\sigma_t = \overline{\sigma}\mathrm{e}^{\frac{-\zeta_t}{2}}$.

- $\eta = (\eta_t)_{0\le t\le T}$ is the temporary price impact process such that $\eta_t = \overline{\eta}\mathrm{e}^{\zeta_t}$.

The cost of trading using implementation shortfall as benchmark is given by

$$\begin{aligned} C &= \int_0^T \hat{S}_t v_t \mathrm{d}t - MS_0 \\ &= \int_0^T S_t v_t \mathrm{d}t + \int_0^T \eta_t v_t^2 \mathrm{d}t - MS_0\,, \end{aligned}$$

then applying integration by parts in the first term and using $v_t = -\frac{\mathrm{d}Q^v}{\mathrm{d}t}$, we get

$$C = \int_0^T \sigma_t v_t \mathrm{d}W_t^s + \int_0^T \eta_t v_t^2 \mathrm{d}t\,,$$

The agent wishes to minimise her trading cost then

$$\min_{v\in\mathcal{A}} \{\mathbb{E}[C] + \lambda Var[C]\}\,, \tag{5.2}$$

Remark 5.2. (5.2) considers also the variability of trading cost.

Using Itô's Isometry, we obtain, so

$$\mathbb{E}[C] = \mathbb{E}\left[\int_t^T \eta_s v_s^2 \mathrm{d}s\right]\,,\ \text{and } Var[C] = \mathbb{E}\left[\int_t^T \sigma_s^2 Q_s^2 \mathrm{d}s\right] + \{\text{terms arising for } \eta_s, \sigma_s, v_s\}\,,$$

assuming that the second term is negligible, and substituting in (5.2), we get

$$H(t,q,\eta,\sigma) = H(t,q,\eta,\zeta) = \min_{v\in\mathcal{A}} \mathbb{E}_{t,q,\eta,\sigma}\left[\int_t^T (\eta_s v_s^2 + \lambda\sigma_s^2 (Q_s^v)^2)\mathrm{d}s\right]\,, \tag{5.3}$$

$\mathbb{E}_{t,q,\eta,\sigma}[.]$ is the conditional expectation given $\eta_t = \eta$, $Q_t = q$, $\sigma_t = \sigma$, $\mathcal{A}$ is the set of admissible strategies, then using 3.11 in (5.3):

$$0 = H_t + \min_v\left[-vH_q + aH_\zeta + \frac{1}{2}b^2 H_{\zeta\zeta} + \eta v^2 + \lambda\sigma^2 q^2\right]\,, \tag{5.4}$$

thus the minimum is attained at

$$v^* = \frac{H_q}{2\eta}\,, \tag{5.5}$$

therefore (5.4) in feedback form is

$$-H_t = aH_\zeta + \frac{1}{2}H_{\zeta\zeta}b^2 + \lambda\sigma^2 q^2 - \frac{(H_q)^2}{4\eta}\,, \tag{5.6}$$

Now, we assume that (5.1) follows a Ornstein-Uhlenbeck process with coefficients

$$a(\zeta) = \frac{-\zeta}{\delta}, b(\zeta) = \frac{\beta}{\sqrt{\delta}}\,.$$

We propose the following ansatz for (5.6)

$$H(t,q,\zeta) = \frac{\overline{\eta}q^2}{\delta} h\left(\frac{T-t}{\delta}, \zeta\right)\,, \tag{5.7}$$

then substituting in (5.6)

$$h_t + \zeta h_\zeta = \mathrm{e}^{-\zeta}\left[\frac{\lambda\overline{\sigma}^2\delta^2}{\overline{\eta}} - h^2\right] + \frac{1}{2}\beta^2 h_{\zeta\zeta}\,. \tag{5.8}$$

Defining $\kappa = \sqrt{\frac{\lambda\overline{\sigma}^2}{\overline{\eta}}}$ and $K = \kappa\delta$, (5.8) is reduced to

$$h_t + \zeta h_\zeta = \mathrm{e}^{-\zeta}\left[K - h^2\right] + \frac{1}{2}\beta^2 h_{\zeta\zeta}\,. \tag{5.9}$$

5.2 Analysis of Partial Differential Equation

In general, to solve a partial differential equation, is necessary to understand its behaviour in some asymptotic cases and determine its final and initial conditions. In this subsection, I analyse (5.9) and also compute its initial condition.

We know that, $\eta_t = (\eta_t)_{0\le t\le T}$ is a stochastic process such that $\eta_t = \overline{\eta}\mathrm{e}^{\zeta_t}$ where $d\zeta_t = a(\zeta)dt + b(\zeta)dW_t^\zeta$; then applying Itô's formula, the process η_t satisfies

$$\mathrm{d}\eta_t = \eta_t\left[(a(\zeta)\mathrm{d}t + \frac{1}{2}b^2(\zeta))\mathrm{d}t + b(\zeta)\mathrm{d}W_t^\zeta\right],$$

then if $t \le s$, thus the expectation of η_s can be approximated by

$$\mathbb{E}[\eta_s] \approx \eta_s\left[1 + \left(a + \frac{1}{2}b^2\right)(s-t)\right],$$

that is to say, on average the value of η_s with s in $[t,T]$ is given by

$$\eta \approx \eta_s\left[1 + \frac{1}{2}\left(a + \frac{1}{2}b^2\right)(T-t)\right]. \tag{5.10}$$

Note that in the constant coefficient case (see Graziano [14]) the cost is:

$$C(t,q,\eta,\sigma) = \eta k q^2 \coth(k(T-t))\,, \tag{5.11}$$

now, if $k(T-t) << 1$ and using Taylor's expansion in (5.11), we get

$$C(t,q,\eta,\sigma) \approx \frac{\eta q^2}{T-t} + \frac{\lambda\sigma^2 q^2}{3}(T-t) + \mathcal{O}((T-t)^3)\,, \tag{5.12}$$

by (5.10), (5.12) becomes into

$$C(t,q,\eta,\sigma) \approx \frac{\overline{\eta}\mathrm{e}^{-\zeta}q^2}{T-t} + \frac{1}{2}\left(a + \frac{1}{2}b^2\right)\overline{\eta}\mathrm{e}^{\zeta}q^2 + \mathcal{O}(T-t) \quad \text{as } T-t \to 0\,, \tag{5.13}$$

therefore $h(t,\zeta)$ satisfies the following relation

$$h(\tau,\zeta) \approx \frac{\mathrm{e}^{\zeta}}{\tau} - \frac{1}{2}\left(\zeta - \frac{1}{2}\beta^2\right) + \mathcal{O}(\tau) \quad \text{as } \tau \to 0\,, \tag{5.14}$$

where ζ is fixed and $\tau = \frac{T-t}{\delta}$. (5.14) is the initial condition for the partial differential equation (5.9).

5.3 Extended model

In the last model, the main assumption in the processes $\eta = (\eta_t)_{0\leq t\leq T}$ and $\sigma = (\sigma_t)_{0\leq t\leq T}$ is that both processes are driven by only one stochastic differential equations. Almgren in [1] proposed an extended model which considers two independent stochastic differential equation for the process $\eta = (\eta_t)_{0\leq t\leq T}$ and $\sigma = (\sigma_t)_{0\leq t\leq T}$. The formulation problem is the following:

- $v = (v_t)_{0\leq t\leq T}$ is the trading rate at which the agent acquires the stock S.
- $Q^v = (Q^v_t)_{0\leq t\leq T}$ is the number of shares remaining to purchase, with $Q^v_0 = M$ and $Q^v_T = 0$.
- $S^v = (S^v_t)_{0\leq t\leq T}$ is the mid price process.
- $\hat{S}^v = (\hat{S}^v_t)_{0\leq t\leq T}$ is the execution price process.
- $\sigma_t = (\sigma_t)_{0\leq t\leq T}$ is the volatility process.
- $\eta_t = (\eta_t)_{0\leq t\leq T}$ is the temporary price impact process.

Remark 5.3. All processes are affected by the trading rate v, these processes satisfy the following equations:

- $\mathrm{d}Q^v_t = -v_t\mathrm{d}t$.
- $\mathrm{d}S^v_t = \sigma_t \mathrm{d}W^s_t$, where $(W^s_t)_{0\leq t\leq T}$ is a Brownian motion.
- $\hat{S}^v = S_t + \eta_t v_t$.
- $\sigma_t = \overline{\sigma}\exp(\zeta_t)$, where $\mathrm{d}\zeta_t = a(\zeta)\mathrm{d}t + b(\zeta)\mathrm{d}W^\zeta_t$ such that $W^\zeta = (W^\zeta_t)_{0\leq t\leq T}$ is a Brownian motion.
- $\eta_t = \overline{\eta}\exp(\psi_t)$, where $\mathrm{d}\psi_t = a(\psi)\mathrm{d}t + b(\psi)\mathrm{d}W^\psi_t$ such that $W^\psi = (W^\psi_t)_{0\leq t\leq T}$ is a Brownian motion.

The correlation structure is given by

$$\mathrm{d}\langle W^s, W^\psi\rangle_t = \mathrm{d}\langle W^s, W^\zeta\rangle_t = 0 \text{ and } \mathrm{d}\langle W^\psi, W^\zeta\rangle_t = \rho\mathrm{d}t\,.$$

The agent wishes

$$H(t,q,\eta,\sigma) = \min_{v\in\mathcal{A}} \mathbb{E}_{q,\eta,\sigma}\left[\int_t^T (\eta_s v_s^2 + \lambda\sigma_s^2(Q^v_s)^2)\mathrm{d}s\right],$$

then applying 3.11:

$$0 = H_t + \min_v\left[-vH_q + a_\zeta H_\zeta + \frac{1}{2}b_\zeta^2 H_{\zeta\zeta} + a_\psi H_\psi + \frac{1}{2}b_\psi^2 H_{\zeta\zeta} + \rho b_\psi b_\zeta H_{\psi\zeta} + \eta v^2 + \lambda\sigma^2 q^2\right], \quad (5.15)$$

then the minimum is attained at

$$v^* = \frac{H_q}{2\eta}\,,$$

(5.15) has a feedback form given by

$$-H_t = a_\zeta H_\zeta + \frac{1}{2} H_{\zeta\zeta} b_\zeta^2 + a_\psi H_\psi + \frac{1}{2} H_{\psi\psi} b_\psi^2 + \lambda\sigma^2 q^2 - \frac{(H_q)^2}{4\eta} + \rho b_\psi b_\zeta H_{\psi\zeta}\,. \tag{5.16}$$

Defining

$$a(\zeta) = \frac{-\zeta}{\delta_L}, a(\psi) = \frac{-\psi}{\delta_V}, b(\zeta) = \frac{\beta_L}{\sqrt{\delta_L}}, b(\psi) = \frac{\beta_V}{\sqrt{\delta_V}}\,,$$

and proposing the following ansatz for (5.16)

$$H(t,q,\zeta,\psi) = \frac{\overline{\eta} q^2}{\delta_L} h\left(\frac{T-t}{\delta_L}, \zeta, \psi\right)\,,$$

then (5.16) is reduced into

$$h_t + \psi h_\psi + \Gamma\zeta h_\zeta = K^2 \mathrm{e}^{2\zeta} - \mathrm{e}^{\psi} h^2 + \frac{1}{2}\beta_L^2 h_{\zeta\zeta} + \rho\sqrt{\Gamma}\beta_L\beta_V h_{\zeta\psi} + \frac{1}{2}\Gamma^2\beta_L^2 h_{\psi\psi}\,, \tag{5.17}$$

where $\Gamma = \frac{\delta_L}{\delta_V}$ and $K = \kappa\delta_L$. It is important to indicate that if $\delta_L = \delta_V$, $\mathrm{d}\langle W^\psi, W^\zeta\rangle_t = 1$, $\beta_L = \beta$ and $\beta_L + 2\beta_V = 0$ then (5.17) becomes into (5.9) in the plane $\psi + 2\zeta = 0$.

5.4 Calibration of Partial Differential Equation using Starbucks Stock

In this section, I proposed a way to estimate the parameters $\overline{\eta}, \overline{\sigma}, \zeta, \delta$ and β for the Partial Differential Equation (5.9) using the SBUX data. (λ is parameter determined by the agent, in this section I assume that is equal to one.)

Firstly, the parameters $\overline{\eta}$ and $\overline{\sigma}$ determine the magnitude of volatility ($\sigma = (\sigma_t)_{0\le t\le T}$) and liquidity ($\eta = (\eta_t)_{0\le t\le T}$) processes, while the stochastic process $\zeta = (\zeta_t)_{0\le t\le T}$ works as a link between both process σ and η, and satisfies the following stochastic differential equation

$$\mathrm{d}\zeta_t = \frac{\zeta}{\delta}\mathrm{d}t + \frac{\beta}{\sqrt{\delta}}\mathrm{d}W_t^\zeta\,,$$

where

- β is the dispersion coefficient for the process $\zeta = (\zeta_t)_{0\le t\le T}$.
- ζ is is the mean coefficient for the process $\zeta = (\zeta_t)_{0\le t\le T}$.
- δ is the relaxation time coefficient.

In section 1, I compute the intraday volatility using the micro-price and also compute the liquidity coefficient using round trip cost; in this section I use these data to estimate the parameters above mentioned.

I normalise both processes σ and η, and compute an average between these normalised processes, in order to generate a “market state” index. This new index captures the behaviour of liquidity and volatility. ζ and β are the mean and the standard deviation of this market state index, respectively.

To estimate the relaxation time coefficient (δ), Bartolozzi et al. in their work [3] propose the following stopping time:

$$\tau(t_k) = \inf\{T > 0 | \hat{\Sigma}(t_k + T) * \hat{\Sigma}(t_k) \leq 0\}\,,$$

as a proxy of relaxation time, where $\hat{\Sigma}(t)$ is a normalised volume imbalance given by

$$\hat{\Sigma}(t) = \frac{\sum_{i=1}^{N_b} V_i^b(t) - \sum_{i=1}^{N_a} V_i^a(t)}{\sum_{i=1}^{N_b} V_i^b(t) + \sum_{i=1}^{N_a} V_i^a(t)}\,,$$

where $V_i^{a,b}(t)$ is the volume posted in the level i in the ask(a) and bid(b) side at time t.

Therefore, the relaxation time is computed as

$$\delta = \frac{1}{T} \sum_{i=0}^{T} \tau(t_i)\,,$$

where t_0 and t_T correspond the beginning and the end of the trading day, respectively.

Secondly, to estimate $\overline{\eta}$ and $\overline{\sigma}$, I use the relationship between these parameters and the processes σ and η which is given by

$$\eta = \overline{\eta} e^{\zeta_t}\,, \sigma = \overline{\sigma} e^{-\frac{\zeta_t}{2}}\,,$$

and $\eta\sigma^2 = \overline{\eta\sigma}^2$.

The results of these estimations are shown in the following table

δ	ζ	β	$\overline{\eta}$	$\overline{\sigma}^2$	λ	κ
$3.25e^{-4}$	-3.2171	0.8795	0.1275	0.01813	1	$1.05e^{-4}$

Table 1: Parameters of stochastic volatility and liquidity model, using SBUX data. The time scale is normalised by 1.

Therefore the partial differential equation (5.9), is fully determined by these parameters. To determine a solution for this parametric partial differential equation, is necessary to implement a numerical method, in appendix A I discuss some alternatives which can be used.

6 Double Liquidation Problem

The models which I discuss in the previous sections do not consider the currency denomination, that is to say, the stochastic cash-flows could be denominated in dollars or pounds. However, these cash-flows, when they are denominated in foreign currency, should be exchanged into local currency. For instance, if the agent is located in the United Kingdom and she liquidates an order of Apple stock, she is going to receive dollars, then the agent would execute another order to exchange her dollars into pounds.

Taking in consideration this problem, Cartea, Jaimungal, and Sánchez in their work [25] analysed this problem in the following setup: the agent is executing an order to liquidate her stock, and at the same time she exchanges her dollars into local currency. That is to say, in this setup is defined two controls, one control which refers to stock liquidation and a second one which comes to foreign currency liquidation. This formulation problem does not have a closed-form solution.

In this work, I propose a variation in the formulation problem. I assume that the stock is liquidated using any trading schedule strategy f_t which is known at time t [10], i.e., f_t is $\mathcal{F}_t-$measurable. For instance, the stock could be liquidated using TWAP, i.e., $f_t = \frac{M}{T}$. The agent at the same time liquidates her cash flows denominated in foreign currency. This new assumption brings the possibility to obtain a closed-form solution.

6.1 Problem formulation

The agent wants to liquidate M shares denominated in foreign currency, also wishes exchange her cash flows into local currency, that is to say, the agent is posting market orders in the stock market and at the same time is posting market orders in the foreign exchange market to exchange her cash-flows. Moreover, I consider that the agent could arrive at time T with a positive stock's inventory, in this case the agent should send a last market order in the foreign exchange market with a penalty α, I also consider a running-inventory penalty ϕ.

- $v = (v_t)_{0\leq t\leq T}$ is the trading rate at which the agent liquidates her cash flows in the foreign exchange market.
- $f = (f_t)_{0\leq t\leq T}$ is the rate at which the agent liquidated her stock, and is defined before the liquidation process.
- $S^f = (S^f_t)_{0\leq t\leq T}$ is the stock price process (in the foreign currency).
- $\hat{S}^f = (\hat{S}^f_t)_{0\leq t\leq T}$ is the stock execution price process (in the foreign currency).
- $Q^f = (Q^f_t)_{0\leq t\leq T}$ is the agent's inventory in the stock, with $Q^f_0 = M$.

[10]this new assumption reduces the problem formulation into only one control

- $E^v = (E_t^v)_{0\leq t\leq T}$ is the foreign exchange rate process.
- $\hat{E}^v = (\hat{E}_t^v)_{0\leq t\leq T}$ is the execution foreign exchange rate process.
- $P^{\overline{v}} = (P_t^{\overline{v}})_{0\leq t\leq T}$ is the agent's inventory in the foreign exchange rate, where $\overline{v} = (v, f)$, and $P_0^{\overline{v}} = 0$.
- $X^v = (X_t^v)_{0\leq t\leq T}$ is the agent's cash process (in the local currency).

Remark 6.1. The processes $S^f, \hat{S}^f$ and Q^f depend on the trading rate f, which is defined previously.

Remark 6.2. The processes X^v, E^v and $\hat{E}^v$ depend on the trading rate v, while $P^{\overline{v}}$ depends on both trading rates v and f, and the processes satisfy the following equations:

- $\mathrm{d}S_t^f = -af_t\mathrm{d}t + \sigma^s\mathrm{d}W_t^s$ where $W^s = (W_t^s)_{0\leq t\leq T}$ is a Brownian Motion. The assumption in this stochastic differential equation is that the permanent price impact is given by $af(t)$ where a is a positive constant.
- $\hat{S}_t^f = S_t^f - bf_t$. In this model, I assumed that the temporary price impact is given by $bf(t)$ where b is a positive.
- $\mathrm{d}Q_t^f = -f_t\mathrm{d}t$.
- $\mathrm{d}E_t^v = E_t^v\sigma^e\mathrm{d}W_t^e$ where $W^e = (W_t^e)_{0\leq t\leq T}$ is a Brownian Motion. I assume that the permanent price impact is zero.
- $\hat{E}_t^v = E_t^v - dv_t$. I consider that the temporary price impact follows a linear model with slope positive constant d.
- $\mathrm{d}P^{\overline{v}} = -v_t\mathrm{d}t + f_t\hat{S}_t\mathrm{d}t = -v_t\mathrm{d}t + f_t(S_t - bf_t)\mathrm{d}t$ where the first term is given by the foreign currency liquidation, and the second term corresponds to stock liquidation.
- $\mathrm{d}X_t^v = \hat{E}_t^v v_t\mathrm{d}t = (E_t - dv_t)v_t\mathrm{d}t$.

The agent wishes to maximise her the expected revenue, i.e.

$$H(t,x,e,p) = \sup_{v\in\mathcal{A}} \mathbb{E}_{t,x,e,p}\left[X_T^v + P_T^v(E_T^v - \alpha P_T^v) - \phi\int_t^T (P_u^v)^2\mathrm{d}u\right] \tag{6.1}$$

where the first term is the terminal wealth, the second term corresponds to the terminal execution value with penalty coefficient α, and the last term is the running inventory penalty with coefficient ϕ ; and $\mathbb{E}_{t,x,e,p}[.]$ is the conditional expectation given that $X_t = x, P_t = p, E_t = e$ and $\mathcal{A}$ is the set of all admissible strategies.

then using 3.11 in (6.1), we get

$$0 = H_t + \sup_v \left[v(e - dv)H_x + \frac{1}{2}\sigma^2 e^2 H_{ee} + (-v + f_t(s - bf_t))H_p \right] - \phi p^2 \,, \tag{6.2}$$

with terminal condition

$$H(T, p, e) = x + p(e - \alpha p) \,, \tag{6.3}$$

then the maximum is attained at

$$v^* = \frac{eH_x - H_p}{2dH_x} \,, \tag{6.4}$$

then (6.2) in the feedback form is

$$0 = H_t - \phi p^2 + \frac{1}{2}\sigma^2 e^2 H_{ee} + f_t[s - bf_t]H_p + \frac{1}{4dH_x}(eH_x - H_p)^2 \,, \tag{6.5}$$

Exploring the terminal condition (6.3), I propose the ansatz $H(t, p, e) = x + pe + h(t, p, e)$ with terminal condition $h(T, p, e) = -\alpha p^2$, substituting it into (6.5)

$$0 = h_t - \phi p^2 + \frac{1}{2}\sigma^2 e^2 h_{ee} + f(t)[s - bf(t)](e + h_p) + \frac{1}{4d}(h_p)^2 \,, \tag{6.6}$$

We observe that (6.6) is a new PDE, independent of (6.5). I propose a new ansatz for (6.6)

$$h(t, p, e) = \gamma(t, e) + \beta(t, e)p + \Gamma(t, e)p^2 \,, \tag{6.7}$$

such that $\gamma(T, e) = \beta(T, e) = 0$ and $\Gamma(T, e) = -\alpha$. Using (6.6) and (6.7) and collecting terms associated to p and p^2

$$p^2 \left[\Gamma_t - \phi + \frac{1}{2}\sigma^2 e^2 \Gamma_{ee} + \frac{1}{d}\Gamma^2 \right] + p \left[\beta_t + \frac{1}{2}\sigma^2 e^2 \beta_{ee} + 2\Gamma f_t(s - bf_t) + \frac{\beta\Gamma}{d} \right] +$$
$$\left[\gamma_t + \frac{1}{2}\sigma^2 e^2 \gamma_{ee} + f_t(s - bf_t)(e + \beta) + \frac{1}{4d}\beta^2 \right] = 0 \,,$$

since this equation should be valid for each p, then each equation in the brackets must be zero individually, i.e.

$$\Gamma_t - \phi + \frac{1}{2}\sigma^2 e^2 \Gamma_{ee} + \frac{1}{d}\Gamma^2 = 0 \,, \text{with } \Gamma(T, e) = -\alpha \,,$$
$$\beta_t + \frac{1}{2}\sigma^2 e^2 \beta_{ee} + 2\Gamma f(t)(s - bf(t)) + \frac{\beta\Gamma}{d} = 0 \,, \text{with } \beta(T, e) = 0 \,,$$
$$\gamma_t + \frac{1}{2}\sigma^2 e^2 \gamma_{ee} + f(t)(s - bf(t))(e + \beta) + \frac{1}{4d}\beta^2 = 0 \,, \text{with } \gamma(T, e) = 0 \,.$$

Remark 6.3. The first equation is a Riccati type ordinary differential equation.

Remark 6.4. $\gamma(t, e)$ is not solved explicitly since does not appear in the optimal trading rate. Therefore the solutions for the partial differential equations are

$$\Gamma(t, e) = \sqrt{d\phi} \tanh \left(\frac{\sqrt{\phi}(t - T) - d \tanh^{-1}\left(\frac{\alpha}{\sqrt{d\phi}}\right)}{\sqrt{d}} \right) , \beta(t, e) = \int_t^T 2\mathrm{e}^{\int_t^y \Gamma(x,e)\mathrm{d}x} f_y(bf_y - s)\Gamma(y, e)\mathrm{d}y \,, \tag{6.8}$$

That is to say, the optimal trading rate v for the double liquidation problem is fully determined by (6.8).

7 Robust Double Liquidation Problem

All execution models are determined by stochastic processes which are characterised by a certain number of constants. However, misspecification in these parameters could generate that execution processes are not close to reality, that is to say, misspecification affects the agents optimal trading strategy directly. The idea of incorporating missecification on the optimisation problem is called ambiguity aversion. Cartea, Donnelly, and Jaimungal in their work [8], proposed an execution model taking in consideration the ambiguity aversion.

In this section, I discuss how Cartea, Donnelly, and Jaimungal in [8] incorporated the ambiguity aversion in the liquidation problem in a more general framework. Due to the primary purpose of this thesis, I do not explain all proofs and results. My contribution in this section is to take into account ambiguity aversion in the double liquidation problem (discussed in the last section). To the extent of my knowledge is the first time to employ ambiguity aversion into a double liquidation problem.

7.1 General model

In [8] is proposed a model where the agent could execute market and limit orders to liquidate Q shares before T, but could arrive at time T with a positive inventory, then she has to post a last market order to complete her liquidation process.

Let us consider the filtered probability space $(\Omega, \mathbb{F}, \mathcal{F} = \{\mathcal{F}_t\}_{0\leq t\leq T}, \mathbb{P})$, where $\mathcal{F}$ is the filtration generated by the mid-price processes $S = (S_t)_{0\leq t\leq T}$ and the process $P = (P_t)_{0\leq t\leq T}$, where $P_t = \int_0^t \int_0^\infty y\mu(\mathrm{d}y, \mathrm{d}s)$ and μ is a Poisson random measure, both μ and $S = (S_t)_{0\leq t\leq T}$ are defined as:

- $S = (S_t)_{0\leq t\leq T}$ is the mid-price processes, and satisfies $\mathrm{d}S_t = \alpha \mathrm{d}t + \sigma \mathrm{d}W_t$ where α, σ are positive constant and $W = (W_t)_{0\leq t\leq T}$ is $\mathbb{P}$-Brownian motion.

- Let μ be a Poisson random measure with $\mathbb{P}$-compensator $v(\mathrm{d}y, \mathrm{d}t) = \lambda F(\mathrm{d}y)\mathrm{d}t$, such that $F(\mathrm{d}y) = ke^{-ky}\mathrm{d}y$, then the number of market buy orders which have arrived up to time t is given by $M_t = \int_0^t \int_0^\infty \mu(\mathrm{d}y, \mathrm{d}s)$.

- The agent posts orders only in the sell side, let δ_t be the distance of limit order from the mid-price. The agent's limit order is only lifted by a market order that has an execution price greater than $S_t + \delta_t$, then number of executed orders is then $N_t = \int_0^t \int_{\delta_s}^\infty \mu(\mathrm{d}y, \mathrm{d}s)$.

- $q = (q_t)_{0\leq t\leq T}$ is the agent's inventory such that $q_0 = Q$ and $q_t = Q - N_t$ and satisfies $\mathrm{d}q_t = -\mathrm{d}N_t$.

- $X = (X_t)_{0\leq t\leq T}$ is the agent's wealth and satisfies $\mathrm{d}X_t = (S_t + \delta_{t^-})\mathrm{d}N_t$.

Then the agent wishes to maximise her wealth, that is to say

$$H(t,x,q,s)=\sup_{\delta_s\in\mathcal{A}}\mathbb{E}^{\mathbb{P}}_{t,x,q,s}\left[X_{\tau\wedge T}+q_{\tau\wedge T}(S_{\tau\wedge T}-l(q_{\tau\wedge T}))\right], \tag{7.1}$$

where τ is a stopping time such that $\tau=\inf\{t: q_t=0\}$, $\mathbb{E}^{\mathbb{P}}_{t,x,q,s}[.]$ is the conditional expectation given $X_{t^-}=x, q_{t^-}=q, S_{t^-}=s$, and $\mathcal{A}$ is the set of all admissible strategies. The first term in (7.1) is the agent's terminal wealth and the second one refers to a terminal liquidation penalty.

The problem formulation (7.1) is given under the reference measure $\mathbb{P}$, and the agent wants to incorporate the fact that her model could be miss-specified, then it is defined a class of new measures $\mathbb{Q}$ equivalent to $\mathbb{P}$ and is evaluated the performance of the trading strategy under this new measure $\mathbb{Q}$.

The agent would penalise a potential deviation of the measure $\mathbb{Q}$ with respect to the measure $\mathbb{P}$, that is to say the cost of rejecting the reference measure $\mathbb{P}$ in favour of the candidate measure $\mathbb{Q}$, therefore the optimisation problem (7.1) is changed to

$$H(t,x,q,s)=\sup_{\delta_s\in A}\inf_{\mathbb{Q}\in\mathcal{Q}}\mathbb{E}^{\mathbb{Q}}_{t,x,q,s}\left[X_{\tau\wedge T}+q_{\tau\wedge T}(S_{\tau\wedge T}-l(q_{\tau\wedge T}))+\mathcal{H}_{t,T}(\mathbb{Q}|\mathbb{P})\right], \tag{7.2}$$

where $\mathcal{Q}$ is the class of all equivalent measures $\mathbb{Q}$ to $\mathbb{P}$, and $\mathcal{H}$ is the relative entropy from t to T.

The link between the measure $\mathbb{P}$ and the measure $\mathbb{Q}$ is given by the Radon-Nikodym derivative

$$\frac{\mathrm{d}\mathbb{Q}^{\alpha,\lambda,k}(\eta,q)}{\mathrm{d}\mathbb{P}}=\frac{\mathrm{d}\mathbb{Q}^{\alpha}(\eta,q)}{\mathrm{d}\mathbb{P}}\frac{\mathrm{d}\mathbb{Q}^{\alpha,\lambda,k}(\eta,q)}{\mathrm{d}\mathbb{Q}^{\alpha}},$$

where

$$\frac{\mathrm{d}\mathbb{Q}^{\alpha}(\eta,q)}{\mathrm{d}\mathbb{P}}=\exp\left[-\frac{1}{2}\int_0^T\left(\frac{\alpha-\eta_t}{\sigma}\right)^2\mathrm{d}t-\int_0^T\left(\frac{\alpha-\eta_t}{\sigma}\right)\mathrm{d}t\right], \tag{7.3}$$

$$\frac{\mathrm{d}\mathbb{Q}^{\alpha,\lambda,k}(\eta,q)}{\mathrm{d}\mathbb{Q}^{\alpha}}=\exp\left[-\int_0^T\int_0^\infty(\mathrm{e}^{g_t(y)}-1)v(\mathrm{d}y,\mathrm{d}t)+\int_0^T\int_0^\infty g_t(y)\mu(\mathrm{d}y,\mathrm{d}t)\right],$$

then the agent would penalise deviations in the reference model with respect to:

- Mid-price dynamic.
- Market order intensity.
- Market order maximal distribution.

Remark 7.1. In (7.2) We compute the infimum between all measures $\mathbb{Q}$ and the reference measure $\mathbb{P}$ to consider the worst deviation between them.

Remark 7.2. The use of ambiguity aversion in the double liquidation problem, is with respect to the mid-price dynamic.

7.2 Double liquidation problem with ambiguity aversion.

In the last subsection I discuss the necessity to incorporate the fact that execution models could be miss-specified; for this reason, the double liquidation problem is extended using ambiguity aversion. The problem formulation is the same; the only change is that $E^v = (E_t^v)_{0\leq t\leq T}$ follows an arithmetic Brownian motion instead of geometric Brownian motion, i.e.

$$\mathrm{d}E_t = -c\mathrm{d}t + \sigma \mathrm{d}W_t \,,$$

this change is made to generate a closed-form solution.

The agent wishes to maximise her wealth taking into account a possible misspecification in the mid-price process' drift c, then

$$H(t,x,p,e) = \sup_{v\in\mathcal{A}} \inf_{\mathbb{Q}\in\mathcal{Q}} \mathbb{E}^{\mathbb{Q}}_{t,x,p,e}\left[X_T + P_T^v(E_T - \alpha P_T^v) + \mathcal{H}_{t,T}(\mathbb{Q}|\mathbb{P})\right] \,, \tag{7.4}$$

where $\mathbb{E}^{\mathbb{Q}}_{t,x,q,s}[.]$ is the conditional expectation given $X_t = x, P_t = p, E_t = e$, $\mathcal{A}$ is the set of all admissible strategies, and $\mathcal{Q}$ is the set of all measures $\mathbb{Q}$ equivalent to $\mathbb{P}$, defined on the filtered space $(\Omega, \mathbb{F}, \mathcal{F} = \{\mathcal{F}_t\}_{0\leq t\leq T})$ where $\mathcal{F}$ is the filtration generated by the foreign exchange rate process $E = (E_t)_{0\leq t\leq T}$ and the process $S = (S_t)_{0\leq t\leq T}$.

Let us define $\mathcal{H}$, the relative entropy between t and T, as

$$\mathcal{H}_{t,T}(\mathbb{Q}|\mathbb{P}) = \frac{1}{\varphi}\log\left(\frac{(d\mathbb{Q}/d\mathbb{P})|_T}{(d\mathbb{Q}/d\mathbb{P})|_t}\right) \,,$$

where φ is a constant and can be thought as "the ambiguity aversion parameter"; if φ tends to zero, the agent is confident about the reference model (measure $\mathbb{P}$). On the other hand, when φ tends to infinity, the agent is ambiguous about the reference model. Now, as [8] suggests, I propose the following Radon-Nikodym derivative:

$$\frac{d\mathbb{Q}(\eta)}{d\mathbb{P}} = \exp\left[-\frac{1}{2}\int_0^T \left(\frac{c-\eta_t}{\sigma}\right)^2 \mathrm{d}t - \int_0^T \left(\frac{c-\eta_t}{\sigma}\right)\mathrm{d}t\right] \,,$$

then using 3.11 in (7.4), we get

$$0 = \partial_t H + \sup_{v\in\mathcal{A}} \inf_{\mathbb{Q}\in\mathcal{Q}} \{\mathcal{L}^{\mathbb{Q}} H - \partial_t \mathbb{E}^{\mathbb{Q}}_{t,x,p,e}\left[\mathcal{H}_{t,T}(\mathbb{Q}|\mathbb{P})\right]\} \,,$$

or

$$0 = H_t + \sup_v \inf_\eta \left[v(e-dv)H_x + \eta H_e + \frac{1}{2}\sigma^2 H_{ee} + [-v + f_t(s - bf_t)]H_p + \frac{1}{2\varphi}\left(\frac{c-\eta}{\sigma}\right)^2\right] \,, \tag{7.5}$$

therefore the critical points are attained at

$$\eta^* = c - \sigma^2\varphi H_e \text{ and } v^* = \frac{aH_x - H_p}{2dH_x} \,, \tag{7.6}$$

then (7.5) in the feedback form is

$$0 = H_t + \frac{(eH_x - H_p)^2}{4dH_x} + cH_e - \frac{1}{2}\sigma^2\varphi H_e^2 + \frac{1}{2}\sigma^2\varphi H_{ee} + f_t(s - bf_t)H_p = 0 \,, \tag{7.7}$$

with terminal condition $H(T,x,p,e;\varphi) = x + p(e - \alpha p)$.

Analysing the terminal condition, I propose the ansatz $H(t,x,p,e;\varphi) = x + pe + h(t,p,e;\varphi)$ such that $h(T,p,e;\varphi) = -\alpha p^2$, then (7.7) is reduced in

$$0 = h_t + \frac{(h_p)^2}{4d} + c(p + h_e) - \frac{1}{2}\sigma^2\varphi(p + h_e)^2 + \frac{1}{2}\sigma^2 h_{ee} + (e + h_p)f(t)(s - bf(t)) . \tag{7.8}$$

Now, (7.8) is a new PDE independent of (7.7). I propose another ansatz for (7.8) given by

$$h(t,p,e;\varphi) = \gamma(t,e;\varphi) + \beta(t,e;\varphi)p + \Gamma(t,e;\varphi)p^2 ,$$

such that $\gamma(T,e;\varphi) = \beta(T,e;\varphi) = 0$ and $\Gamma(T,e;\varphi) = -\alpha$. Substituting this ansatz in (7.8) and collecting terms associated to p, p^2, p^3 and p^4.

$$\begin{aligned}
0 =& p^4\left[-\frac{1}{2}\sigma^2\varphi\beta_e^2\right] + p^3\left[-\frac{1}{2}\sigma^2\varphi(2\beta_e + \beta_e\Gamma_e)\right] + \\
& p^2\left[\beta_t + \frac{\beta^2}{d} + c\beta_e - \frac{1}{2}\sigma^2\varphi - \sigma^2\varphi\Gamma_e - \frac{1}{2}\sigma^2\varphi\Gamma_e^2 - \sigma^2\varphi\beta_e\gamma_e + \frac{1}{2}\sigma^2\beta_{ee}\right] + \\
& p\left[\Gamma_t + \frac{\Gamma\beta}{d} + c + c\Gamma_e - \sigma^2\varphi\gamma_e - \sigma^2\Gamma_{ee} + 2\beta f(t)(s - bf(t))\right] + \\
& \left[\gamma_t + \frac{1}{4d}\Gamma^2 + c\gamma_e - \frac{1}{2}\sigma^2\varphi\gamma_e^2 + \frac{1}{2}\sigma^2\gamma_{ee} + (e + \Gamma)f(t)(s - bf(t))\right] ,
\end{aligned} \tag{7.9}$$

then as (7.9) should be valid for each p, and then the equations in the brackets must be zero. It is sufficient to solve each partial differential equation in each bracket taking in consideration their terminal conditions. The solutions for each partial differential equation are

$$\begin{aligned}
\beta(t,e) &= k\sigma\sqrt{d\phi}\tanh\left(\frac{k\sqrt{\phi}\sigma(t-T)}{\sqrt{d}}\right) \quad \text{with } k = 0.707107 , \\
\Gamma(t,e) &= -\alpha e^{\int_t^T \frac{\beta(y,e)}{d}\mathrm{d}y} + \int_t^T \frac{1}{d}\left(e^{\int_t^T \frac{\beta(y,e)}{d}\mathrm{d}y}\right)(-cd + 2d\sigma\beta(x,e)f(x) + 2bd\beta(x,e) + f^2(x))\mathrm{d}x , \\
\gamma(t,e) &= \int_t^T \frac{1}{4d}\left(-4de\sigma f(y) + 4bdef^2(y) - 4d\sigma f(y)\Gamma(y,e) + 4bdf^2(y)\Gamma(y,e) - \Gamma^2(y,e)\right)\mathrm{d}y .
\end{aligned}$$

Therefore the optimal trading rate with ambiguity aversion to the mid-price drift is fully determined by the last equations.

Conclusion

The role of mathematics in finance has been crucial during recent years, papers such as Black et al. [6] changed the way of trading in the financial markets, and also motivated that researchers worked on new theories about asset pricing and replication theory. However, in the last years, practitioners and scientists have been more focused on a state-of-the-art topic: high-frequency trading.

High-frequency trading has emerged due to the necessity to trade quickly, optimising the potential profits for a bank, a house brokerage, or any market participant. Recent research indicates that in the equity market the participation of high-frequency traders has increased dramatically, some statistics show that approximately fifty percent of the traded volume is executed using an algorithms.

In this thesis I analyse the optimal way to execute an order under different scenarios, the vast majority of these models could be applied into the financial markets; especially the execution model using stochastic volatility and liquidity, and the double liquidation problem. Also, from the theoretical point of view, I extend the double liquidation problem assuming ambiguity aversion, a technique which involves topics such as change of measures and Itô calculus. As far as I concern, the incorporation of ambiguity aversion in the double liquidation problem has not been solved before and could be analysed in more detail in the future.

Finally, I would like to mention that all models which had been discussed in this thesis are a parametrization of the market; a proper calibration and execution of these models do not depend entirely on the quant's skills.It is always necessary to understand what happened in the financial market out of the quantitative point of view, if the agent could incorporate a qualitative and economic knowledge into her execution, her models would be more accurate.

A Numerical Solution of Partial Differential Equation

Many partial differential equations can be found in finance, as an example, the price of a put option can be derived solving

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0\,. \tag{A.1}$$

Feynman and Kac established this link between partial differential equations and stochastic processes in the 1940s. However, to get an exact solution for any partial differential equation is not an easy task, in the vast majority of cases is necessary to implement some numerical methods to get a solution.

In this appendix, I discuss the main numerical methods to solve a partial differential equation (PDE); In the topic of execution models, solve PDE is a common task, this is why is necessary to bear in mind these methods.

A.1 Finite difference method

Finite difference is the main method to approximate the solution of a PDE, this method consists in approximate derivatives through finite difference; let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a function such that f in $\mathcal{C}^\infty$. Then by Taylor's theorem, we get

$$f(x_0 + h) - f(x_0) = \sum_{i=1}^{n} \frac{1}{i!} f^{(i)}(x_0)h^i + E_n(x)\,, \tag{A.2}$$

where $E_n(x)$ is an error function.

To approximate the first derivative of f, let us consider (A.2) in this form

$$f(x_0 + h) - f(x_0) = f'(x_0)h + E_1(x)$$

therefore $f'(x_0) = \frac{f(x_0+h)-f(x_0)}{h} + \frac{E_1(x)}{h}$, and assuming that $\frac{E_1(x)}{h}$ vanishes when h goes to 0.

We can conclude that

$$f'(x_0) \approx \frac{f(x_0 + h) - f(x_0)}{h}\,.$$

To generalise this idea, consider (A.1) as the partial differential equation associated to an American put option with maturity T.

Consider the interval $[0, T]$ and define $\delta t = \frac{T}{N}$, then we get $(N+1)-$steps, i.e. $\{0, \delta t, 2\delta t, ..., T\}$. Second, let us suppose the existence of S_{max} as the maximum value that the stock price can reach, then defining the interval $[0, S_{max}]$ and $\delta S = \frac{S_{max}}{M}$, we obtain $(M+1)-$steps, i.e. $\{0, \delta S, 2\delta S, ..., S_{max}\}$.

That is to say, we have generated a grid over the space $[0, T] \times [0, S_{max}]$, where the stock price is defined. In the literature, there are three common ways to approximate (A.1)

- The implicit method.

- The explicit method.
- Crank-Nicholson method.

The implicit method: Let us define $V_j^n := V(S_j, t_n)$ then the derivative with respect to time can be approximate by

$$\frac{\partial V(S_j,t_n)}{\partial t} \approx \frac{V_j^{n+1} - V_j^n}{\delta t},$$

the first derivative with respect to the stock can be approximate by a central difference (n,j) across the nodes $(n, j-1)$ to $(n, j+1)$, i.e.

$$\frac{\partial V(S_j,t_n)}{\partial S} \approx \frac{V_{j+1}^n - V_{j-1}^n}{2\delta S},$$

then the second derivative with respect to the stock is

$$\frac{\partial^2 V(S_j,t_n)}{\partial S^2} \approx \frac{V_{j+1}^n - 2V_j^n + V_{j-1}^n}{\delta S^2},$$

then (A.1) becomes

$$\frac{V_j^{n+1} - V_j^n}{\delta t} + rj\delta S\frac{V_{j+1}^n - V_{j-1}^n}{2\delta S} + \frac{1}{2}\sigma^2 j^2 \delta S^2 \frac{V_{j+1}^n - 2V_j^n + V_{j-1}^n}{\delta S^2} = rV_j^n .$$

The explicit method: In this method is assumed that the value of the first derivative with respect to the stock in the node (n,j) is equivalent to its value in $(n+1,j)$, we then get

$$\frac{\partial V(S_j,t_n)}{\partial S} \approx \frac{V_{j+1}^{n+1} - V_{j-1}^{n+1}}{2\delta S},$$

and

$$\frac{\partial^2 V(S_j,t_n)}{\partial S^2} \approx \frac{V_{j+1}^{n+1} - 2V_j^{n+1} + V_{j-1}^{n+1}}{\delta S^2},$$

then (A.1) becomes into

$$\frac{V_j^{n+1} - V_j^n}{\delta t} + rj\delta S\frac{V_{j+1}^{n+1} - V_{j-1}^{n+1}}{2\delta S} + \frac{1}{2}\sigma^2 j^2 \delta S^2 \frac{V_{j+1}^{n+1} - 2V_j^{n+1} + V_{j-1}^{n+1}}{\delta S^2} = rV_j^n .$$

The Crank-Nicholson method: Taking an average of the explicit and the implicit method, then (A.1) is reduced into:

$$\frac{V_j^{n+1} - V_j^n}{\delta t} + \frac{1}{2}rj\delta S\left(\frac{V_{j+1}^n - V_{j-1}^n}{2\delta S} + \frac{V_{j+1}^{n+1} - V_{j-1}^{n+1}}{2\delta S}\right) +$$
$$\frac{1}{2^2}\sigma^2 j^2 \delta S^2\left(\frac{V_{j+1}^n - 2V_j^n + V_{j-1}^n}{\delta S^2} + \frac{V_{j+1}^{n+1} - 2V_j^{n+1} + V_{j-1}^{n+1}}{\delta S^2}\right) = rV_j^n .$$

Finally, boundary conditions play a crucial roll when is estimated a solution for a PDE. In the example (A.1), boundary conditions are given by

$$\begin{cases} V_j^N = (k - j\delta S)_+ & ,\forall\ j \in [0, M] \\ V_M^n = 0 & ,\forall\ n \in [0, N] \\ V_0^n = k & ,\forall\ n \in [0, N] \end{cases}$$

.

B Code

This section contains all Python and Wolfram Mathematica scripts, which are used during this thesis.

B.1 Code for section 2

I use these libraries

```
import pandas as pd
%matplotlib inline
import numpy as np
import matplotlib.pyplot as plt
plt.style.use('seaborn-whitegrid')
import os
import random
import math
import datetime
import time
import scipy
import matplotlib as mpl
import seaborn as sns
from sklearn.linear_model import LinearRegression
from sklearn import linear_model, datasets
```

To build a limit order book, and the demand and supply graph.

```
ASK = np.zeros((20,2))
BID = np.zeros((20,2))
TIME=LOB['Time']
H=math.trunc(TIME/(60*60))
M=math.trunc((TIME/(60*60)-H)*60)
S=round(TIME-M*60-H*60*60,2)
for i in range(0,20):
    ASK[i,0]=LOB[i*4] #Price
    ASK[i,1]=LOB[i*4+1] #Size
    BID[i,0]=LOB[i*4+2] #Price
    BID[i,1]=LOB[i*4+3] #Size

BID=pd.DataFrame(BID,columns=["price","volume"])
ASK=pd.DataFrame(ASK,columns=["price","volume"])
#Micro-Price and mid-Price
MP=((BID['price']*BID['volume']).sum()+(ASK['price']*ASK['volume']).sum())/(BID['
    volume'].sum()+ASK['volume'].sum())
Mid=0.5*(BID['price'][0]+ASK['price'][0])

```

```
B=np.arange(0,20,dtype=float)
A=np.arange(0,20,dtype=float)
for i in range(0,20):
    auxB=0
    auxA=0
    j=0
    while(j<=i):
        auxB=BID["volume"][j]+auxB
        auxA=ASK["volume"][j]+auxA
        j=j+1

    B[i]=auxB
    A[i]=auxA
```

To compute the round trip cost curve for any limit order book.

```
def Rt_curve(j):

    LOB=result.loc[j,:]

    ASK = np.zeros((20,2))
    BID = np.zeros((20,2))
    for i in range(0,20):
        ASK[i,0]=LOB[i*4] #Price
        ASK[i,1]=LOB[i*4+1] #Size
        BID[i,0]=LOB[i*4+2] #Price
        BID[i,1]=LOB[i*4+3] #Size

    BID=pd.DataFrame(BID,columns=["price","volume"])
    ASK=pd.DataFrame(ASK,columns=["price","volume"])

    auxRT=int(min(BID['volume'].sum(),ASK['volume'].sum()))
    RT=np.arange(0,auxRT,dtype=float)

    for i in range(0,auxRT):

        auxBid=np.zeros((20,1))
        auxAsk=np.zeros((20,1))
        auxVA=i+1
        auxVB=i+1
        auxlA=0
        auxlB=0

        while((auxVA>=0 and auxlA<=19)):
            auxAsk[auxlA]=min(auxVA,ASK['volume'][auxlA])*ASK['price'][auxlA]
```

```
            auxVA=auxVA-ASK['volume'][auxlA]
            auxlA=auxlA+1

        while((auxVB>=0 and auxlB<=19)):
            auxBid[auxlB]=min(auxVB,BID['volume'][auxlB])*BID['price'][auxlB]
            auxVB=auxVB-BID['volume'][auxlB]
            auxlB=auxlB+1

        RT[i]=(auxAsk.sum()-auxBid.sum())/(i+1)

    return(RT)
```

To compute the traded volume by minute.

```
Volume=MSG[MSG.Type>3]
Volume=Volume[Volume.Type<6]
Volume['Time']*=1/(60*60)
Volume['Price']*=1/(10000)
Volume.index=range(Volume.shape[0])
auxV=math.ceil(round((Volume['Time'][Volume.shape[0]-1]-Volume['Time'][0])/(1/60)))
V=np.zeros((auxV,3))

V[0,0]=volAc(0,1/60)[0]
V[0,1]=volAc(0,1/60)[1]
V[0,2]=volAc(0,1/60)[2]

for i in range(1,auxV):
    aux=int(V[i-1,1])
    V[i,0]=volAc(aux,1/60)[0]
    V[i,1]=volAc(aux,1/60)[1]
    V[i,2]=volAc(aux,1/60)[2]

V=pd.DataFrame(V,columns=["Time","position","volume"])

ban=0
j=0
while(V["volume"][j]!=0):
    ban=j
    j=j+1
ban=ban+1
```

To compute the round trip cost associated to 99-percentile traded volume, micro-price, mid-price and bid-ask spread.

```
#Daily Roundtrip cost(using mean, max P95V), micro-price,Bid-Ask
```

```
AuxR=result.shape[0]
RTD=np.zeros((AuxR,2))
MP=np.zeros((AuxR,2))
Mid=np.zeros((AuxR,2))
BA=np.zeros((AuxR,2))

for i in range(0,AuxR):
    LOB=result.loc[i,:]

    ASK = np.zeros((20,2))
    BID = np.zeros((20,2))
    TIME=LOB['Time']*1/(60*60)

    for j in range(0,20):
        ASK[j,0]=LOB[j*4] #Price
        ASK[j,1]=LOB[j*4+1] #Size
        BID[j,0]=LOB[j*4+2] #Price
        BID[j,1]=LOB[j*4+3] #Size

    BID=pd.DataFrame(BID,columns=["price","volume"])
    ASK=pd.DataFrame(ASK,columns=["price","volume"])

    MP[i,0]=TIME
    MP[i,1]=((BID['price']*BID['volume']).sum()+(ASK['price']*ASK['volume']).sum())
        /(BID['volume'].sum()+ASK['volume'].sum())

    Mid[i,0]=TIME
    Mid[i,1]=0.5*(BID['price'][0]+ASK['price'][0])

    BA[i,0]=TIME
    BA[i,1]=(ASK['price'][0]-BID['price'][0])

    auxBid=np.zeros((20,1))
    auxAsk=np.zeros((20,1))
    auxVA=P99V
    auxVB=P99V
    auxlA=0
    auxlB=0

    while((auxVA>=0 and auxlA<=19)):
        auxAsk[auxlA]=min(auxVA,ASK['volume'][auxlA])*ASK['price'][auxlA]
        auxVA=auxVA-ASK['volume'][auxlA]
        auxlA=auxlA+1

    while((auxVB>=0 and auxlB<=19)):
        auxBid[auxlB]=min(auxVB,BID['volume'][auxlB])*BID['price'][auxlB]
```

```
            auxVB=auxVB-BID['volume'][auxlB]
            auxlB=auxlB+1

    RTD[i,1]=(auxAsk.sum()-auxBid.sum())/(P99V)

    RTD[i,0]=TIME
```

To compute the volatility by minute.

```
#Vol by minute
Dlog=np.log(MicroPrice["Microprice"])-np.log(MicroPrice["Microprice"].shift(1))
Dlog=pd.DataFrame(Dlog)
RendLog=pd.concat([MicroPrice["Time"],Dlog],axis=1,join_axes=[Dlog.index])
RendLog=RendLog.iloc[1:,]
RendLog.index=range(RendLog.shape[0])

def volMin(startPoint,space):
    auxI=startPoint
    auxT=RendLog["Time"][auxI]+space

    while(RendLog["Time"][auxI]<=auxT and auxI<(RendLog.shape[0]-1)):
        auxI=auxI+1

    return (RendLog["Time"][auxI-1],auxI,RendLog["Microprice"][startPoint:auxI].std()*math.sqrt(252)*100)
Vol=np.zeros((int(RendLog.shape[0]),3))

for i in range(0,int(RendLog.shape[0])):
    Vol[i,0]=volMin(i,1/60)[0]
    Vol[i,1]=volMin(i,1/60)[1]
    Vol[i,2]=volMin(i,1/60)[2]

Vol=pd.DataFrame(Vol,columns=["Time","position","Vol"])
```

To compute the market impact by minute using the round trip cost.

```
RT_Av=roudTrip.iloc[1:,]
RT_Av.index=range(RT_Av.shape[0])
def RTMin(startPoint,space):
    auxI=startPoint
    auxT=RT_Av["Time"][auxI]+space
    cont=0
    aux=0

    while(RT_Av["Time"][auxI]<=auxT and auxI<(RT_Av.shape[0]-1)):
```

```
            auxI=auxI+1
            cont=cont+1
            aux=aux+RT_Av["99  Percentile"][auxI]


      return  (RT_Av["Time"][auxI-1],auxI,aux/cont)
auxRT=math.ceil(round((RT_Av['Time'][RT_Av.shape[0]-1]-RT_Av['Time'][0])/(1/60)))
RT_A=np.zeros((auxRT,3))

RT_A[0,0]=RTMin(0,1/60)[0]
RT_A[0,1]=RTMin(0,1/60)[1]
RT_A[0,2]=RTMin(0,1/60)[2]

for i in range(1,auxRT):
      aux=int(RT_A[i-1,1])
      RT_A[i,0]=RTMin(aux,1/60)[0]
      RT_A[i,1]=RTMin(aux,1/60)[1]
      RT_A[i,2]=RTMin(aux,1/60)[2]

RT_A=pd.DataFrame(RT_A,columns=["Time","position","RT"])
```

B.2 Code for section 5

To compute the relaxation time parameter (δ).

```
AuxR=result.shape[0]
Vol=np.zeros((AuxR,2))

for i in range(0,AuxR):
      LOB=result.loc[i,:]

      ASK = np.zeros((20,2))
      BID = np.zeros((20,2))
      TIME=LOB['Time']*1/(60*60)

      for j in range(0,20):
            ASK[j,0]=LOB[j*4] #Price
            ASK[j,1]=LOB[j*4+1] #Size
            BID[j,0]=LOB[j*4+2] #Price
            BID[j,1]=LOB[j*4+3] #Size

      BID=pd.DataFrame(BID,columns=["price","volume"])
      ASK=pd.DataFrame(ASK,columns=["price","volume"])

      Vol[i,0]=(TIME-9.5)/6.5
```

```
    Vol[i,1]=((BID['volume']).sum()-(ASK['volume']).sum())/(BID['volume'].sum()+ASK
        ['volume'].sum())

Vol=pd.DataFrame(Vol,columns=["time","volume"])

S=Vol["volume"]
S[S>=0]=1.0
S[S<0]=-1.0
ind_pos=np.where(np.roll(S,1)!=S)[0]
A=np.array(ind_pos).tolist()
T=Vol["time"]
x = np.array(T[A])
Tau=np.diff(x)
Tau.mean()
```

To compute β and ζ. The inputs in this code are the volatility and market impact calculated before.

```
Vola=pd.read_csv("Vol.csv")
liq=pd.read_csv("MI.csv")
sigma2=(Vola["Vol"]-Vola["Vol"].mean())/Vola["Vol"].std()
eta=(liq["MI"]-liq["MI"].mean())/liq["MI"].std()
avIndex=(eta+sigma2)*0.5
print(avIndex.mean(),avIndex.std())
```

To compute $\overline{\eta}$ and $\overline{\sigma}^2$. The inputs in this code are the volatility and market impact calculated before.

```
eta_Hat=(liq["MI"]/np.exp(avIndex)).mean() #eta hat
sigma2_Hat=(liq["MI"]*Vola["Vol"]/eta_Hat).mean() #sigma hat
print(eta_Hat,sigma2_Hat)
```

B.3 Code for section 6

To solve the partial differential equations

```
pde1 = D[g[t, s, e], t] - f + (1/d) (g[t, s, e]*g[t, s, e]) == 0
DSolve[{pde1, g[T, s, e] == -a}, {g[t, s, e]}, {t}]

pde2 = D[B[t, s, e], t] +
   2*g[t, s, e]*f[t] (s - b*f[t]) + (1/d)*g[t, s, e]*B[t, s, e] == 0
DSolve[{pde2, B[T, s, e] == 0}, {B[t, s, e]}, {t}]
```

B.4 Code for section 7

To solve the partial differential equations

```
pde3 = D[B[t, s, e], t] + (1/d)*B[t, s, e]*B[t, s, e] - 0.5*h*s*s == 0
DSolve[{pde3, B[T, s, e] == 0}, {B[t, s, e]}, {t}]

pde4 = D[g[t, s, e], t] + (1/d)*g[t, s, e]*b[t, s, e] + c +
   2*b[t, s, e]*f[t] (s - b*f[t]) == 0
DSolve[{pde4, g[T, s, e] == -a}, {g[t, s, e]}, {t}]

pde5 = D[h[t, s, e], t] + (1/(4 d))*g[t, s, e]*
    g[t, s, e] + (e + g[t, s, e])*f[t]*(s - b*f[t]) == 0
DSolve[{pde5, h[T, s, e] == 0}, {h[t, s, e]}, {t}]
```

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