Stochastic Optimisation under Probability Distortion

by Justin Gwee

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Justin Gwee (CID: 01547802)

Supervisor: Dr. Alex Tse

Department of Mathematics
Imperial College London
London SW7 2AZ
United Kingdom

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Declaration

The work contained in this thesis is my own work unless otherwise stated.

Signature and date:

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1 Introduction

1.1 Expected Utility Maximisation

The maximisation of an investor's expected utility is one of the primary criteria governing the investor's decision rules in several financial contexts, such as portfolio selection or determining the optimal time to sell an asset. In Expected Utility Theory (EUT) developed by Von Neumann and Morgenstern (1944), it is assumed that decision makers are rational and uniformly risk averse. However, through extensive experimental studies, it was observed that such features of EUT are inconsistent with real human behaviour. Specifically, it was concluded that:

- People are *not* uniformly risk averse: they are risk-averse on gains and risk-taking on losses, and substantially more sensitive to losses than to gains (known as *loss aversion*). Gains and losses are defined with respect to a reference point, that is, amounts above a reference point are considered gains and those below are considered losses. In the context of asset allocation, people evaluate assets based on gains and losses (with respect to a reference point), instead of the final wealth position of the asset - this is known as the *framing effect*[2].

- People are *not* completely rational and thus *not* able to objectively evaluate probabilities: they tend to overweigh small probabilities and underweigh large probabilities. As described in [15], overweighing small probabilities associated with extremely large gains corresponds to the human emotion of *hope* (which prompts people to buy lotteries) and overweighing small probabilities associated with extremely large losses corresponds to *fear* (which prompts people to buy insurance).

There are also several well known paradoxes that cannot be explained by EUT such as the Allais paradox (Allais 1953) and the Ellesberg paradox (Ellesberg 1961)[7].

1.2 Tversky and Kahneman's Cumulative Prospect Theory

Kahneman and Tversky (1979) [1] proposed the *prospect theory* (PT) to address the drawbacks of EUT, by incorporating human emotions and psychology. This was later modified in Tversky and Kahneman (1992) [3] into the Nobel prize-winning *cumulative prospect theory* (CPT), whose main features are:

- A *reference point* that distinguishes gains from losses

- A *value function* (which replaces EUT's notion of a utility function) that is *concave for gains* (to represent risk aversion) and *convex for losses* (to represent a risk-taking attitude) and *steeper* for losses than for gains (to represent loss aversion).

- A *probability distortion function* that is a *non-linear* transformation of cumulative probabilities, enlarging small cumulative probabilities and diminishing large cumulative probabilities.

The main difference between PT and CPT is that probability distortion is applied to cumulative probabilities (in the latter) instead of all probabilities (in the former). This modification was made to rectify the violation of first order stochastic dominance in PT, defined in [4] as follows:

Definition 1.1 (First Order Stochastic Dominance)**.** Random variable A has first order stochastic dominance over random variable B if $\mathbb{P}(A \geqslant x) \geqslant \mathbb{P}(B \geqslant x)$ for all x and $\mathbb{P}(A \geqslant x) > \mathbb{P}(B \geqslant x)$ for some x. In other words, $F_A(x) \leqslant F_B(x)$ for all x and $F_A(x) < F_B(x)$ for some x, where F_A and F_B are the cumulative distribution functions of A and B respectively.

To be precise, in PT framework, it is possible for random variable B to be preferred over random variable A even though A has first order stochastic dominance over B. This is not even violated in EUT, because if A has first order stochastic dominance over B, then $\mathbb{E}[U(A)] > \mathbb{E}[U(B)]$, where U is a non-decreasing utility function, and hence A is preferred to B.

In EUT framework, we aim to maximise expected utility $\mathbb{E}[U(X)]$ of an individual with non-decreasing concave utility function $U : \mathbb{R} \to \mathbb{R}$ over random variables $X : \Omega \to \mathbb{R}$ which represents the outcome (the set of outcomes is denoted by Ω) of the decision made by the individual. This expected utility can be written in terms of integrals of cumulative probabilities:

$$\mathbb{E}[U(X)] = \mathbb{E}\left[U_+\left(X^+\right)\right] - \mathbb{E}\left[U_-\left(X^-\right)\right] = \int_0^\infty \mathbb{P}\left(U_+\left(X^+\right) > x\right) \mathrm{d}x - \int_0^\infty \mathbb{P}\left(U_-\left(X^-\right) > x\right) \mathrm{d}x$$

where $U_+ : \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ is concave and $U_- : \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ is convex, and applied to the positive and negative parts X^+ and X^- of X respectively, that is, $U(x) = \begin{cases} U_+(x), & x \geqslant 0 \\ -U_-(-x), & x \leqslant 0 \end{cases}$

In CPT framework, U is instead *S-shaped* (concave on $\mathbb{R}_{\geqslant 0}$ and convex on $\mathbb{R}_{\leqslant 0}$) to capture risk aversion on gains and risk-taking attitude on losses, U_- should be steeper than U_+ to capture loss aversion, and non-linear probability distortion functions $w_+ : [0,1] \to [0,1]$ and $w_- : [0,1] \to [0,1]$ are applied to cumulative probabilities (enlarging small ones and decreasing large ones) associated with $U_+\left(X^+\right)$ and $U_-\left(X^-\right)$ and the following *value function* is maximised instead:

$$V(X) := \int_0^\infty w_+\left(\mathbb{P}\left(U_+\left(X^+\right) > x\right)\right) \mathrm{d}x - \int_0^\infty w_-\left(\mathbb{P}\left(U_-\left(X^-\right) > x\right)\right) \mathrm{d}x$$

which is a difference of nonlinear expectations (a generalisation of expectation), called the *Choquet expectation* or *Choquet integral* of the random variables $U_+\left(X^+\right)$ under the *capacity* $w_+(\mathbb{P}(\cdot))$ and $U_-\left(X^-\right)$ under the *capacity* $w_-(\mathbb{P}(\cdot))$. w_+ and w_- both have a *reverse S-shape* in CPT framework, that is, concave on $[0, q]$ and convex on $[q, 1]$ where $q \in (0, 1)$ is the inflection point. Figures 1 and

2 below illustrate the shape of the utility functions and probability distortion functions in CPT framework.

Figure 1: CPT S-shaped Utility Function

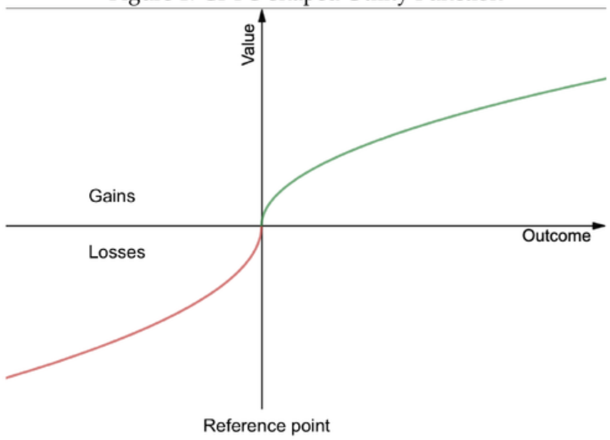
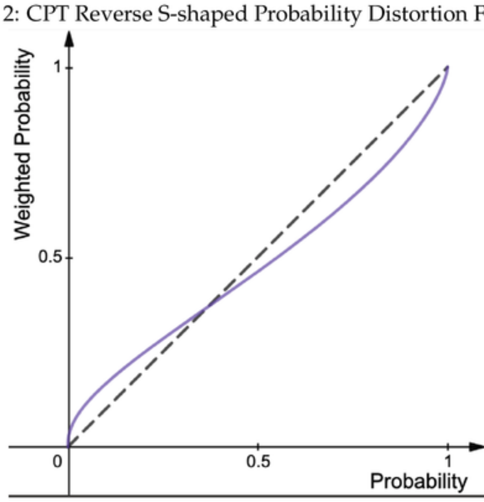


Figure 2: CPT Reverse S-shaped Probability Distortion Function



1.3 Convex Optimisation

Convex optimisation techniques are crucial in solving the problems in this thesis. It is useful to identify convex optimisation problems, due to their fundamental property that any locally optimal point is globally optimal, and that solving these problems is *equivalent* to solving their corresponding Lagrange dual problem (since the optimal value of both problems are equal) - this is also known as *strong duality*. This section defines the required terminology and outlines the techniques used to solve such problems, and is adapted from [5].

Definition 1.2 (Convex Optimisation Problem). A *convex optimisation problem* is a minimisation problem of the form:

$$\begin{aligned} \underset{x \in \mathbb{R}^n}{\text{minimize}} \quad & f_0(x) \\ \text{subject to} \quad & f_i(x) \leqslant 0, \quad i = 1, ..., m, \\ & h_j(x) := a_j^\intercal x - b_j = 0, \quad j = 1, ..., p \end{aligned} \tag{1.1}$$

where $f_0, f_1, ..., f_m$ are convex functions and $h_1, ..., h_p$ are affine functions. $x \in \mathbb{R}^n$ is called the *optimisation variable*, $f_0 : \mathbb{R}^n \to \mathbb{R}$ the *objective function*, $f_i : \mathbb{R}^n \to \mathbb{R}$ are the *inequality constraint functions* and $h_i : \mathbb{R}^n \to \mathbb{R}$ are the *equality constraint functions*. If there are no constraints, that is, $m = p = 0$, the problem is *unconstrained*. The *domain* of the optimisation problem (denoted by $\mathcal{D}$) is the set of points for which the objective and all the constraint functions are defined:

$$\mathcal{D} := \left\{ \bigcap_{i=0}^{m} \mathbf{dom} f_i \right\} \cap \left\{ \bigcap_{j=1}^{p} \mathbf{dom} h_j \right\}$$

A point $x \in \mathcal{D}$ is *feasible* it satisfies the constraints $f_i(x) \leqslant 0, i = 1, ..., m$ and $h_j(x) = 0, j = 1, ..., p$. Problem (1.1) is said to be *feasible* if there exists at least one feasible point, and *infeasible* otherwise. The set of all feasible points is called the *feasible set*. The *optimal value* (denoted by p^*) to problem (1.1) is defined as

$$p^* := \inf \{ f_0(x) | f_i(x) \leqslant 0, i = 1, ..., m, h_j(x) = 0, j = 1, ..., p \}$$

If the problem is infeasible, $p^* = \inf \emptyset = \infty$. If there are feasible points x_k such that $f_0(x_k) \to -\infty$ as $k \to \infty$, then $p^* = -\infty$ and problem (1.1) is said to be *unbounded*.

Remark 1.3. The following maximisation problem is also a convex optimisation problem:

$$\begin{aligned} \underset{x \in \mathbb{R}^n}{\text{maximize}} \quad & f_0(x) \\ \text{subject to} \quad & f_i(x) \leqslant 0, \quad i = 1, ..., m, \\ & h_j(x) := a_j^\intercal x - b_j = 0, \quad j = 1, ..., p \end{aligned}$$

where f_0 is concave, $f_1, ..., f_m$ are convex and $h_1, ..., h_p$ are affine, since maximising $f_0(x)$ is equivalent to minimising $-f_0(x)$, and f_0 is concave if and only if $-f_0$ is convex.

Definition 1.4 (Lagrangian/Lagrange Dual Function/Lagrange Dual Problem). The *Lagrangian* associated with the convex optimisation problem above is defined as $\mathcal{L} : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R}$,

$$\mathcal{L}(x, \lambda, \nu) := f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x) + \sum_{j=1}^{p} \nu_j h_j(x)$$

The *Lagrange Dual Function* is $g : \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R}$,

$$g(\lambda, \nu) := \inf_{x \in \mathcal{D}} \mathcal{L}(x, \lambda, \nu) = \inf_{x \in \mathcal{D}} \left(f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x) + \sum_{j=1}^{p} \nu_j h_j(x) \right)$$

and the *Lagrange Dual Problem* is

$$\begin{aligned} \underset{\lambda, \nu}{\text{maximize}} \quad & g(\lambda, \nu) \\ \text{subject to} \quad & \lambda_i \geqslant 0, \quad i = 1, ..., m \end{aligned}$$

This thesis involves *stochastic optimisation problems*, which involves optimising over *random variables* instead of deterministic variables. We define a *convex stochastic optimisation problem* similarly to its previously defined deterministic version:

$$\begin{aligned} \underset{X}{\text{minimize}} \quad & \mathbb{E}[f_0(X)] \\ \text{subject to} \quad & \mathbb{E}[f_i(X)] \leqslant 0, \quad i = 1, ..., m, \\ & \mathbb{E}\left[h_j(X)\right] = 0, \quad j = 1, ..., p \end{aligned}$$

where X is a random variable, $f_0, f_1, ..., f_m$ are convex functions and $h_1, ..., h_p$ are affine functions. This problem is considerably harder to solve than its deterministic version. A possible approach is to consider converting the problem into a convex optimisation problem over G, the quantile function of X. Since $X \overset{\Delta}{=} G(U)$, where $U \sim \mathrm{U}[0,1]$ the problem can be written as

$$\begin{aligned} \underset{G}{\text{minimize}} \quad & \mathbb{E}[f_0(G(U))] \\ \text{subject to} \quad & \mathbb{E}[f_i(G(U))] \leqslant 0, \quad i = 1, ..., m, \\ & \mathbb{E}\left[h_j(G(U))\right] = 0, \quad j = 1, ..., p \end{aligned}$$

and the corresponding Lagrangian and Lagrange dual function are:

$$\mathcal{L}(G, \lambda, \nu) := \mathbb{E}\left[f_0(G(U)) + \sum_{i=1}^{m} \lambda_i f_i(G(U)) + \sum_{j=1}^{p} \nu_j h_j(G(U)) \right], \quad g(\lambda, \nu) := \inf_{G} \mathcal{L}(G, \lambda, \nu)$$

By linearity and monotonicity of expectation, we can solve the following problem, for any given $x \in [0,1]$:

$$\begin{aligned} \underset{G}{\text{minimize}} \quad & f_0(G(x)) \\ \text{subject to} \quad & f_i(G(x)) \leqslant 0, \quad i = 1, ..., m, \\ & h_j(G(x)) = 0, \quad j = 1, ..., p \end{aligned}$$

However, in the CPT framework, stochastic optimisation problems are non-convex, since utility functions are S-shaped. Furthermore, under probability distortion, the objective function or the constraint functions are no longer linear and the problem becomes much harder to simplify and solve. Moreover, in dynamic problems, time consistency structure is lost in the presence of probability distortion and dynamic programming principle cannot be applied. However, certain techniques have been developed to overcome such difficulties: in particular, a problem may be decomposed into sub-problems that are solved separately, and a series of transformations are applied to convert non-convex optimisation problems into convex ones.

1.4 Outline of Thesis

This thesis aims to first review two stochastic optimisation problems under probability distortion, namely optimal stopping and behavioural portfolio selection, that have been formulated and solved in [6] and [7] respectively using similar techniques. The optimal stopping chapter also considers several extensions to the problem and discusses some limitations of the techniques involved. Similar techniques will then be applied to solve a new problem, that of determining the optimal contract function in employee stock options, motivated by a model proposed by Spalt in [9], and we will see that the optimisation involved is more closely related to that in the behavioural portfolio selection problem.

2 Optimal Stopping under Probability Distortion

There have been several established approaches to solving classical optimal stopping problems that do not incorporate probability distortion, most notably the probabilistic approach involving martingale theory and the PDE approach involving dynamic programming principles or variational inequalities [6]. However, these approaches crucially rely on the time consistency structure of the underlying problem, which is lost in the presence of probability distortion, and the well-known approaches cannot be applied [6]. This motivated Xu and Zhou [6] to develop a novel approach to tackle the challenges posed by probability distortion, and this chapter aims to review these techniques. Prior to this, Barberis had studied optimal exit strategies in casino gambling in the presence of probability distortion, and only managed to obtain numerical solutions via exhaustive enumeration [6].

The main ideas of Xu and Zhou's methods can be summarised as follows: suppose that $(X_t)_{t\geqslant 0}$ is the process we wish to stop optimally. Then we first determine the probability distribution of the optimally stopped state X_{τ^*} and then recover the corresponding optimal stopping time τ^* either in a clear way in certain important cases or generally via Skorokhod embedding. The original objective function, a function of stopping times τ, is converted into either a functional of distribution or quantile functions of stopped states (distribution and quantile formulation), and once the optimal distribution or quantile function is determined, the corresponding optimal stopping time can be derived.

2.1 Formulation of Problem

The paper considers an optimal stopping problem for a stock in a Black-Scholes model, whose price process, which we denote by $(X_t)_{t\geqslant 0}$, follows a geometric Brownian motion

$$\frac{\mathrm{d}X_t}{X_t} = \mu \mathrm{d}t + \sigma \mathrm{d}W_t \iff X_t = X_0 \exp\left(\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W_t\right)$$

where $\mu \in \mathbb{R}, \sigma, X_0 > 0$ are constants, $(W_t)_{t\geqslant 0}$ is a standard one-dimensional Brownian motion in a complete filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geqslant 0}, \mathbb{P})$.

Denote by $\mathcal{T}$ the set of all $\{\mathcal{F}_t\}_{t\geqslant 0}$-adapted stopping times τ satisfying $\mathbb{P}(\tau < +\infty) = 1$. If the process $(X_t)_{t\geqslant 0}$ is stopped at $\tau \in \mathcal{T}$, a payoff of $U(X_\tau)$ is obtained, where $U : \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ is a given non-decreasing and continuous function. We want to determine the optimal time τ which maximises the payoff $U(X_\tau)$. In a financial context, an agent decides on the optimal time τ to liquidate the asset, and U represents the agent's utility function, and this utility is determined by the amount she sells the asset for. Alternatively, the agent could be determining the optimal time to exercise an American option (with payoff function U) written on an asset with price process $(X_t)_{t\geqslant 0}$. The standard optimal stopping problem (which does not assume probability distortion by the agent) is:

$$\begin{aligned} &\underset{\tau}{\text{maximize}} \quad \mathcal{J}(\tau) := \mathbb{E}_{\mathbb{P}}[U(X_\tau)] = \int_0^\infty \mathbb{P}(U(X_\tau) > x)\,\mathrm{d}x \\ &\text{subject to} \quad \tau \in \mathcal{T} \end{aligned}$$

where $\mathbb{E}_{\mathbb{P}}$ denotes the (linear) expectation under probability measure $\mathbb{P}$ and the second equality above holds because $U(X_\tau)$ is a strictly positive random variable for all $\tau \in \mathcal{T}$ (since $(X_t)_{t\geqslant 0}$ is a geometric Brownian motion which implies X_t is a strictly positive random variable at all times t, and $U(x) \geqslant 0$ for all $x \geqslant 0$). Probability distortion is incorporated as follows: define a weighting function $w : [0,1] \to [0,1]$ which is strictly increasing and absolutely continuous, with $w(0) = 0$ and $w(1) = 1$. The function w is applied to the exceedance probabilities $\mathbb{P}(U(X_\tau) > x)$ (in the absence of probability distortion, $w(x) = x$ for all $x \in [0,1]$) as stipulated in CPT, and the problem becomes:

$$\begin{aligned} &\underset{\tau}{\text{maximize}} \quad \mathcal{J}(\tau) := \int_0^\infty w(\mathbb{P}(U(X_\tau) > x))\,\mathrm{d}x \\ &\text{subject to} \quad \tau \in \mathcal{T} \end{aligned}$$

The above objective function is a nonlinear expectation (a generalisation of expectation), called the *Choquet expectation* or *Choquet integral* of the random variable $U(X_\tau)$ under the *capacity* $w(\mathbb{P}(\cdot))$.

2.2 Solving the Problem

The case when $\mu = \frac{1}{2}\sigma^2 \implies X_t = X_0 \exp(\sigma W_t)$ can be solved easily. Define, for all $x \in (0, +\infty)$,

$$\tau_x := \inf \left\{ t \geqslant 0 : W_t = \frac{1}{\sigma} \log\left(\frac{x}{X_0}\right) \right\}$$

Then for all $x \in (0, +\infty)$, $\mathbb{P}(\tau_x < +\infty) = 1$ since a standard one-dimensional Brownian motion hits any deterministic level in finite time almost surely and therefore $\tau_x \in \mathcal{T}$. Furthermore, $X_{\tau_x} = x$ almost surely (by construction) and

$$\mathcal{J}(\tau_x) = \int_0^\infty w\left(\mathbb{P}\left(U\left(X_{\tau_x}\right) > y\right)\right) \mathrm{d}y = \int_0^\infty w\left(\mathbb{P}\left(U(x) > y\right)\right) \mathrm{d}y = \int_0^{U(x)} w(1)\mathrm{d}y = U(x)$$

However, for any $\tau \in \mathcal{T}$, denoting $\overline{U} := \sup_{x>0} U(x)$,

$$\mathcal{J}(\tau) = \int_0^{\overline{U}} w\left(\mathbb{P}\left(U\left(X_\tau\right) > x\right)\right) \mathrm{d}x \leqslant \int_0^{\overline{U}} w(1)\mathrm{d}x = \overline{U} = \sup_{x>0} \mathcal{J}(\tau_x)$$

Therefore, the optimal value is $\overline{U}$ and the optimal stopping time, if it exists, is of the form τ_x. If there exists at least one $x^* > 0$ such that $U(x^*) = \overline{U}$, then τ_{x^*} is an optimal stopping time. However, if on the other hand $U(y) < \overline{U}$ for all $y > 0$, then for any stopping time $\tau \in \mathcal{T}$, $U(X_\tau) < \overline{U}$. Then since w is strictly increasing,

$$\mathcal{J}(\tau) = \int_0^\infty w\left(\mathbb{P}\left(U\left(X_\tau\right) > x\right)\right) \mathrm{d}x < \int_0^\infty w\left(\mathbb{P}\left(\overline{U} > x\right)\right) \mathrm{d}x = \overline{U}$$

which means that the optimal value is not achievable by any stopping time. Having solved the case when $\mu = \frac{1}{2}\sigma^2$, we will subsequently assume that $\mu \neq \frac{1}{2}\sigma^2$.

2.2.1 Transformation with Scale Functions

In order to solve the problem for the case $\mu \neq \frac{1}{2}\sigma^2$, we will eventually apply the Skorokhod embedding theorem, which requires the underlying price process to be a martingale. We will transform the continuous $\mathbb{P}$-semimartingale $(X_t)_{t\geqslant 0}$ into a continuous (local) $\mathbb{P}$-martingale $(S_t)_{t\geqslant 0}$ via a scale function, which is defined as follows:

Definition 2.1 (Scale Function)**.** Let $(X_t)_{t\geqslant 0}$ be a one-dimensional continuous semimartingale under $\mathbb{P}$. The continuous and strictly increasing function $s \in C^2(\mathbb{R})$ (twice continuously differentiable) is called a scale function if the process $(S_t)_{t\geqslant 0} := (s(X_t))_{t\geqslant 0}$ is a one-dimensional continuous local martingale under $\mathbb{P}$.

The following proposition gives the form of the scale function [11] when $(X_t)_{t\geqslant 0}$ is specified as a solution to some stochastic differential equation.

Proposition 2.2. *Let $(X_t)_{t\geqslant 0}$ taking values in some interval I be the solution to the one-dimensional SDE*

$$\mathrm{d}X_t = b(X_t)\,\mathrm{d}t + \sigma(X_t)\,\mathrm{d}W_t$$

where $(W_t)_{t\geqslant 0}$ is a standard one-dimensional Brownian motion in a complete filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geqslant 0}, \mathbb{P})$ and $b, \sigma \in C(\mathbb{R})$. Then its scale function is given by

$$s(x) = \int_{x_0}^{x} \exp\left(-\int_{y_0}^{y}\left(\frac{2b(z)}{\sigma(z)^2}\right) \mathrm{d}z\right) \mathrm{d}y$$

where x_0 and y_0 are points in I arbitrarily fixed.

Proof. Let $(Y_t)_{t\geqslant 0} := (s(X_t))_{t\geqslant 0}$ be a continuous local martingale under $\mathbb{P}$ and assume $s \in C^2(\mathbb{R})$. Then by Itô's formula,

$$\mathrm{d}Y_t = s'(X_t)\,\sigma(X_t)\,\mathrm{d}W_t + \left[s'(X_t)\,b(X_t) + \frac{1}{2}s''(X_t)\,\sigma(X_t)^2\right]\mathrm{d}t$$

Since Y is a local martingale, we set the drift term to zero:

$$s'(X_t)\,b(X_t) + \frac{1}{2}s''(X_t)\,\sigma(X_t)^2 = 0 \iff s''(X_t) = \left(-\frac{2b(X_t)}{\sigma(X_t)^2}\right)s'(X_t)$$

Integrating twice, we obtain the required form of the scale function. □

Remark 2.3. The scale function is an increasing function, since for some fixed $x_0 \in I$,

$$s'(x) = \exp\left(-\int_{x_0}^{x}\left(\frac{2b(y)}{\sigma(y)^2}\right)\mathrm{d}y\right) > 0$$

for all $x \in I$. The scale function is also *unique up to affine transformation*, which is clear from the two integration steps, and can also be illustrated through an application of the optional stopping theorem in the following example:

Example 2.1. *Let $(M_t)_{t\geqslant 0}$ be a martingale starting at $M_0 = x \in \mathbb{R}$ almost surely, $a > 0$, $b < 0$ and define the stopping times*

$$T_a := \inf\{t \geqslant 0 : M_t = a\}, \quad T_b := \inf\{t \geqslant 0 : M_t = b\}, \quad \tau := T_a \wedge T_b = \inf\{t \geqslant 0 : M_t \in \{a, b\}\}$$

By the optional stopping theorem,

$$\begin{aligned}\mathbb{E}[M_\tau] = \mathbb{E}[M_0] = x &= a\mathbb{P}(T_a < T_b) + b\mathbb{P}(T_a > T_b)\\ &= a\mathbb{P}(T_a < T_b) + b(1 - \mathbb{P}(T_a < T_b)) \implies \mathbb{P}(T_a < T_b) = \frac{x-b}{a-b}\end{aligned}$$

If $(X_t)_{t\geqslant 0}$ is a semimartingale starting at $X_0 = x \in \mathbb{R}$ almost surely such that $(s(X_t))_{t\geqslant 0}$ is a martingale, then we can define the stopping times

$$\begin{aligned}&\widetilde{T}_a := \inf\{t \geqslant 0 : X_t = a\}, \quad \widetilde{T}_b := \inf\{t \geqslant 0 : X_t = b\},\\ &\widetilde{\tau} := \widetilde{T}_a \wedge \widetilde{T}_b = \inf\{t \geqslant 0 : s(X_t) \in \{s(a), s(b)\}\}\end{aligned}$$

and obtain

$$\mathbb{P}\left(\widetilde{T}_a < \widetilde{T}_b\right) = \frac{s(x) - s(b)}{s(a) - s(b)}$$

If we apply an affine transformation to the scale function s and define a new function $\widetilde{s}$ by

$$\widetilde{s}(x) := ms(x) + c$$

for some $m > 0, c \in \mathbb{R}$ then clearly we also have

$$\mathbb{P}\left(\widetilde{T}_a < \widetilde{T}_b\right) = \frac{(ms(x)+c) - (ms(b)+c)}{(ms(a)+c) - (ms(b)+c)} = \frac{\widetilde{s}(x) - \widetilde{s}(b)}{\widetilde{s}(a) - \widetilde{s}(b)}$$

Therefore $\widetilde{s}$ is also a scale function for $(X_t)_{t\geqslant 0}$ such that $(\widetilde{s}(X_t))_{t\geqslant 0}$ is a martingale.

Applying the proposition to our problem, we have

$$\begin{aligned} s(x) = \int_{x_0}^{x} \exp\left(-\int_{y_0}^{y}\left(\frac{2\mu z}{\sigma^2 z^2}\right)\mathrm{d}z\right)\mathrm{d}y &= \int_{x_0}^{x} \exp\left(-\left(\frac{2\mu}{\sigma^2}\right)\int_{y_0}^{y}\left(\frac{1}{z}\right)\mathrm{d}z\right)\mathrm{d}y \\ &= \int_{x_0}^{x}\left(\frac{y_0}{y}\right)^{\left(\frac{2\mu}{\sigma^2}\right)}\mathrm{d}y \\ &= \left(\frac{y_0^{\left(\frac{2\mu}{\sigma^2}\right)}}{1-\frac{2\mu}{\sigma^2}}\right)\left(x^{\left(1-\frac{2\mu}{\sigma^2}\right)} - x_0^{\left(1-\frac{2\mu}{\sigma^2}\right)}\right) \end{aligned}$$

Since the scale function is unique up to affine transformation, if $\beta := 1 - \frac{2\mu}{\sigma^2} > 0$, we may divide by $\dfrac{y_0^{\left(\frac{2\mu}{\sigma^2}\right)}}{1-\frac{2\mu}{\sigma^2}}$ and then add $x_0^{\left(1-\frac{2\mu}{\sigma^2}\right)}$ or equivalently, taking $x_0 = 0$ and $y_0 = \left(1-\frac{2\mu}{\sigma^2}\right)^{\frac{\sigma^2}{2\mu}}$, to obtain the increasing scale function

$$s(x) = x^{\left(1-\frac{2\mu}{\sigma^2}\right)} = x^{\beta}, \quad x > 0 \iff s^{-1}(x) = x^{\frac{1}{\beta}}, \quad x > 0$$

However, if $\beta < 0 \iff \mu > \frac{1}{2}\sigma^2$, we define the increasing scale function as

$$s(x) = -x^{\beta}, \quad x > 0 \iff s^{-1}(x) = (-x)^{\frac{1}{\beta}}, \quad x < 0$$

In other words, $(S_t)_{t\geqslant 0} := \left(X_t^{\beta}\right)_{t\geqslant 0}$ is an exponential martingale under $\mathbb{P}$ which is the solution to the SDE

$$\mathrm{d}S_t = s'(X_t)\,\sigma X_t \mathrm{d}W_t = \beta X_t^{\beta-1}\sigma X_t \mathrm{d}W_t = \beta\sigma S_t \mathrm{d}W_t \iff S_t = S_0 \exp\left(\beta\sigma W_t - \frac{1}{2}\beta^2\sigma^2 t\right)$$

Now we define a new utility function $u : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ by

$$u(x) := U\left(x^{\frac{1}{\beta}}\right) = \left(U \circ s^{-1}\right)(x) \quad \forall x \in (0, +\infty)$$

Since s is an increasing function, s^{-1} is also an increasing function, and since u is a composition of increasing functions U and s^{-1}, u is an increasing function. The problem may be rewritten as:

$$\begin{aligned} &\underset{\tau}{\text{maximize}} \quad \mathcal{J}(\tau) := \int_0^{\infty} w\left(\mathbb{P}\left(U\left(X_\tau\right) > x\right)\right)\mathrm{d}x = \int_0^{\infty} w\left(\mathbb{P}\left(u\left(S_\tau\right) > x\right)\right)\mathrm{d}x \\ &\text{subject to} \quad \tau \in \mathcal{T} \end{aligned}$$

which we will solve in the remainder of this section. It is worth noting that u may have a different shape from U, due to the different shapes of s^{-1} (which depends on the value of β). If both U and s^{-1} are both concave, then u is concave, and if U and s^{-1} are both convex, then u is convex. This is formalised in the lemma below:

Lemma 2.4. *Let $f, g \in C^2$ be increasing functions, and $h := f \circ g$. Then*

(i) If f and g are both convex, then h is convex.

(ii) If f and g are both concave, then h is concave.

(iii) If f is convex and g is concave, or if f is concave and g is convex, then the shape of h is inconclusive.

Proof. By the chain rule,

$$h'(x) = f'(g(x))\, g'(x) \implies h''(x) = f''(g(x))\left(g'(x)\right)^2 + f'(g(x))\, g''(x)$$

By the above equation, the conclusions are obvious. □

Remark 2.5. The case when f and g are not twice differentiable can be analysed similarly. If only the first derivatives of f and g exist, we can use the fact that f convex implies f' is increasing and g concave implies g' is decreasing, and arrive at the same conclusions. The case where f and g are not differentiable is presented in [5].

Note that s^{-1} is convex for all $\beta \leqslant 1 \iff \mu \geqslant 0$ (the stock performs well) and concave for $\beta > 1 \iff \mu < 0$ (the stock performs badly). The results of the lemma is illustrated in the following examples:

Example 2.2. *(i) If U is the payoff function of a call option with strike price $K > 0$, that is,*

$$U(x) = (x-K)^+ \implies u(x) = \begin{cases} \left(x^{\frac{1}{\beta}} - K\right)^+, & \beta > 0 \\ \left((-x)^{\frac{1}{\beta}} - K\right)^+, & \beta < 0 \end{cases}$$

Since U is convex, u is convex for $\beta \leqslant 1$, that is, for $\beta < 0$ and $0 < \beta < 1$. For $\beta \geqslant 1$, $u(x) = 0$ for all $x \leqslant K^\beta$ and $u(x) = x^{\frac{1}{\beta}} - K$ for all $x \geqslant K^\beta$, and therefore u is convex on $\left(-\infty, K^\beta\right]$ and concave on $\left(K^\beta, \infty\right)$, that is, u is S-shaped.

(ii) If U is a power function given by $U(x) = \frac{1}{\gamma}x^\gamma$, $\gamma \in (0,1)$, then $u(x) = \begin{cases} \frac{1}{\gamma}x^{\frac{\gamma}{\beta}}, & \beta > 0 \\ \frac{1}{\gamma}(-x)^{\frac{\gamma}{\beta}}, & \beta < 0 \end{cases}$

U is concave and therefore u is concave if $\beta > 1$. If $\beta \leqslant 1$ we have three possible cases, $\beta < 0$, $0 < \beta \leqslant \gamma$ and $\gamma < \beta \leqslant 1$. In the first two cases, u is convex and in the third, u is concave. Therefore u is concave if $\beta > \gamma$ and is convex if $\beta \leqslant \gamma$.

(iii) *If* U *is a log utility function given by* $U(x) = \log(x+1)$ *then* $u(x) = \begin{cases} \log\left(x^{\frac{1}{\beta}}+1\right), & \beta > 0 \\ \log\left((-x)^{\frac{1}{\beta}}+1\right), & \beta < 0 \end{cases}$

U is concave and therefore u is concave if $\beta > 1$. *If* $\beta \leqslant 1$, *we can determine the shape of u by examining its second order derivative: if* $0 < \beta \leqslant 1$,

$$\frac{\mathrm{d}^2u}{\mathrm{d}x^2} = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\frac{1}{\beta}x^{\frac{1}{\beta}-1}}{x^{\frac{1}{\beta}}+1}\right) = \frac{\left(x^{\frac{1}{\beta}}+1\right)\frac{1}{\beta}\left(\frac{1}{\beta}-1\right)\left(x^{\frac{1}{\beta}-2}\right) - \frac{1}{\beta}x^{\frac{1}{\beta}-1}\left(\frac{1}{\beta}x^{\frac{1}{\beta}-1}\right)}{\left(x^{\frac{1}{\beta}}+1\right)^2}$$

$$= \frac{\frac{1}{\beta}\left(\frac{1}{\beta}-1\right)x^{\frac{1}{\beta}-2} - \frac{1}{\beta}x^{\frac{2}{\beta}-2}}{\left(x^{\frac{1}{\beta}}+1\right)^2} = \frac{x^{\frac{1}{\beta}-2}\left(\frac{1-\beta}{\beta} - x^{\frac{1}{\beta}}\right)}{\beta\left(x^{\frac{1}{\beta}}+1\right)^2}$$

then $\frac{\mathrm{d}^2u}{\mathrm{d}x^2} \geqslant 0 \iff x \leqslant \left(\frac{1-\beta}{\beta}\right)^{\beta}$. *Therefore u is convex on* $\left(0, \left(\frac{1-\beta}{\beta}\right)^{\beta}\right]$ *and concave on* $\left(\left(\frac{1-\beta}{\beta}\right)^{\beta}, \infty\right)$. *Similarly, if* $\beta < 0$, *we obtain*

$$\frac{\mathrm{d}^2u}{\mathrm{d}x^2} = \frac{(-x)^{\frac{1}{\beta}-2}\left(\frac{1-\beta}{\beta} - (-x)^{\frac{1}{\beta}}\right)}{\beta\left((-x)^{\frac{1}{\beta}}+1\right)^2}$$

and $\frac{\mathrm{d}^2u}{\mathrm{d}x^2} \geqslant 0 \iff x \leqslant -\left(\frac{1-\beta}{\beta}\right)^{\beta}$ *and therefore u is convex on* $\left(-\infty, -\left(\frac{1-\beta}{\beta}\right)^{\beta}\right]$ *and concave on* $\left(-\left(\frac{1-\beta}{\beta}\right)^{\beta}, 0\right)$. *In other words, if* $\beta \leqslant 1$, *u is S-shaped.*

(iv) *If* U *is an exponential utility function given by* $U(x) = 1 - \mathrm{e}^{-\alpha x}$, $\alpha > 0$, *then*

$$u(x) = \begin{cases} 1 - \mathrm{e}^{-\alpha x^{\frac{1}{\beta}}}, & \beta > 0 \\ 1 - \mathrm{e}^{-\alpha(-x)^{\frac{1}{\beta}}}, & \beta < 0 \end{cases}$$

U is concave and therefore u is concave if $\beta > 1$. *If* $\beta \leqslant 1$, *we can determine the shape of u by examining its second order derivative: if* $0 < \beta \leqslant 1$,

$$\frac{\mathrm{d}^2u}{\mathrm{d}x^2} = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\alpha}{\beta}x^{\frac{1}{\beta}-1}\mathrm{e}^{-\alpha x^{\frac{1}{\beta}}}\right) = \frac{\alpha}{\beta}\left(x^{\frac{1}{\beta}-1}\left(-\frac{\alpha}{\beta}x^{\frac{1}{\beta}-1}\mathrm{e}^{-\alpha x^{\frac{1}{\beta}}}\right) + \mathrm{e}^{-\alpha x^{\frac{1}{\beta}}}\left(\frac{1}{\beta}-1\right)x^{\frac{1}{\beta}-2}\right)$$

$$= \frac{\alpha}{\beta}\left(\left(\frac{1}{\beta}-1\right)x^{\frac{1}{\beta}-2}\mathrm{e}^{-\alpha x^{\frac{1}{\beta}}} - \frac{\alpha}{\beta}\mathrm{e}^{-\alpha x^{\frac{1}{\beta}}}x^{\frac{2}{\beta}-2}\right)$$

then $\frac{\mathrm{d}^2u}{\mathrm{d}x^2} \geqslant 0 \iff x \leqslant \left(\frac{1-\beta}{\alpha}\right)^{\beta}$. *Therefore u is convex on* $\left(0, \left(\frac{1-\beta}{\alpha}\right)^{\beta}\right]$ *and concave on* $\left(\left(\frac{1-\beta}{\alpha}\right)^{\beta}, \infty\right)$. *Similarly, if* $\beta < 0$, *we obtain*

$$\frac{\mathrm{d}^2u}{\mathrm{d}x^2} = \frac{\alpha}{\beta}\left(\left(\frac{1}{\beta}-1\right)(-x)^{\frac{1}{\beta}-2}\mathrm{e}^{-\alpha(-x)^{\frac{1}{\beta}}} - \frac{\alpha}{\beta}\mathrm{e}^{-\alpha(-x)^{\frac{1}{\beta}}}(-x)^{\frac{2}{\beta}-2}\right)$$

and $\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} \geqslant 0 \iff x \leqslant -\left(\frac{1-\beta}{\alpha}\right)^{\beta}$ *and therefore u is convex on* $\left(-\infty, -\left(\frac{1-\beta}{\alpha}\right)^{\beta}\right]$ *and concave on* $\left(-\left(\frac{1-\beta}{\alpha}\right)^{\beta}, 0\right)$. *In other words, if* $\beta \leqslant 1$, *u is S-shaped.*

Finally, we may assume without loss of generality that $u(0) = 0$, since one can consider $\overline{u}(\cdot) := u(\cdot) - u(0)$ if $u(0) \neq 0$.

2.2.2 Distribution and Quantile Formulation

The problem will be solved by reformulating it into its corresponding distribution or quantile formulation, in which one optimally chooses the probability distribution or quantile function of the stopped state.

Definition 2.6 (Cumulative Distribution Function (CDF))**.** A function $F : \mathbb{R} \to [0,1]$ is called a *cumulative distribution function* (CDF) if:

- $F(-\infty) \equiv \lim_{x \to -\infty} F(x) = 0$, $F(+\infty) \equiv \lim_{x \to +\infty} F(x) = 1$
- F is non-decreasing and cádlág (right-continuous with left limits)

By right continuity, $F(x) = F(x^+)$ where $x^+ := x + \varepsilon$ for some $\varepsilon > 0$ small.

Definition 2.7 (Quantile Function)**.** A function $G : [0,1] \to \mathbb{R}$ is called a *quantile function* if it is non-decreasing and can be written as a left-continuous inverse function of a CDF F as follows:

$$G(x) = \inf\{y \in \mathbb{R} : F(y) \geqslant x\} =: F^{-1}(x)$$

for all $x \in [0,1)$. By left-continuity, $G(x) = G(x^-)$ where $x^- := x - \varepsilon$ for some $\varepsilon > 0$ small.

In our problem, $(S_t)_{t \geqslant 0}$ is an exponential martingale and hence only takes values in $\mathbb{R}_{>0}$ at any time t. Therefore we will restrict ourselves to CDFs satisfying $F(x) = 0$ for all $x \leqslant 0$, and quantile functions satisfying $G(0) = 0$, $G(x) > 0$ for all $x \in (0,1)$. We define the distribution set $\mathcal{D}$ and quantile set $\mathcal{Q}$ for this problem as follows:

$$\mathcal{D} := \{F : \mathbb{R}_{\geqslant 0} \to [0,1] | F \text{ is the CDF of } S_\tau, \text{ for some } \tau \in \mathcal{T}\}$$
$$\mathcal{Q} := \left\{G : [0,1] \to \mathbb{R}_{\geqslant 0} \middle| G = F^{-1} \text{ for some } F \in \mathcal{D}\right\}$$

Lemma 2.8. *For any* $\tau \in \mathcal{T}$,

$$\mathcal{J}(\tau) = \mathcal{J}_D(F) := \int_0^\infty w\,(1 - F(x))\, u'(x) \mathrm{d}x$$
$$\mathcal{J}(\tau) = \mathcal{J}_Q(G) := \int_0^1 u\,(G(x))\, w'(1 - x) \mathrm{d}x$$

where F and G are the CDF and quantile function of S_τ *rsepectively. Moreover,*

$$\sup_{\tau \in \mathcal{T}} \mathcal{J}(\tau) = \sup_{F \in \mathcal{D}} \mathcal{J}_D(F) = \sup_{G \in \mathcal{Q}} \mathcal{J}_Q(G)$$

Proof. First assume that u is a strictly increasing C^∞ function with $u(0) = 0$. Then we have

$$\begin{aligned}\mathcal{J}(\tau) = \int_0^\infty w\left(\mathbb{P}\left(u\left(S_\tau\right) > x\right)\right) \mathrm{d}x &= \int_0^\infty w\left(\mathbb{P}\left(u\left(S_\tau\right) > u(y)\right)\right) \mathrm{d}u(y) \\ &= \int_0^\infty w\left(\mathbb{P}\left(S_\tau > x\right)\right) \mathrm{d}u(x) \\ &= \int_0^\infty w\left(1 - F(x)\right) \mathrm{d}u(x) \\ &= \int_0^\infty u(x) \mathrm{d}\left[-w\left(1 - F(x)\right)\right] \\ &= \int_0^\infty u(x) w'\left(1 - F(x)\right) \mathrm{d}F(x) = \int_0^1 u\left(G(x)\right) w'\left(1 - x\right) \mathrm{d}x\end{aligned}$$

where the second and final equality follows by the appropriate substitution, the third equality follows from the fact that u is strictly increasing, the fourth equality follows by definition of F, and the fifth equality follows by Fubini's theorem.

Now assume u is absolutely continuous and non-decreasing with $u(0) = 0$. Then for each $\varepsilon > 0$ there exists a strictly increasing C^∞ function u_ε such that $|u_\varepsilon(x) - u(x)| < \varepsilon$ for all $x \in \mathbb{R}_{\geqslant 0}$. Then

$$\begin{aligned}&\left|\int_0^1 u_\varepsilon\left(G(x)\right) w'\left(1 - x\right) \mathrm{d}x - \int_0^1 u\left(G(x)\right) w'\left(1 - x\right) \mathrm{d}x\right| \\ &= \left|\int_0^1 \left(u_\varepsilon\left(G(x)\right) - u\left(G(x)\right)\right) w'\left(1 - x\right) \mathrm{d}x\right| \\ &\leqslant \int_0^1 \left|u_\varepsilon\left(G(x)\right) - u\left(G(x)\right)\right| w'\left(1 - x\right) \mathrm{d}x \\ &< \varepsilon \int_0^1 w'\left(1 - x\right) \mathrm{d}x = \varepsilon\left(w(1) - w(0)\right) = \varepsilon\end{aligned}$$

It can also be verified similarly that

$$\left|\int_0^\infty w\left(\mathbb{P}\left(u_\varepsilon\left(S_\tau\right) > x\right)\right) \mathrm{d}x - \int_0^\infty w\left(\mathbb{P}\left(u\left(S_\tau\right) > x\right)\right) \mathrm{d}x\right| < \varepsilon$$

We also have by the above result that

$$\begin{aligned}&\int_0^\infty w\left(\mathbb{P}\left(u_\varepsilon\left(S_\tau\right) > x\right)\right) \mathrm{d}x = \int_0^1 u_\varepsilon\left(G(x)\right) w'\left(1 - x\right) \mathrm{d}x \\ &\implies \left|\int_0^\infty w\left(\mathbb{P}\left(u_\varepsilon\left(S_\tau\right) > x\right)\right) \mathrm{d}x - \int_0^\infty w\left(\mathbb{P}\left(u\left(S_\tau\right) > x\right)\right) \mathrm{d}x\right| \\ &\quad = \left|\int_0^1 u_\varepsilon\left(G(x)\right) w'\left(1 - x\right) \mathrm{d}x - \int_0^\infty w\left(\mathbb{P}\left(u\left(S_\tau\right) > x\right)\right) \mathrm{d}x\right| < \varepsilon\end{aligned}$$

Therefore, by the triangle inequality,

$$\begin{aligned}&\left|\int_0^1 u\left(G(x)\right) w'\left(1 - x\right) \mathrm{d}x - \int_0^\infty w\left(\mathbb{P}\left(u\left(S_\tau\right) > x\right)\right) \mathrm{d}x\right| \\ &\leqslant \left|\int_0^1 u\left(G(x)\right) w'\left(1 - x\right) \mathrm{d}x - \int_0^1 u_\varepsilon\left(G(x)\right) w'\left(1 - x\right) \mathrm{d}x\right| \\ &\quad + \left|\int_0^1 u_\varepsilon\left(G(x)\right) w'\left(1 - x\right) \mathrm{d}x - \int_0^\infty w\left(\mathbb{P}\left(u\left(S_\tau\right) > x\right)\right) \mathrm{d}x\right| < 2\varepsilon\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we have shown the second equation. To show the first equation, we have by change of variables and Fubini's theorem,

$$\begin{aligned}\mathcal{J}(\tau) = \int_0^1 u\left(G(x)\right) w'\left(1-x\right) \mathrm{d}x &= \int_0^1 u\left(G(x)\right) \mathrm{d}\left[-w(1-x)\right] \\ &= \int_0^\infty u\left(x\right) \mathrm{d}\left[-w(1-F(x))\right] \\ &= \int_0^\infty w(1-F(x))\mathrm{d}u(x) = \int_0^\infty w(1-F(x))u'(x)\mathrm{d}x\end{aligned}$$

The final assertion is clear by the two equations. □

Remark 2.9. We observe that u and w play symmetric roles in the two formulations. Therefore we may choose the formulation that is convenient in solving the original stopping problem: if u is known to be concave or convex while w has an arbitrary shape, it is more convenient to work with quantile formulation, and vice versa.

The following lemma provides necessary and sufficient conditions for a distribution function to belong to our distribution set and for a quantile function to belong to our quantile set. This provides an explicit characterisation for the distribution and quantile sets.

Lemma 2.10. *Let $s := S_0 = s\left(X_0\right)$. Then we have the following assertions:*

$$\begin{aligned}F \in \mathcal{D} &\iff \int_0^\infty \left(1-F(x)\right) \mathrm{d}x \leqslant s \\ G \in \mathcal{Q} &\iff \int_0^1 G(x)\mathrm{d}x \leqslant s\end{aligned}$$

Proof. Suppose $F \in \mathcal{D}$, i.e. F is the CDF of S_τ for some $\tau \in \mathcal{T}$, and equivalently, $G := F^{-1} \in \mathcal{Q}$. Then, since $(S_t)_{t\geqslant 0}$ is a non-negative (local) martingale, it is a supermartingale (by Fatou's lemma), and therefore by the optional stopping theorem,

$$\int_0^\infty (1-F(x))\mathrm{d}x = \int_0^\infty \mathbb{P}\left(S_\tau > x\right) \mathrm{d}x = \mathbb{E}\left[S_\tau\right] \leqslant \mathbb{E}\left[S_0\right] = s$$

and by the change of variable $y = G(x)$,

$$\int_0^1 G(x)\mathrm{d}x = \int_0^\infty y\mathrm{d}F(y) = \mathbb{E}\left[S_\tau\right] \leqslant s$$

For the converse implication, assume F is some function satisfying

$$\begin{aligned}\int_0^\infty \left(1-F(x)\right) \mathrm{d}x \leqslant s \iff \int_0^\infty x\mathrm{d}F(x) \leqslant s &\iff \int_{-\infty}^\infty \left(s\mathrm{e}^{\beta\sigma y}\right) \mathrm{d}F\left(s\mathrm{e}^{\beta\sigma y}\right) \leqslant s \\ &\iff \int_{-\infty}^\infty \left(\mathrm{e}^{\beta\sigma x}\right) \mathrm{d}F\left(s\mathrm{e}^{\beta\sigma x}\right) \leqslant 1\end{aligned}$$

First assume $\beta > 0$. Then we can write

$$S_t = s\exp\left(\beta\sigma W_t - \frac{1}{2}\beta^2\sigma^2 t\right) =: s\exp\left(\beta\sigma\widetilde{W}_t\right)$$

where $\left(\widetilde{W}_t\right)_{t\geqslant 0} := \left(W_t - \frac{1}{2}\beta\sigma t\right)_{t\geqslant 0}$ is a Brownian motion with negative drift. Denote by $F_{\widetilde{W}_\tau}$ and F_{S_τ} as the CDFs of $\widetilde{W}_\tau$ and S_τ respectively. Then for any $\tau \in \mathcal{T}$,

$$F_{\widetilde{W}_\tau}(x) = \mathbb{P}\left(\widetilde{W}_\tau \leqslant x\right) = \mathbb{P}\left(S_\tau \leqslant s\mathrm{e}^{\beta\sigma x}\right) = F_{S_\tau}\left(s\mathrm{e}^{\beta\sigma x}\right)$$

Therefore,

$$\int_{-\infty}^{\infty}\left(\mathrm{e}^{\beta\sigma x}\right)\mathrm{d}F_{S_\tau}\left(s\mathrm{e}^{\beta\sigma x}\right) = \int_{-\infty}^{\infty}\left(\mathrm{e}^{\beta\sigma x}\right)\mathrm{d}F_{\widetilde{W}_\tau}(x) = \mathbb{E}\left[\mathrm{e}^{\beta\sigma\widetilde{W}_\tau}\right] \leqslant 1$$

Therefore, since F and F_{S_τ} satisfy the same inequality, by (theorem) $F = F_{S_\tau}$ for some $\tau \in \mathcal{T}$ and therefore $F \in \mathcal{D}$, and equivalently, $G := F^{-1} \in \mathcal{Q}$, and satisfies

$$\int_0^1 G(x)\mathrm{d}x = \int_0^\infty (1 - F(x))\,\mathrm{d}x \leqslant s$$

If $\beta < 0$, then can write

$$S_t =: s\exp\left(-\beta\sigma\widehat{W}_t\right)$$

where $\left(\widehat{W}_t\right)_{t\geqslant 0} := \left(-W_t + \frac{1}{2}\beta\sigma t\right)_{t\geqslant 0} = \left(-\left(W_t - \frac{1}{2}\beta\sigma t\right)\right)_{t\geqslant 0}$ is another Brownian motion with negative drift, and so the same arguments in the above analysis can be used to arrive at the same conclusion. □

Corollary 2.11. *$\mathcal{D}$ and $\mathcal{Q}$ are convex sets.*

Proof. This is a direct application of the lemma above. Let $F_1, F_2 \in \mathcal{D}$, $\lambda \in [0,1]$ and define $F := \lambda F_1 + (1-\lambda) F_2$. Then,

$$\begin{aligned}\int_0^\infty (1 - F(x))\,\mathrm{d}x &= \int_0^\infty (1 - (\lambda F_1(x) + (1-\lambda) F_2(x)))\,\mathrm{d}x \\ &= \lambda\int_0^\infty (1 - F_1(x))\,\mathrm{d}x + (1-\lambda)\int_0^\infty (1 - F_2(x))\,\mathrm{d}x \leqslant \lambda s + (1-\lambda)s = s\end{aligned}$$

Similarly, let $G_1, G_2 \in \mathcal{Q}$, $\lambda \in [0,1]$ and define $G := \lambda G_1 + (1-\lambda) G_2$. Then,

$$\begin{aligned}\int_0^1 G(x)\mathrm{d}x &= \int_0^1 (\lambda G_1(x) + (1-\lambda) G_2(x))\,\mathrm{d}x \\ &= \lambda\int_0^1 G_1(x)\mathrm{d}x + (1-\lambda)\int_0^1 G_2(x)\mathrm{d}x \leqslant \lambda s + (1-\lambda)s = s\end{aligned}$$

□

2.2.3 Determining The Optimal Quantile Function

Xu and Zhou solve for the optimal quantile function for different shapes of u and w, and for certain cases, easily deduce the optimal stopping time through an application of the optional stopping theorem. We present their results (readers can refer to [6] for the proofs) for convex u, concave u and S-shaped u, with w reverse S-shaped in all three cases, as these cases are most relevant to the CPT framework, and describe their financial implications.

Theorem 2.12 (Convex u). *If u is convex, then*

$$\sup_{G\in\mathcal{Q}} \mathcal{J}_{Q}(G) = \sup_{G\in\mathcal{Q}_2} \mathcal{J}_{Q}(G)$$

where $\mathcal{Q}_2$ is defined as

$$\mathcal{Q}_2 := \left\{G \in \mathcal{Q} : G = a\mathbb{1}_{(0,c]} + b\mathbb{1}_{(c,1)}, \quad 0 < a \leqslant b, \quad 0 < c \leqslant 1\right\}$$

and

$$\sup_{\tau\in\mathcal{T}} \mathcal{J}(\tau) = \sup_{0<a\leqslant s\leqslant b} \left[\left(1 - w\left(\frac{s-a}{b-a}\right)\right) u(a) + w\left(\frac{s-a}{b-a}\right) u(b)\right] = \sup_{x\in(0,1]} \left[w(x)u\left(\frac{s}{x}\right)\right]$$

Furthermore, if (a^*, b^*) satisfy

$$(a^*, b^*) = \operatorname*{arg\,max}_{0<a\leqslant s\leqslant b} \left[\left(1 - w\left(\frac{s-a}{b-a}\right)\right) u(a) + w\left(\frac{s-a}{b-a}\right) u(b)\right]$$

then

$$\tau_{(a^*,b^*)} := \begin{cases} \inf\left\{t \geqslant 0 : S_t \notin (a^*, b^*)\right\}, & \text{if} \quad a^* < b^* \\ 0, & \text{if} \quad a^* = b^* \end{cases}$$

is an optimal stopping time.

Corollary 2.13 (Convex u). *If u is convex, then $\tau^* \equiv 0$ is an optimal stopping time if and only if*

$$u(s) = \sup_{x\in(0,1]} \left[w(x)u\left(\frac{s}{x}\right)\right]$$

Furthermore, if

$$u(s) < \sup_{x\in(0,1]} \left[w(x)u\left(\frac{s}{x}\right)\right]$$

then the maximum in (eq) is not achievable.

Depending on the shape of U, u could be convex depending on whether the asset performs well or badly, which is based on the shape of s^{-1} (see Example 2.2). The theorem states that the optimal strategy is of a "take-profit-or-cut-loss" form. The corollary states that if the maximum is attained at time 0, it is optimal to sell immediately, otherwise the maximum is not achievable. It is worth noting that Theorem 2.12 and Corollary 2.13 hold for any shape of w, and therefore probability distortion does not affect the optimal stopping time.

Theorem 2.14 (Concave u, Reverse S-shaped w). *Assume u is concave and w is reverse S-shaped, that is, w is concave on $[0, 1-q]$ and convex on $[1-q, 1]$ for some $q \in (0, 1)$. If (a^*, λ^*) with $a^* > 0$ is a solution to the optimisation problem*

$$\begin{aligned} \underset{a,\lambda}{\text{maximize}} \quad & (1 - w(1-q))u(a) + \int_q^1 u\left(a \vee (u')_l^{-1}\left(\frac{\lambda}{w'(1-x)}\right)\right) w'(1-x)\mathrm{d}x \\ \text{subject to} \quad & aq + \int_q^1 a \vee (u')_l^{-1}\left(\frac{\lambda}{w'(1-x)}\right) \mathrm{d}x = s, \\ & a, \lambda \geqslant 0 \end{aligned}$$

where $(u')_l^{-1}$ *is defined by*

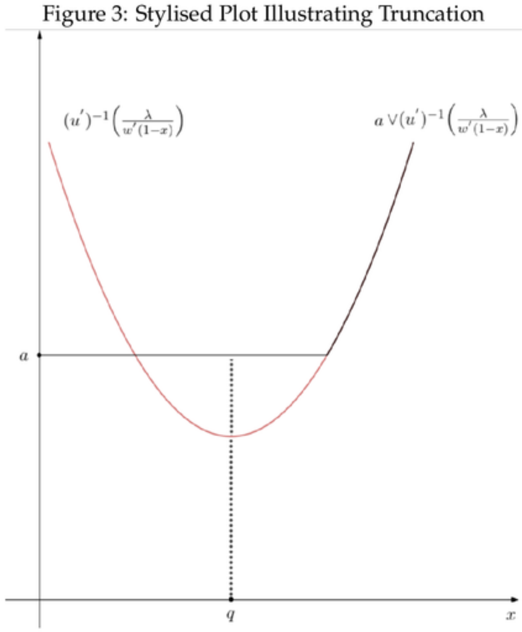
$$(u')_l^{-1}(x) := \inf\{y \geqslant 0 : u'(y) \leqslant x\}$$

Then the optimal quantile function is

$$G^*(x) = a^*\mathbb{1}_{(0,q]}(x) + \left(a^* \vee (u')_l^{-1}\left(\frac{\lambda^*}{w'(1-x)}\right)\right)\mathbb{1}_{(q,1)}(x)$$

Remark 2.15. The above expression is a truncation of the function $x \mapsto (u')_l^{-1}\left(\frac{\lambda^*}{w'(1-x)}\right)$ which is required for G^* to be increasing and satisfy the property of a quantile function. Since u is concave, $(u')_l^{-1}$ is a decreasing function and therefore $w'(1-x)$ should be increasing in x, that is, w' should be a decreasing function, in order for $(u')_l^{-1}\left(\frac{\lambda^*}{w'(1-x)}\right)$ to be increasing in x. However, this is not the case as w is reverse S-shaped, and therefore the truncation is required for G^* to be increasing. In particular, if w is reverse S-shaped and concave on $[0, 1-q]$ and convex on $[1-q, 1]$ for some $q \in (0,1)$, then w' is decreasing on $[0, 1-q]$ and increasing on $[1-q, 1]$, and therefore $x \mapsto \frac{1}{w'(1-x)}$ is decreasing on $[q, 1]$, and increasing on $[0, q]$. As a result, since $(u')_l^{-1}$ is a decreasing function, $x \mapsto (u')_l^{-1}\left(\frac{\lambda^*}{w'(1-x)}\right)$, which is a composition of $(u')_l^{-1}$ and $x \mapsto \frac{1}{w'(1-x)}$, is decreasing on $[0, q]$ and increasing on $[q, 1]$. In other words, $x \mapsto (u')_l^{-1}\left(\frac{\lambda^*}{w'(1-x)}\right)$ has a "U-shape". The proof of the optimality of this truncated function can be found in Appendix E of [6]. Figure 1 below illustrates this procedure, in which the black curve is a truncation of the red curve.

Figure 3: Stylised Plot Illustrating Truncation



Finally, in the case of S-shaped u, it was concluded in [6] that the optimal quantile function is a combination of optimal quantile functions corresponding to convex u and concave u, as formalised in Theorem 2.16 below:

Theorem 2.16 (S-shaped u, Reverse S-shaped w)**.** *Assume u is S-shaped: convex on $[0,\theta]$ and concave on $[\theta,\infty)$, and w is reverse S-shaped: concave on $[0,1-q]$ and convex on $[1-q,1]$ for some $q\in(0,1)$. If $a_1^*,a_2^*,a_3^*,c_1^*,c_2^*$ and λ^* are the solution to the optimisation problem*

$$\begin{aligned}
\underset{a_1,a_2,a_3,c_1,c_2,\lambda}{\text{maximize}} \quad & (1-w(1-c_1))u(a_1)+(w(1-c_1)-w(1-c_2))u(a_2) \\
& +(w(1-c_2)-w(1-q))u(a_3)+\int_q^1 u\left(a_3\vee(u')_l^{-1}\left(\frac{\lambda}{w'(1-x)}\right)\right)w'(1-x)\mathrm{d}x \\
\text{subject to} \quad & a_1c_1+a_2(c_2-c_1)+a_3(q-c_2)+\int_q^1 a_3\vee(u')_l^{-1}\left(\frac{\lambda}{w'(1-x)}\right)\mathrm{d}x\leqslant s, \\
& \lambda\geqslant 0,\quad 0<a_1\leqslant a_2\leqslant a_3\leqslant\theta,\quad 0\leqslant c_1\leqslant c_2\leqslant q
\end{aligned}$$

Then the optimal quantile function is

$$G^*(x)=a_1^*\mathbb{1}_{(0,c_1^*]}(x)+a_2^*\mathbb{1}_{(c_1^*,c_2^*]}(x)+a_3^*\mathbb{1}_{(c_2^*,q]}(x)+\left(a_3^*\vee(u')_l^{-1}\left(\frac{\lambda^*}{w'(1-x)}\right)\right)\mathbb{1}_{(q,1)}(x)$$

2.2.4 Determining The Optimal Stopping Time

Having determined the optimal quantile function, the optimal stopping time can be determined through the optimal quantile function by solving a Skorokhod embedding problem, if it cannot be deduced easily (such as the case when u is convex). The Skorokhod embedding problem is as follows: for a given probability measure μ on $\mathbb{R}$, find a stopping time τ such that $W_\tau\sim\mu$, where W is a standard real-valued Brownian motion and $(W_{t\wedge\tau})_{t\geqslant 0}$ is a uniformly integrable martingale.

Theorem 2.17 (Skorokhod's Embedding Theorem)**.** *Let X be a real-valued random variable with $\mathbb{E}[X]=0$ and $\mathrm{var}(X)<\infty$, and W a standard real-valued Brownian motion. Then there exists a stopping time (with respect to the natural filtration of W) τ such that $W_\tau\overset{\Delta}{=}X$ (W_τ and X have the same distribution) satisfying:*

$$\mathbb{E}[\tau]=\mathbb{E}\left[X^2\right],\quad \mathbb{E}\left[\tau^2\right]\leqslant 4\mathbb{E}\left[X^4\right]$$

Azéma and Yor developed a stopping time solving the Skorokhod embedding problem, and it relies on the Hardy-Littlewood maximal function for centered probability measures, as defined below [20]:

Definition 2.18 (Hardy-Littlewood Maximal Function for Centered Probability Measures)**.** Let μ be a centered probability measure on $\mathbb{R}$, and F be the distribution function associated with μ, i.e. $F(x)=\mu((-\infty,x])$ for all $x\in\mathbb{R}$. Then the Hardy-Littlewood maximal function (also known as the barycenter function) for μ is an increasing function given by:

$$\Psi_\mu(x) := \begin{cases} 0, & x \leqslant m \\ \frac{1}{\mu([x,\infty))} \int_{[x,\infty)} y \mathrm{d}\mu(y), & m < x < M \\ x, & x \geqslant M \end{cases} = \begin{cases} 0, & x \leqslant m \\ \frac{1}{1-F(x-)} \int_{[x,\infty)} y \mathrm{d}F(y), & m < x < M \\ x, & x \geqslant M \end{cases}$$

where $m := \inf\{x : F(x) > 0\}$ and $M := \sup\{x : F(x) < 1\}$.

We first recall the Dubins-Schwarz theorem [11], which states that every continuous local martingale can be represented as a time-changed Brownian motion, indexed by the quadratic variation of the local martingale:

Theorem 2.19 (Dubins-Schwarz). *Let $(M_t)_{t\geqslant 0}$ be a continuous local martingale starting at 0 such that $[M]_\infty = \infty$ almost surely. If we define the stopping time*

$$\tau_s := \inf\{t > 0 : [M]_t > s\}$$

then $W_s := M_{\tau_s}$ is a $(\mathcal{F}_{\tau_s})_{s\geqslant 0}$ Brownian motion and $M_t = W_{[M]_t}$.

Therefore, by the Dubins-Schwarz theorem, the distribution function of our stopped process S_τ is associated with a centered Gaussian measure with mean zero. We now state the theorem below characterising the Azéma-Yor stopping time [20], which is the first time that the martingale hits a moving level that is a function of the martingale's running maximum.

Theorem 2.20 (Azéma-Yor). *Let $(M_t)_{t\geqslant 0}$ be a continuous martingale satisfying $M_0 = 0$ almost surely and $\langle M, M\rangle_\infty = \infty$ almost surely. For any centered probability measure μ on $\mathbb{R}$,*

$$\tau_{AY} := \inf\left\{t \geqslant 0 : \Psi_\mu(M_t) \leqslant \sup_{s\in[0,t]} M_s\right\} = \inf\left\{t \geqslant 0 : M_t \leqslant \Psi_\mu^{-1}\left(\sup_{s\in[0,t]} M_s\right)\right\}$$

is a stopping time in the natural filtration of $(M_t)_{t\geqslant 0}$ and $M_{\tau_{AY}} \sim \mu$. Furthermore, $(M_{t\wedge\tau_{AY}})_{t\geqslant 0}$ is a uniformly integrable martingale.

This stopping time is not a unique solution to the Skorokhod embedding problem, however it is the most convenient one in our context. This is because, having determined the optimal quantile function, we can easily find the corresponding optimal distribution function, its Hardy-Littlewood maximal function and hence the associated Azéma-Yor stopping time.

2.3 Variants of Problem

As we have seen Section 2.2.3, the nature of the optimal stopping time depends on the shape of $u = U \circ s^{-1}$, we consider variants of the problem, involving processes different from the geometric Brownian motion with different scale functions.

2.3.1 Dividend Paying Stock

Assume as before that the price process of a stock denoted by $(X_t)_{t\geqslant 0}$ follows the geometric Brownian motion

$$\frac{\mathrm{d}X_t}{X_t} = \mu \mathrm{d}t + \sigma \mathrm{d}W_t$$

We may extend the Black-Scholes model by assuming additionally that the stock pays a constant dividend yield δ. We then consider the dividend-paying stock's price process

$$\left(\widehat{X}_t\right)_{t\geqslant 0} := \left(\mathrm{e}^{\delta t} X_t\right)_{t\geqslant 0}$$

By Itô's formula,

$$\mathrm{d}\widehat{X}_t = \mathrm{e}^{\delta t}\mathrm{d}X_t + \delta \mathrm{e}^{\delta t} X_t \mathrm{d}t = \mathrm{e}^{\delta t}\left(\mu X_t \mathrm{d}t + \sigma X_t \mathrm{d}W_t\right) + \delta \mathrm{e}^{\delta t} X_t \mathrm{d}t = (\mu+\delta)\,\widehat{X}_t \mathrm{d}t + \sigma \widehat{X}_t \mathrm{d}W_t$$

Then by similar computations, $\left(\widehat{S}_t\right)_{t\geqslant 0} := \left(s\left(\widehat{X}_t\right)\right)_{t\geqslant 0} := \left(\widehat{X}_t^{\widehat{\beta}}\right)_{t\geqslant 0}$ is an exponential martingale under $\mathbb{P}$, where

$$\widehat{\beta} := 1 - \frac{2(\mu+\delta)}{\sigma^2}$$

Therefore, compared with a non-dividend paying stock the only difference in the scale function is the exponent. Furthermore, s^{-1} is convex for $\mu \geqslant -\delta$ and concave for $\mu < -\delta$, instead of $\mu \geqslant 0$ and $\mu < 0$ respectively for a non-dividend paying stock.

2.3.2 Constant Elasticity of Variance

We may consider the *constant elasticity of variance* (CEV) model, a local volatility model in which the instantaneous volatility is a power function of the underlying spot price [17], and the SDE is given by:

$$\mathrm{d}X_t = \mu X_t \mathrm{d}t + \sigma X_t^{\gamma} \mathrm{d}W_t$$

where $\mu \in \mathbb{R}, \sigma, \gamma \geqslant 0$. This is a generalisation of the geometric Brownian motion, which is the particular case $\gamma = 1$, where the instantaneous volatility is linear in the underlying spot price. The parameter γ governs the relationship between the price and volatility. The case $\gamma < 1$ corresponds to the *leverage effect* where the volatility increases as price decreases, a feature commonly observed in equity markets [17]. The case $\gamma > 1$ corresponds to the *inverse leverage effect* where the volatility increases as price increases, a feature commonly observed in commodity markets [17]. However, the scale function is not always available explicitly:

$$\begin{aligned} s'(x) &= \exp\left(-\int_{x_0}^{x}\left(\frac{2\mu y}{\sigma^2 y^{2\gamma}}\right)\mathrm{d}y\right) = \exp\left(-\left(\frac{2\mu}{\sigma^2}\right)\int_{x_0}^{x} y^{1-2\gamma}\mathrm{d}y\right) \\ &= \exp\left(-\left(\frac{\mu}{\sigma^2(1-\gamma)}\right)\left(x^{2-2\gamma} - x_0^{2-2\gamma}\right)\right) \end{aligned}$$

which cannot be integrated explicitly in general, and therefore the optimal stopping problem cannot be solved explicitly. Nevertheless, we can deduce the shape of s^{-1}:

$$s''(x) = -\left(\frac{2\mu}{\sigma^2}\right) x^{1-2\gamma} s'(x)$$

Since s is increasing, s is convex when $\mu > 0$ and concave when $\mu < 0$, and therefore s^{-1} is concave when $\mu > 0$ and convex when $\mu < 0$, which is the same shape as in the case of a geometric Brownian motion. Therefore u has the same shape as in the case of a geometric Brownian motion if both U and s^{-1} are convex, or both U and s^{-1} are concave and therefore the nature of the stopping time would be the same, even though the parameters would differ. However, if U is convex and s^{-1} is concave, or U is concave and s^{-1} is convex, u may have a different shape from that in the case of a geometric Brownian motion (for example, it could be S-shaped with a different inflection point). The scale function is available explicitly when $\gamma = \frac{1}{2}$, where the SDE is given by

$$\mathrm{d}X_t = \mu X_t \mathrm{d}t + \sigma\sqrt{X_t}\mathrm{d}W_t$$

$$\begin{aligned}
\implies s(x) &= \int_{x_0}^{x} \exp\left(-\int_{y_0}^{y}\left(\frac{2\mu z}{\sigma^2 z}\right)\mathrm{d}z\right)\mathrm{d}y \\
&= \int_{x_0}^{x} \exp\left(-\left(\frac{2\mu}{\sigma^2}\right)(y-y_0)\right)\mathrm{d}y \\
&= \exp\left(\left(\frac{2\mu}{\sigma^2}\right)y_0\right)\int_{x_0}^{x}\exp\left(-\left(\frac{2\mu}{\sigma^2}\right)y\right)\mathrm{d}y \\
&= \exp\left(\left(\frac{2\mu}{\sigma^2}\right)y_0\right)\left(\frac{\sigma^2}{2\mu}\right)\left(\exp\left(-\left(\frac{2\mu}{\sigma^2}\right)x_0\right) - \exp\left(-\left(\frac{2\mu}{\sigma^2}\right)x\right)\right)
\end{aligned}$$

Since the scale function is increasing and unique up to affine transformation, we may take the scale function to be, for $\mu > 0$,

$$s(x) = -\exp\left(-\left(\frac{2\mu}{\sigma^2}\right)x\right), \quad x > 0 \iff s^{-1}(x) = -\left(\frac{\sigma^2}{2\mu}\right)\log(-x), \quad x < 0$$

and for $\mu < 0$,

$$s(x) = \exp\left(-\left(\frac{2\mu}{\sigma^2}\right)x\right), \quad x > 0 \iff s^{-1}(x) = -\left(\frac{\sigma^2}{2\mu}\right)\log(x), \quad x > 0$$

We can verify that s^{-1} is convex when $\mu > 0$ and concave when $\mu < 0$: when $\mu > 0$,

$$\left(s^{-1}\right)''(x) = \left(\frac{\sigma^2}{2\mu}\right)\left(\frac{1}{x^2}\right) > 0 \quad \forall x < 0$$

and when $\mu < 0$,

$$\left(s^{-1}\right)''(x) = \left(\frac{\sigma^2}{2\mu}\right)\left(\frac{1}{x^2}\right) < 0 \quad \forall x > 0$$

The scale function is also available explicitly when $\gamma = 0$, however we must also have $\mu > 0$:

$$\begin{aligned} s(x) &= \int_{x_0}^{x} \exp\left(-\int_{y_0}^{y}\left(\frac{2\mu z}{\sigma^2}\right)\mathrm{d}z\right)\mathrm{d}y \\ &= \exp\left(\left(\frac{\mu}{\sigma^2}\right)y_0^2\right)\int_{x_0}^{x}\exp\left(-\left(\frac{\mu}{\sigma^2}\right)y^2\right)\mathrm{d}y \\ &= \exp\left(\left(\frac{\mu}{\sigma^2}\right)y_0^2\right)\sqrt{2\pi\left(\frac{\sigma^2}{2\mu}\right)}\left(\Phi\left(\frac{x}{\sqrt{\frac{\sigma^2}{2\mu}}}\right)-\Phi\left(\frac{x_0}{\sqrt{\frac{\sigma^2}{2\mu}}}\right)\right) \end{aligned}$$

where Φ is the standard normal cumulative distribution function. Since the scale function is increasing and unique up to affine transformation, we may take the scale function to be

$$s(x) = \Phi\left(\left(\frac{\sqrt{2\mu}}{\sigma}\right)x\right)$$

2.3.3 Stochastic Reference Level

In reality, the reference level is not constant and could be the price of another stock or a benchmark index. Consider a stochastic reference level denoted by $(R_t)_{t\geqslant 0}$, and suppose that the price process $(X_t)_{t\geqslant 0}$ and the reference level are solutions to the following one-dimensional geometric Brownian motions:

$$\frac{\mathrm{d}X_t}{X_t} = \mu\mathrm{d}t + \sigma\mathrm{d}W_t, \quad \frac{\mathrm{d}R_t}{R_t} = \alpha\mathrm{d}t + \eta\mathrm{d}\widetilde{W}_t, \quad \mathrm{d}\left\langle W, \widetilde{W}\right\rangle_t = \rho\mathrm{d}t$$

where $(W_t)_{t\geqslant 0}$ and $\left(\widetilde{W}_t\right)_{t\geqslant 0}$ are correlated standard one-dimensional Brownian motions under $\mathbb{P}$, $\mu, \alpha \in \mathbb{R}$, $\sigma, \eta, X_0, R_0 > 0$ and $\rho \in (-1, 1)$. We may first consider the process

$$(P_t)_{t\geqslant 0} := (X_t - R_t)_{t\geqslant 0}$$

which represents the difference between the price and reference level. An agent would want to stop the process at time τ such that P_τ is as positive as possible. To convert $(P_t)_{t\geqslant 0}$ into a martingale, we apply the appropriate scale functions to $(X_t)_{t\geqslant 0}$ and $(R_t)_{t\geqslant 0}$ and define the martingale $(S_t)_{t\geqslant 0}$ (a difference of two martingales) by

$$S_t := X_t^{\beta} - R_t^{\widetilde{\beta}} \quad \forall t \geqslant 0, \quad \beta := 1 - \frac{2\mu}{\sigma^2}, \quad \widetilde{\beta} := 1 - \frac{2\alpha}{\eta^2}$$

By Itô's formula, we can determine the dynamics of $(S_t)_{t\geqslant 0}$ as follows, where $(Z_t)_{t\geqslant 0}$ is a standard one-dimensional Brownian motion independent of $(W_t)_{t\geqslant 0}$:

$$\begin{aligned} \mathrm{d}S_t = \mathrm{d}X_t^{\beta} - \mathrm{d}R_t^{\widetilde{\beta}} = \beta\sigma X_t^{\beta}\mathrm{d}W_t - \widetilde{\beta}\eta R_t^{\widetilde{\beta}}\mathrm{d}\widetilde{W}_t &= \beta\sigma X_t^{\beta}\mathrm{d}W_t - \widetilde{\beta}\eta R_t^{\widetilde{\beta}}\left(\rho\mathrm{d}W_t + \sqrt{1-\rho^2}\mathrm{d}Z_t\right) \\ &= \left(\beta\sigma X_t^{\beta} - \widetilde{\beta}\eta\rho R_t^{\widetilde{\beta}}\right)\mathrm{d}W_t - \widetilde{\beta}\eta\sqrt{1-\rho^2}R_t^{\widetilde{\beta}}\mathrm{d}Z_t \\ &= \begin{pmatrix} \beta\sigma & -\widetilde{\beta}\eta\rho \\ 0 & -\widetilde{\beta}\eta\sqrt{1-\rho^2} \end{pmatrix}\begin{pmatrix} X_t^{\beta} \\ R_t^{\widetilde{\beta}} \end{pmatrix}\cdot\begin{pmatrix} \mathrm{d}W_t \\ \mathrm{d}Z_t \end{pmatrix} \end{aligned}$$

Therefore, the dynamics of $(S_t)_{t\geqslant 0}$ are based on that of the two-dimensional process $\left(X_t^{\beta}, R_t^{\tilde{\beta}}\right)_{t\geqslant 0}$ and therefore we are required to determine the optimal stopping time of $\left(X_t^{\beta}, R_t^{\tilde{\beta}}\right)_{t\geqslant 0}$. In the presence of probability distortion, this would involve determining an optimal two-dimensional distribution of $\left(X_\tau^{\beta}, R_\tau^{\tilde{\beta}}\right)$ and then determining τ via Skorokhod embedding. However, it is difficult to optimise over multivariate distributions and furthermore at the moment there is no solution to a multi-dimensional version of the Skorokhod embedding problem.

Consider instead the following process

$$(Y_t)_{t\geqslant 0} := \left(\frac{X_t - R_t}{R_t}\right)_{t\geqslant 0} = \left(\frac{X_t}{R_t} - 1\right)_{t\geqslant 0}$$

which represents the relative difference between the price of the stock and the reference level, with respect to the reference level. By Itô's formula, we can determine the dynamics of $(Y_t)_{t\geqslant 0}$:

$$\begin{aligned}
\mathrm{d}Y_t = \mathrm{d}\left(\frac{X_t}{R_t}\right) &= \left(\frac{1}{R_t}\right)\mathrm{d}X_t - \left(\frac{X_t}{(R_t)^2}\right)\mathrm{d}R_t - 2\left(\frac{1}{2}\right)\left(\frac{1}{(R_t)^2}\right)\mathrm{d}\left\langle X, R\right\rangle_t + \frac{1}{2}\left(\frac{2X_t}{(R_t)^3}\right)\mathrm{d}\left\langle R\right\rangle_t \\
&= \left(\frac{X_t}{R_t}\right)\left((\mu - \alpha)\,\mathrm{d}t + \left(\sigma \mathrm{d}W_t - \eta \mathrm{d}\widetilde{W}_t\right) - \sigma\eta\rho \mathrm{d}t + \eta^2 \mathrm{d}t\right) \\
&= \left(\frac{X_t}{R_t}\right)\left(\left(\mu - \alpha - \sigma\eta\rho + \eta^2\right)\mathrm{d}t + \sqrt{\sigma^2 - 2\sigma\eta\rho + \eta^2}\mathrm{d}Z_t\right) \\
\Longrightarrow\ Y_t &= \left(\frac{X_0}{R_0}\right)\exp\left(\left(\mu - \alpha + \frac{1}{2}\eta^2 - \frac{1}{2}\sigma^2\right)t + \sqrt{\sigma^2 - 2\sigma\eta\rho + \eta^2}Z_t\right) - 1
\end{aligned}$$

where $(Z_t)_{t\geqslant 0}$ is a standard one-dimensional Brownian motion under $\mathbb{P}$. Therefore $(Y_t)_{t\geqslant 0}$ is another one-dimensional geometric Brownian motion and its scale function is also of the form $s(x) = x^{\beta}$, where

$$\beta = 1 - \frac{2\left(\mu - \alpha - \sigma\eta\rho + \eta^2\right)}{\sigma^2 - 2\sigma\eta\rho + \eta^2} = 1 - \frac{2\left(\mu - \alpha + \left(\eta - \frac{1}{2}\sigma\rho\right)^2 - \frac{1}{4}\sigma^2\rho^2\right)}{(\sigma - \eta)^2 + 2\sigma\eta\,(1 - \rho)}$$

From the above expression for β, we can deduce the effect of the coefficients $\mu, \alpha, \sigma, \eta$ and ρ on the convexity of s^{-1}. Recall that s^{-1} is convex for all $\beta \leqslant 1$ and concave for $\beta > 1$. Since $\rho \in (-1, 1)$, $(\sigma - \eta)^2 + 2\sigma\eta\,(1 - \rho) > 0$, and we have

$$\beta \leqslant 1 \iff \mu - \alpha + \left(\eta - \frac{1}{2}\sigma\rho\right)^2 - \frac{1}{4}\sigma^2\rho^2 \geqslant 0$$

Therefore, higher values of μ, lower values of σ, lower values of α and higher values of σ contribute to the convexity of s^{-1}. In other words, the higher mean of a stock or the lower its volatility compared to the reference level (that is, the performance of the stock is good compared to the reference level) the more convex s^{-1} is. The lower the correlation ρ, the more convex s^{-1} is, which indicates the benefits of diversification.

2.3.4 Foreign Asset

We could also consider optimal time to sell a foreign asset, whose price process (in the foreign currency) we denote by $(S_t)_{t\geqslant 0}$, and we denote the foreign exchange rate by $(X_t)_{t\geqslant 0}$ and we assume they are solutions to the following one-dimensional geometric Brownian motions:

$$\frac{\mathrm{d}S_t}{S_t} = \mu \mathrm{d}t + \sigma \mathrm{d}W_t, \quad \frac{\mathrm{d}X_t}{X_t} = \alpha \mathrm{d}t + \eta \mathrm{d}\widetilde{W}_t, \quad \mathrm{d}\left\langle W, \widetilde{W}\right\rangle_t = \rho \mathrm{d}t$$

where $(W_t)_{t\geqslant 0}$ and $\left(\widetilde{W}_t\right)_{t\geqslant 0}$ are correlated standard one-dimensional Brownian motions under $\mathbb{P}$, $\mu, \alpha \in \mathbb{R}$, $\sigma, \eta, S_0, X_0 > 0$ and $\rho \in (-1,1)$. The price process of the asset in local currency is represented by

$$(Y_t)_{t\geqslant 0} := (S_t X_t)_{t\geqslant 0}$$

In this problem the agent wants to sell this foreign asset at a time that maximises his payoff, which depends on the price of the asset in local currency. By Itô's formula, we can determine the dynamics of $(Y_t)_{t\geqslant 0}$:

$$\begin{aligned}
\mathrm{d}Y_t &= S_t \mathrm{d}X_t + X_t \mathrm{d}S_t + \mathrm{d}\left\langle S, X\right\rangle_t \\
&= Y_t \left((\mu + \alpha)\, \mathrm{d}t + \left(\sigma \mathrm{d}W_t + \eta \mathrm{d}\widetilde{W}_t\right) + \sigma\eta\rho \mathrm{d}t \right) \\
&= Y_t \left((\mu + \alpha + \sigma\eta\rho)\, \mathrm{d}t + \sqrt{\sigma^2 + 2\sigma\eta\rho + \eta^2} \mathrm{d}Z_t \right) \\
\implies Y_t &= Y_0 \exp\left(\left(\mu + \alpha - \frac{1}{2}\sigma^2 - \frac{1}{2}\eta^2 \right) t + \sqrt{\sigma^2 + 2\sigma\eta\rho + \eta^2} Z_t \right)
\end{aligned}$$

where $(Z_t)_{t\geqslant 0}$ is a standard one-dimensional Brownian motion under $\mathbb{P}$. Therefore $(Y_t)_{t\geqslant 0}$ is another one-dimensional geometric Brownian motion and its scale function is also of the form $s(x) = x^{\beta}$, where

$$\beta = 1 - \frac{2(\mu + \alpha + \sigma\eta\rho)}{\sigma^2 + 2\sigma\eta\rho + \eta^2} \leqslant 1 \iff \mu + \alpha + \sigma\eta\rho \geqslant 0$$

Higher values of μ and α contribute to the convexity of s^{-1}, which is clear since the return of a foreign asset in local currency is better if the return of the foreign asset in its own currency is better and the exchange rate between the currency improves (that is, the same amount of foreign currency can be exchanged for a higher amount of local currency). More interestingly, if ρ is positive, higher values of σ and η contribute to the convexity of s^{-1}, but if ρ is negative, higher values of σ and η decrease the convexity of s^{-1}. In the case of positive ρ, if the foreign asset price increases, the exchange rate improves hence the price in local currency increases. On the other hand, if the foreign asset price decreases, the exchange rate weakens, and this leads to a smaller decrease in price in local currency. Therefore, higher σ or η would either lead to a larger increase or a smaller decrease in local currency, and hence a better performance of the foreign asset. Conversely, if ρ is negative, higher σ or η leads to either a smaller increase or larger decrease in local currency.

2.3.5 Mean-Reverting Processes

Asset prices may exhibit the *mean-reversion* property, which means that it tends to move to its average over time [22]. The most common mean-reverting processes are the *Ornstein-Uhlenbeck* (OU) process and the *Cox-Ingersoll-Ross* (CIR) process [22], respectively given by

$$\mathrm{d}X_t = a\,(b - X_t)\,\mathrm{d}t + \sigma \mathrm{d}W_t$$
$$\mathrm{d}X_t = a\,(b - X_t)\,\mathrm{d}t + \sigma\sqrt{X_t}\mathrm{d}W_t$$

where $a > 0$ is the *speed of mean-reversion*, $b \in \mathbb{R}$ is the *level of mean-reversion* and $\sigma > 0$ is the volatility parameter. We first attempt to compute the scale function for the OU process:

$$s'(x) = \exp\left(-\int_{x_0}^{x}\left(\frac{2a(b-y)}{\sigma^2}\right)\mathrm{d}y\right) = \exp\left[\left(\frac{a}{\sigma^2}\right)x^2 - \left(\frac{2ab}{\sigma^2}\right)x - \left(\left(\frac{a}{\sigma^2}\right)x_0^2 - \left(\frac{2ab}{\sigma^2}\right)x_0\right)\right]$$

Since the scale function is unique up to affine transformation, we can take

$$s'(x) = \exp\left[\left(\frac{a}{\sigma^2}\right)x^2 - \left(\frac{2ab}{\sigma^2}\right)x\right] = \exp\left[\left(\frac{a}{\sigma^2}\right)\left((x-b)^2 - b^2\right)\right] \propto \exp\left[\left(\frac{a}{\sigma^2}\right)\left((x-b)^2\right)\right]$$

For the CIR process, the special case $b = 0$ is equivalent to the CEV model in Section 2.3.2 with $\gamma = \frac{1}{2}$ and $\mu < 0$. We attempt to compute the scale function for a general CIR process:

$$s'(x) = \exp\left(-\int_{x_0}^{x}\left(\frac{2a(b-y)}{\sigma^2 y}\right)\mathrm{d}y\right) \propto x^{-\frac{2ab}{\sigma^2}}\,\mathrm{e}^{\left(\frac{2a}{\sigma^2}\right)x}$$

The expressions for s' for both the OU and CIR processes cannot be integrated explicitly, and therefore the optimal stopping problem cannot be solved explicitly. However, we can deduce some properties about the optimal stopping time based on the shape of s. For both the OU and CIR processes, s' is decreasing on $(-\infty, b]$ and increasing on $[b, \infty)$, and therefore s is concave on $(-\infty, b]$ and convex on $[b, \infty)$. Since s is increasing, s^{-1} is convex on $(-\infty, b]$ and concave on $[b, \infty)$, that is, s^{-1} is reverse S-shaped.

2.4 Discussion

Having reviewed Xu and Zhou's methods in overcoming the difficulties in solving optimal stopping problems involving probability distortion, and considered several variants of the problem, we find that there are some limitations:

Firstly, in order for the optimal stopping problem to be solved explicitly, the scale function of the underlying process needs to be known explicitly. As we saw in Section 2.3.5, this is not the case for OU and CIR processes in general. Furthermore, a scale function is only defined for time homogeneous processes, and therefore the techniques are not applicable for processes with time-dependent deterministic or stochastic coefficients.

Secondly, only optimal stopping problems involving one-dimensional processes can be readily solved, and problems involving multidimensional processes can only be solved in particular case - when we have a one-dimensional process that is a product(s) or quotient(s) of correlated geometric Brownian motions under the same measure $\mathbb{P}$, as we saw in Sections 2.3.3 and 2.3.4.

Thirdly, in this problem it is assumed that we are in an infinite time horizon, and that the underlying process can be stopped at any time. However, in reality, in many financial contexts there is a maturity date T and the process cannot be stopped beyond this time. However, the Azéma-Yor stopping time is a solution to the Skorokhod embedding problem where it is not assumed that stopping times are bounded by T. Ankirchner, Strack and Hobson [18][19] have derived necessary and sufficient conditions for the existence of such stopping times but have not obtained explicit solutions as yet.

Finally, the functions U considered are only state-dependent. If the payoff or utility function was time-dependent as well, for example, if we have, for all $t \geqslant 0$,

$$V(t, S_t) := \mathrm{e}^{-rt} U(S_t)$$

where V is a *discounted* utility function, then the problem becomes considerably more difficult, because probability distortion causes the time consistency structure to be lost.

3 Behavioural Portfolio Selection

This chapter reviews the main results obtained in [7] by Jin and Zhou, who solved the portfolio selection problem for an individual with CPT preferences, in a complete market consisting of assets having general Itô price processes. Solving this problem involves determining the optimal wealth position, and then the solution is the portfolio replicating this position. The optimal terminal wealth position that was derived has a simple structure differentiating between two types of states: favourable and unfavourable, and has a straight forward interpretation, that of a gambling policy betting on favourable states of the economy, and accepting a fixed known loss in the case of an unfavourable state.

Zhang, Jin and Zhou then followed up on their approach to behavioural portfolio selection in [8] by introducing an additional requirement that losses should be bounded by a known constant. The resulting optimal terminal wealth position is similar to the one previously obtained, the only difference being that it distinguishes between three states: good, intermediate and bad states, where a gambling policy bets on good states, a constant moderate loss in intermediate states and the maximal loss in the bad state.

3.1 Formulation of Problem

Assume filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geqslant 0}, \mathbb{P})$ which supports a standard $\{\mathcal{F}_t\}_{t\geqslant 0}$-adapted m-dimensional Brownian motion $(W_t)_{t\geqslant 0} = \left(\left(W_t^1, ..., W_t^m\right)^\intercal\right)_{t\geqslant 0}$ Assume that the price process $(B_t)_{t\geqslant 0}$ of the bank account evolves according to

$$\mathrm{d}B_t = r_t B_t \mathrm{d}t, \quad B_0 = b > 0 \iff B_t = b\exp\left(\int_0^t r_s \mathrm{d}s\right)$$

where the $\{\mathcal{F}_t\}_{t\geqslant 0}$-adapted real-valued one-dimensional stochastic process $(r_t)_{t\geqslant 0}$ is the short-term interest rate satisfying $\int_0^T |r_s|\,\mathrm{d}s < +\infty$ almost surely. We also assume that there are m assets in this economy, whose price processes are denoted by $(S_t^i)_{t\geqslant 0}$, $i = 1, ..., m$ and satisfy the following stochastic differential equations for $t \in [0, T]$:

$$\mathrm{d}S_t^i = S_t^i\left[\mu_t^i \mathrm{d}t + \sum_{j=1}^m \sigma_t^{ij} \mathrm{d}W_t^j\right], \quad S_0^i = s_i > 0$$

where $(\mu_t^i)_{t\geqslant 0}$, $i = 1, ..., m$, and $\left(\sigma_t^{ij}\right)_{t\geqslant 0}$, $i, j = 1, ..., m$ are the *appreciation* and *dispersion* (or volatility) rates respectively, are $\{\mathcal{F}_t\}_{t\geqslant 0}$-adapted and real-valued one-dimensional stochastic processes satisfying

$$\int_0^T \left[\sum_{i=1}^m \left|\mu_s^i\right| + \sum_{i,j=1}^m \left|\sigma_s^{ij}\right|^2\right] \mathrm{d}s < +\infty$$

almost surely. Define the m-dimensional *excess rate of return vector process*

$$b_t := \left(\left(\mu_t^1 - r_t\right), ..., \left(\mu_t^m - r_t\right)\right)^\intercal = \mu_t - r_t \mathbb{1}_m$$

where $(\mu_t)_{t\geqslant 0} := \left(\left(\mu_t^1, ..., \mu_t^m\right)^\intercal\right)_{t\geqslant 0}$ and $\mathbb{1}_m := (1, ..., 1)^\intercal \in \mathbb{R}^m$, and define the *volatility matrix process* $(\sigma_t)_{t\geqslant 0}$ by $\sigma_t := \left(\sigma_t^{ij}\right)_{i,j=1,...,m}$ for $t \in [0, T]$. The following basic assumptions are imposed on the market parameters:

(i) There exists $c \in \mathbb{R}$ such that $\int_0^T |r_s|\,\mathrm{d}s \geqslant c$ almost surely. This ensures that b_t is bounded almost surely for almost every $t \in [0, T]$.

(ii) $\operatorname{rank}(\sigma_t) = m$ for almost every $t \in [0, T]$ almost surely. This ensures that σ_t is almost surely invertible for almost every $t \in [0, T]$.

(iii) There exists an $\mathbb{R}^m$-valued, uniformly bounded, $\{\mathcal{F}_t\}_{t\geqslant 0}$-adapted process $\theta := (\theta_t)_{t\geqslant 0}$ such that $\sigma_t\theta_t = b_t \iff \theta_t = (\sigma_t)^{-1} b_t$ for almost every $t \in [0, T]$ almost surely, where $(\sigma_t)^{-1}$ is the inverse of the matrix σ_t. $(\theta_t)_{t\geqslant 0}$ can be thought of as a "multivariate Sharpe ratio process".

Under these assumptions, the economy is arbitrage-free and complete and therefore there exists a unique equivalent martingale measure $\mathbb{Q}$ (or risk-neutral measure) defined by the Radon-Nikodym derivative

$$\left.\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right|_{\mathcal{F}_t} = \exp\left(-\int_0^t (\theta_s)^\intercal \,\mathrm{d}W_s - \frac{1}{2}\int_0^t |\theta_s|^2 \,\mathrm{d}s\right)$$

for $t \in [0, T]$. Define the *pricing kernel* or *state density price* process $(\rho_t)_{t\in[0,T]}$ by

$$\rho_t := \exp\left(-\int_0^t r_s \mathrm{d}s\right) \frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\bigg|_{\mathcal{F}_t} = \exp\left(-\int_0^t (\theta_s)^\intercal \mathrm{d}W_s - \int_0^t \left[r_s + \frac{1}{2}|\theta_s|^2\right] \mathrm{d}s\right)$$

and denote $\rho := \rho_T$. Since ρ_t is inversely related to $(\theta_s)_{s\in[0,t]}$ for all $t \in [0, T]$, higher values of θ correspond to lower values of ρ, which means that lower values of ρ represent favourable states of the economy and vice versa. Clearly $0 < \rho < +\infty$ almost surely, and by Hölder's inequality,

$$\begin{aligned} 0 < \mathbb{E}_\mathbb{P}(\rho) &\leqslant \left(\mathbb{E}_\mathbb{P}\left[\exp\left(-\int_0^T r_s \mathrm{d}s\right)\right]\right)^{\frac{1}{2}} \left(\mathbb{E}_\mathbb{P}\left[\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\bigg|_{\mathcal{F}_T}\right]\right)^{\frac{1}{2}} \\ &= \left(\mathbb{E}_\mathbb{P}\left[\exp\left(-\int_0^T r_s \mathrm{d}s\right)\right]\right)^{\frac{1}{2}} < +\infty \end{aligned}$$

by assumption (i), where $\mathbb{E}_\mathbb{P}$ denotes expectation under $\mathbb{P}$. The paper imposes the following additional assumption, which, although not essential, avoids undue technicality: *ρ admits no atom* (i.e. it is a continuous random variable), formally, $\mathbb{P}(\rho = a) = 0$ for all $a \in \mathbb{R}$. In particular, this assumption is satisfied when $(r_t)_{t\geqslant 0}$ and $(\theta_t)_{t\geqslant 0}$ are deterministic functions of time (and so $(\sigma_t)_{t\geqslant 0}$ is also deterministic) with $\int_0^T |\theta_s|^2 \mathrm{d}s \neq 0$, in which case ρ is a non-degenerate log-normal random variable, where

$$\log(\rho) \sim \mathcal{N}\left(-\int_0^T \left[r_s + \frac{1}{2}|\theta_s|^2\right] \mathrm{d}s, \int_0^T |\theta_s|^2 \mathrm{d}s\right), \quad \mathbb{E}_\mathbb{P}(\rho) = \exp\left(-\int_0^T r_s \mathrm{d}s\right)$$

The following notation is defined:

$$\begin{aligned} \overline{\rho} &:= \operatorname{ess\,sup} \rho := \sup\{a \in \mathbb{R} : \mathbb{P}(\rho > a) > 0\} \\ \underline{\rho} &:= \operatorname{ess\,inf} \rho := \inf\{a \in \mathbb{R} : \mathbb{P}(\rho < a) > 0\} \end{aligned}$$

which define the end points of the domain of ρ. In the case of log-normal ρ, $\underline{\rho} = 0$ and $\overline{\rho} = \infty$. We now consider an agent with a fixed initial endowment $x_0 \in \mathbb{R}$ (difference between initial wealth and a discounted reference wealth) whose total wealth at time $t \geqslant 0$ is denoted by $X_t \in \mathbb{R}$. Assume that the trading of shares is self-financing and takes place continuously, and that there are no transaction costs. Denoting $(S_t)_{t\geqslant 0} := \left((S_t^1, ..., S_t^m)^\intercal\right)_{t\geqslant 0}$ and the portfolio of the agent by $(\pi_t)_{t\geqslant 0} := \left((\pi_t^1, ..., \pi_t^m)^\intercal\right)_{t\geqslant 0}$ where π_t^i denotes the total value of the agent's wealth in the i^{th} asset at time t, the one-dimensional wealth process $(X_t)_{t\geqslant 0}$ satisfies the following stochastic differential equation:

$$\begin{aligned} \mathrm{d}X_t = r_t (X_t - \pi_t \cdot \mathbb{1}_m) \mathrm{d}t + \pi_t \cdot \mathrm{d}S_t &= (r_t X_t - (r_t \mathbb{1}_m) \pi_t) \mathrm{d}t + \pi_t \cdot (\mu_t \mathrm{d}t + \sigma_t \mathrm{d}W_t) \\ &= (r_t X_t + (\mu_t - r_t \mathbb{1}_m) \cdot \pi_t) \mathrm{d}t + \pi_t \cdot \sigma_t \mathrm{d}W_t \\ &= (r_t X_t + \pi_t \cdot b_t) \mathrm{d}t + \pi_t \cdot \sigma_t \mathrm{d}W_t \end{aligned}$$

Definition 3.1 (Tame Portfolios). A portfolio $(\pi_t)_{t\geqslant 0}$ is said to be *admissible* if it is $\mathbb{R}^m$-valued, $\{\mathcal{F}_t\}_{t\geqslant 0}$-adapted and satisfies the following almost surely:

$$\int_0^T |\sigma_t \cdot \pi_t|^2 \mathrm{d}t < +\infty, \quad \int_0^T |b_t \cdot \pi_t| \mathrm{d}t < +\infty$$

An admissible portfolio $(\pi_t)_{t\geqslant 0}$ is said to be *tame* if the corresponding discounted wealth process $\left(\frac{X_t}{B_t}\right)_{t\geqslant 0} = \left(\exp\left(-\int_0^t r_s \mathrm{d}s\right) X_t\right)_{t\geqslant 0}$ (assuming $B_0 = b = 1$ without loss of generality) is almost surely bounded from below, where the bound may depend on $(\pi_t)_{t\geqslant 0}$.

Under $\mathbb{Q}$, $\left(\exp\left(-\int_0^t r_s \mathrm{d}s\right) X_t\right)_{t\geqslant 0}$ is a martingale, and we can verify this by determining its dynamics. By Girsanov's Theorem,

$$\widetilde{W}_t := W_t + \int_0^t \theta_s \mathrm{d}s$$

is a standard m-dimensional Brownian motion under $\mathbb{Q}$. Then by Itô's formula,

$$\begin{aligned}
\mathrm{d}\left(\exp\left(-\int_0^t r_s \mathrm{d}s\right) X_t\right) &= -r_t \exp\left(-\int_0^t r_s \mathrm{d}s\right) X_t \mathrm{d}t + \exp\left(-\int_0^t r_s \mathrm{d}s\right) \mathrm{d}X_t \\
&= \exp\left(-\int_0^t r_s \mathrm{d}s\right) \left(\pi_t \cdot b_t \mathrm{d}t + \pi_t \cdot \sigma_t \mathrm{d}W_t\right) \\
&= \exp\left(-\int_0^t r_s \mathrm{d}s\right) \left(\pi_t \cdot b_t \mathrm{d}t + \pi_t \cdot \sigma_t \left(\mathrm{d}\widetilde{W}_t - \theta_t \mathrm{d}t\right)\right) \\
&= \exp\left(-\int_0^t r_s \mathrm{d}s\right) \left(\pi_t \cdot b_t \mathrm{d}t + \pi_t \cdot \sigma_t \left(\mathrm{d}\widetilde{W}_t - (\sigma_t)^{-1} b_t \mathrm{d}t\right)\right) \\
&= \exp\left(-\int_0^t r_s \mathrm{d}s\right) \pi_t \cdot \sigma_t \mathrm{d}\widetilde{W}_t
\end{aligned}$$

$$\begin{aligned}
\implies \exp\left(-\int_0^t r_s \mathrm{d}s\right) X_t &= x_0 + \int_0^t \exp\left(-\int_0^s r_u \mathrm{d}u\right) \pi_s \cdot \sigma_s \mathrm{d}\widetilde{W}_s \\
\implies X_t &= x_0 \exp\left(\int_0^t r_s \mathrm{d}s\right) + \exp\left(\int_0^t r_s \mathrm{d}s\right) \int_0^t \exp\left(-\int_0^s r_u \mathrm{d}u\right) \pi_s \cdot \sigma_s \mathrm{d}\widetilde{W}_s
\end{aligned}$$

Since we assume $(r_t)_{t\geqslant 0}$ and $(\sigma_t)_{t\geqslant 0}$ are deterministic, it is clear that $\left(\exp\left(-\int_0^t r_s \mathrm{d}s\right) X_t\right)_{t\geqslant 0}$ is a $\mathbb{Q}$-martingale with constant expectation x_0 under $\mathbb{Q}$.

Proposition 3.2. *For any $\mathcal{F}_T$-measurable random variable ξ that is almost surely bounded from below and satisfies $\mathbb{E}_{\mathbb{P}}[\rho\xi] = x_0$, there exists a tame admissible portfolio $(\pi_t)_{t\geqslant 0}$ such that the corresponding wealth process $(X_t)_{t\geqslant 0}$ satisfies $X_T = \xi$.*

Proof. By the property of the Radon-Nikodym derivative,

$$\mathbb{E}_{\mathbb{P}}[\rho\xi] = \mathbb{E}_{\mathbb{Q}}\left[\exp\left(-\int_0^T r_s \mathrm{d}s\right)\xi\right] = x_0$$

where $\mathbb{E}_{\mathbb{Q}}$ denotes expectation under $\mathbb{Q}$. Let $(X_t)_{t\geqslant 0}$ be the wealth process satisfying $X_T = \xi$, then $\left(\exp\left(-\int_0^t r_s \mathrm{d}s\right) X_t\right)_{t\geqslant 0}$ is a $\mathbb{Q}$-martingale and therefore by the martingale representation theorem, there exists a predictable process $(H_t)_{t\geqslant 0}$ (where $H_t \in \mathbb{R}^m$ for all t) such that

$$\exp\left(-\int_0^T r_s \mathrm{d}s\right) X_T = \mathbb{E}_{\mathbb{Q}}\left[\exp\left(-\int_0^T r_s \mathrm{d}s\right) X_T\right] + \int_0^T H_s \cdot \mathrm{d}\widetilde{W}_s = x_0 + \int_0^T H_s \cdot \mathrm{d}\widetilde{W}_s$$

where $\left(\widetilde{W}_t\right)_{t\geqslant 0}$ is a standard m-dimensional Brownian motion under $\mathbb{Q}$. Then by the equation (...) above, we can take $(\pi_t)_{t\geqslant 0}$ satisfying

$$\exp\left(-\int_0^t r_s \mathrm{d}s\right) \pi_t \cdot \sigma_t = H_t \iff \pi_t = \exp\left(\int_0^t r_s \mathrm{d}s\right) H_t \cdot (\sigma_t)^{-1}$$

for all $t \in [0, T]$. □

In a conventional portfolio selection problem, the agent wants to find the optimal tame admissible portfolio $(\pi_t)_{t\geqslant 0}$ maximising the expected utility of her terminal wealth $X_T =: X$. Assume that her utility function u is S-shaped, i.e. concave on $\mathbb{R}_{\geqslant 0}$ and convex on $\mathbb{R}_{<0}$ and define

$$u := u_+ - u_-$$

where $u_+ : \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ and $u_- : \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ are concave functions associated with gains and losses respectively. The objective function in this maximisation problem is

$$\begin{aligned}\mathbb{E}_{\mathbb{P}}\left[u\left(X\right)\right] &= \mathbb{E}_{\mathbb{P}}\left[u_+\left(X^+\right)\right] - \mathbb{E}_{\mathbb{P}}\left[u_-\left(X^-\right)\right] \\ &= \int_0^\infty \mathbb{P}\left(u_+\left(X^+\right) > x\right) \mathrm{d}x - \int_0^\infty \mathbb{P}\left(u_-\left(X^-\right) > x\right) \mathrm{d}x\end{aligned}$$

In behavioural portfolio selection, where we work within the CPT framework of Tversky and Kahneman, in which probability distortion is involved, the objective function becomes a non-linear Choquet integral, a generalisation of expected utility. Assume that probabilities associated with random variable $u_+\left(X^+\right)$ are distorted with the function $w_+ : [0,1] \to [0,1]$ and those associated with $u_-\left(X^-\right)$ are distorted with the function $w_- : [0,1] \to [0,1]$, where X^+ and X^- denote the (almost surely positive) positive and negative parts of $X = X^+ - X^-$ respectively. Then the objective function is

$$V\left(X\right) := V_+\left(X^+\right) - V_-\left(X^-\right) := \int_0^\infty w_+\left(\mathbb{P}\left(u_+\left(X^+\right) > x\right)\right) \mathrm{d}x - \int_0^\infty w_-\left(\mathbb{P}\left(u_-\left(X^-\right) > x\right)\right) \mathrm{d}x$$

The optimisation problem is:

$$\begin{aligned}&\underset{(\pi_t)_{t\geqslant 0}}{\text{maximize}} && V\left(X\right) \\ &\text{subject to} && \mathrm{d}X_t = \left(r_t X_t + \pi_t \cdot b_t\right) \mathrm{d}t + \pi_t \cdot \sigma_t \mathrm{d}W_t \text{ (self-financing condition)}, \\ &&& (\pi_t)_{t\geqslant 0} \text{ is admissible and tame}\end{aligned}$$

However, by the proposition, we only need to solve the following optimisation problem:

$$\begin{aligned}&\underset{X}{\text{maximize}} && V\left(X\right) \\ &\text{subject to} && \mathbb{E}_{\mathbb{P}}\left[\rho X\right] = x_0, \\ &&& X \text{ is an almost surely lower bounded } \mathcal{F}_T\text{-measurable random variable}\end{aligned}$$

The optimal portfolio $(\pi_t^*)_{t\geqslant 0}$ is then the one replicating X^*, the solution to the above problem. As mentioned in Section 1.3, dynamic programming principles are inapplicable under probability distortion due to the loss of time conistency. However, since this is a maximisation problem that only involves the terminal wealth, this difficulty is avoided.

3.2 Solving the Problem

Jin and Zhou proved in [7] that a probability distortion on losses is necessary for well-posedness of the problem, that is, the objective function cannot grow arbitrarily large, and discussions on well-posedness can be found in [7]. We present the methods used to solve well-posed problems, which are split into its positive and negative part problem, by observing the following:

(i) Maximising $\int_0^\infty w_+ \left(\mathbb{P}\left(u_+\left(X^+\right) > x\right)\right) \mathrm{d}x - \int_0^\infty w_- \left(\mathbb{P}\left(u_-\left(X^-\right) > x\right)\right) \mathrm{d}x$ is equivalent to simultaneously maximising $\int_0^\infty w_+ \left(\mathbb{P}\left(u_+\left(X^+\right) > x\right)\right) \mathrm{d}x$ and minimising $\int_0^\infty w_- \left(\mathbb{P}\left(u_-\left(X^-\right) > x\right)\right) \mathrm{d}x$.

(ii) The constraint $\mathbb{E}_{\mathbb{P}}\left[\rho X\right] = \mathbb{E}_{\mathbb{P}}\left[\rho X^+\right] - \mathbb{E}_{\mathbb{P}}\left[\rho X^-\right] = x_0$ is equivalent to $\mathbb{E}_{\mathbb{P}}\left[\rho X^+\right] = x_+$ and $\mathbb{E}_{\mathbb{P}}\left[\rho X^-\right] = x_+ - x_0$, where $x_+ \geqslant 0$.

(iii) X is almost surely lower bounded is equivalent to X^- being almost surely upper bounded.

This problem is similar to the optimal stopping problem in that the objective function is of the same form, but the main difference is the presence of the pricing kernel in the constraint. However, we will solve this and see that the solution is not substantially different from that in the optimal stopping problem.

3.2.1 Positive Part Problem

$$\begin{aligned}
\underset{X^+}{\text{maximize}} \quad & V_+\left(X^+\right) := \int_0^\infty w_+\left(\mathbb{P}\left(u_+\left(X^+\right) > x\right)\right) \mathrm{d}x \\
\text{subject to} \quad & \mathbb{E}_{\mathbb{P}}\left[\rho X^+\right] = x_+, \\
& X^+ \geqslant 0 \quad a.s., \\
& X^+ = 0 \quad a.s. \quad \text{on} \quad A^{\complement} := \{X \geqslant 0\}^{\complement} := \{X < 0\}
\end{aligned}$$

For any random variable Y with quantile function G_Y,

$$Y \geqslant 0 \quad a.s. \iff G_Y(x) \geqslant 0$$

for all $x \in [0,1]$. Define the probability $p := \mathbb{P}\left(X \geqslant 0\right)$. Then, $X^+ = 0$ almost surely on $A^{\complement} := \{X \geqslant 0\}^{\complement} = \{X < 0\}$ is equivalent to $X^+\left(\omega\right) = 0$ for all $\omega \in \{X < 0\}$, which is equivalent to $G_{X^+}\left[U^+\left(\omega\right)\right] = 0$ for all $\omega \in \{X < 0\}$, where $U^+ := F_{X^+}\left(X^+\right) \sim \mathrm{U}[0,1]$, where F_{X^+} and G_{X^+} are the cumulative distribution function and quantile function of X^+ respectively. Since

$$\{X < 0\} = \{G_X(U) < 0\} = \{U < F_X(0)\} = \{U < 1 - p\}$$

where $U := F_{X^+}\left(X^+\right) \sim \mathrm{U}[0,1]$ and F_X and G_X are the cumulative distribution function and quantile function of X respectively,

$$\mathbb{P}\left(X < 0\right) = 1 - p = \mathbb{P}\left(U < 1 - p\right) = \ell\left([0, 1 - p]\right)$$

where ℓ denotes the Lebesgue measure on $[0,1]$. Therefore, $G_{X^+}(x) = 0$ for all $x \in [0, 1-p]$. Similarly, $X^- = 0$ almost surely on $A = \{X \geqslant 0\}$ is equivalent to $G_{X^-}(x) = 0$ for all $x \in [0, p]$. Then A^* is related to p^* via $p^* = \mathbb{P}(A^*) = \mathbb{P}(X^* \geqslant 0)$. By writing the objective function in terms of its quantile formluation and the fact that

$$G_{X^+}(x) = 0 \implies u_+(G_{X^+}(x)) = u_+(0) = 0$$

the positive part problem becomes

$$\begin{aligned}
\underset{G_{X^+}}{\text{maximize}} \quad & \mathbb{E}_{\mathbb{P}}\left[u_+(G_{X^+}(U))\, w_+'(1-U)\right] = \int_{1-p}^{1} u_+(G_{X^+}(x))\, w_+'(1-x)\mathrm{d}x \\
\text{subject to} \quad & \mathbb{E}_{\mathbb{P}}\left[\rho X^+\right] = x_+, \\
& G_{X^+}(x) \geqslant 0 \quad \forall x \in [0,1], \\
& G_{X^+}(x) = 0 \quad \forall x \in [0, 1-p]
\end{aligned}$$

If $x_+ = 0$, the only feasible (and hence optimal) solution is $(X^+)^* = 0$ almost surely, in other words, $G_{(X^+)^*}(x) = 0$ for all $x \in [0,1]$. Assuming $x_+ > 0$, the following lemma characterises an optimal solution $(X^+)^*$, given in terms of its quantile function and ρ. In particular, $(X^+)^*$ must be anti-comonotonic with ρ.

Lemma 3.3. *An optimal solution $(X^+)^*$ to the above problem with distribution function and quantile function $F_{(X^+)^*}$ and $G_{(X^+)^*}$ respectively must satisfy*

$$(X^+)^* = F^{-1}_{(X^+)^*}(1 - F_\rho(\rho)) = G_{(X^+)^*}(1 - F_\rho(\rho)) \quad a.s.$$

where F_ρ is the distribution function of ρ.

Proof. Let $\overline{X}^+ := G_{(X^+)^*}(1 - F_\rho(\rho))$. Since $1 - F_\rho(\rho) \sim \mathrm{U}(0,1)$, $\overline{X}^+$ has the same distribution as $(X^+)^*$, and therefore $F_{\overline{X}^+} \equiv F_{(X^+)^*}$ and $G_{\overline{X}^+} \equiv G_{(X^+)^*}$. Furthermore,

$$\mathbb{E}\left[\rho \overline{X}^+\right] = \mathbb{E}\left[\rho G_{(X^+)^*}(1 - F_\rho(\rho))\right] > 0$$

since ρ is a strictly positive random variable. If $(X^+)^* = \overline{X}^+$ almost surely is not true, then by the uniqueness result in (theorem),

$$\mathbb{E}\left[\rho \overline{X}^+\right] < \mathbb{E}\left[\rho\,(X^+)^*\right] = x_+$$

Define $\widehat{X}^+ := k\overline{X}^+$, where $k := \frac{x_+}{\mathbb{E}\left[\rho \overline{X}^+\right]} > 1$. Then $\mathbb{E}\left[\rho \widehat{X}^+\right] = x_+$ (i.e. $\widehat{X}^+$ is feasible). We also have $\widehat{X}^+ > \overline{X}^+$ almost surely, and therefore $G_{\widehat{X}^+}(x) > G_{\overline{X}^+}(x)$ for all $x \in [0,1]$. Since u_+ is an increasing function,

$$\begin{aligned}
\int_{1-p}^{1} u_+(G_{\widehat{X}^+}(x))\, w_+'(1-x)\mathrm{d}x &> \int_{1-p}^{1} u_+\left(G_{\overline{X}^+}(x)\right) w_+'(1-x)\mathrm{d}x \\
&= \int_{1-p}^{1} u_+\left(G_{(X^+)^*}(x)\right) w_+'(1-x)\mathrm{d}x
\end{aligned}$$

which contradicts the optimality of $(X^+)^*$. □

By the lemma and the fact that $F_\rho(\rho)$ is uniformly distributed on $[0,1]$, and that ρ is equal in distribution to $G_\rho(U)$, where $U \sim \mathrm{U}[0,1]$, the left hand side of the constraint becomes

$$\mathbb{E}_{\mathbb{P}}\left[\rho X^+\right] = \mathbb{E}_{\mathbb{P}}\left[G_\rho(U) G_{X^+}(1-U)\right] = \int_0^1 G_\rho(y) G_{X^+}(1-y)\,\mathrm{d}y = \int_0^1 G_\rho(1-x) G_{X^+}(x)\,\mathrm{d}x$$

Therefore, the problem that we will solve is

$$\begin{aligned}
\underset{G_{X^+}}{\text{maximize}} \quad & \mathbb{E}_{\mathbb{P}}\left[u_+\left(G_{X^+}(U)\right) w_+'(1-U)\right] = \int_{1-p}^1 u_+\left(G_{X^+}(x)\right) w_+'(1-x)\mathrm{d}x \\
\text{subject to} \quad & \mathbb{E}_{\mathbb{P}}\left[G_\rho(1-U) G_{X^+}(U)\right] = \int_{1-p}^1 G_\rho(1-x) G_{X^+}(x)\,\mathrm{d}x = x_+, \\
& G_{X^+}(x) \geqslant 0 \quad \forall x \in [0,1], \\
& G_{X^+}(x) = 0 \quad \forall x \in [0, 1-p]
\end{aligned}$$

Since u_+ is a concave function, the objective function which we are maximising over G_{X^+} is a concave functional of G_{X^+}. Furthermore, the left hand side of the first constraint (an equality constraint) is an affine functional of G_{X^+}. Therefore the problem is a convex optimisation problem and we may use the Lagrange multiplier method and solve the following unconstrained convex optimisation problem for a given $\lambda \geqslant 0$:

$$\underset{G_{X^+}}{\text{maximize}} \quad \int_{1-p}^1 u_+\left(G_{X^+}(x)\right) w_+'(1-x)\mathrm{d}x + \lambda\left(x_+ - \int_{1-p}^1 G_\rho(1-x) G_{X^+}(x)\,\mathrm{d}x\right)$$

By monotonicity of expectation, it is sufficient to maximise the integrand, therefore we may solve the following unconstrained convex optimisation problem for each x:

$$\underset{G_{X^+}}{\text{maximize}} \quad \mathcal{L}\left(G_{X^+}(x), \lambda\right) := \left(u_+\left(G_{X^+}(x)\right) w_+'(1-x)\right) - \lambda G_\rho(1-x) G_{X^+}(x)$$

Since we have a convex optimisation problem, the following First Order Conditions are both necessary and sufficient:

$$\begin{aligned}
\frac{\partial \mathcal{L}\left(G_{X^+}, \lambda\right)}{\partial G_{X^+}} &= u_+'\left(G_{X^+}(x)\right) w_+'(1-x) - \lambda G_\rho(1-x) = 0 \\
\iff G_{X^+}(x) &= \left(u_+'\right)^{-1}\left(\frac{\lambda G_\rho(1-x)}{w_+'(1-x)}\right)
\end{aligned}$$

We then determine λ by substituting the above expression for G_{X^+} into the original constraint:

$$\int_{1-p}^1 G_\rho(1-x)\left(u_+'\right)^{-1}\left(\frac{\lambda G_\rho(1-x)}{w_+'(1-x)}\right)\mathrm{d}x = x_+$$

Since u_+ is concave, u_+' as well as $\left(u_+'\right)^{-1}$ are decreasing functions. Therefore, we require $\left(\frac{G_\rho(1-x)}{w_+'(1-x)}\right)$ to be non-increasing in $x \in (0,1]$ (i.e. $\left(\frac{G_\rho(y)}{w_+'(y)}\right)$ should be non-decreasing in $y \in [0,1)$), in order for $G_{X^+}(x)$ to be non-decreasing in $x \in (0,1]$ (to satisfy the property of being a quantile function). Jin and Zhou [7] derived conditions required for $\left(\frac{G_\rho(x)}{w_+'(x)}\right)$ to be non-decreasing in

$x \in [0,1)$, and also derived conditions under which a reverse S-shaped w_+ satisfies this monotonicity property:

Monotonicity of G_ρ/w'_+

By directly differentiating, we may observe that the monotonicity condition is satisfied if:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{G_\rho(x)}{w'_+(x)}\right) = \frac{w'_+(x)\frac{\mathrm{d}}{\mathrm{d}x}\left(G_\rho(x)\right) - w''_+(x)G_\rho(x)}{\left(w'_+(x)\right)^2} \geqslant 0$$
$$\implies \left(-\frac{w''_+(x)}{w'_+(x)}\right) \geqslant \left(-\frac{\frac{\mathrm{d}}{\mathrm{d}x}\left(G_\rho(x)\right)}{G_\rho(x)}\right) \iff \frac{\mathrm{d}}{\mathrm{d}x}\left(\log\left(w'_+(x)\right)\right) \leqslant \frac{\mathrm{d}}{\mathrm{d}x}\left(\log\left(G_\rho(x)\right)\right)$$

where $-\frac{w''_+(x)}{w'_+(x)}$ can be regarded as the *Arrow-Pratt measure of absolute risk aversion* (ARA) [7] of w_+. Since G_ρ is an increasing function (since it is the inverse of increasing function F_ρ), $-\frac{\frac{\mathrm{d}}{\mathrm{d}x}(G_\rho(x))}{G_\rho(x)} \leqslant 0$ for all $x > 0$. If w_+ is reverse S-shaped, that is, concave on $[0,q]$ and convex on $[q,1]$ for some $q \in (0,1)$, then the inequality is satisfied for all $x \in [0,q]$, since the ARA is non-negative for concave function (as the second order derivative is negative). For the inequality to be satisfied for $x \in [q,1]$, w_+ should not be "too convex" (or too steep), that is, its positive second order derivative should not be too high (resulting in the ARA not being too negative). To summarise, the ARA of w_+ should be sufficiently high. Alternatively, we can also consider the analysis in [7], which begins by observing that $\frac{G_\rho(x)}{w'_+(x)}$ being non-decreasing in x is equivalent to $\frac{w'_+(F_\rho(x))}{x}$ being non-increasing in $x > 0$. Define

$$H(x) := w_+\left(F_\rho(x)\right), \quad I(x) := \frac{w'_+\left(F_\rho(x)\right)}{x} = \frac{H'(x)}{xF'_\rho(x)}$$

Then $I(x)$ is non-increasing in x if and only if, for all $x > 0$

$$I'(x) = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{H'(x)}{xF'_\rho(x)}\right) = \frac{xF'_\rho(x)H''(x) - H'(x)\left(xF''_\rho(x) + F'_\rho(x)\right)}{x^2\left(F'_\rho(x)\right)^2}$$
$$= \frac{xH''(x)F'_\rho(x) - xH'(x)F''_\rho(x) - H'(x)F'_\rho(x)}{x^2\left(F'_\rho(x)\right)^2} \leqslant 0$$
$$\iff \frac{xH''(x)}{H'(x)} - \frac{xF''_\rho(x)}{F'_\rho(x)} = \left(-\frac{xF''_\rho(x)}{F'_\rho(x)}\right) - \left(-\frac{xH''(x)}{H'(x)}\right) \leqslant 1$$

where $-\frac{xF''_\rho(x)}{F'_\rho(x)}$ and $-\frac{xH''(x)}{H'(x)}$ can be regarded as the *Arrow-Pratt measure of relative risk aversion* (RRA) [7] of F_ρ and $H = w_+ \circ F_\rho$ respectively. The RRA of a *concave* utility function u (corresponding to a risk averse attitude) is non-negative (since the utility function is non-decreasing and by concavity its second order derivative is negative). Therefore, higher risk aversion corresponds to a more negative second order derivative and higher RRA. Denoting

$$\mu_\rho := -\int_0^T \left[r_s + \frac{1}{2}\left|\theta_s\right|^2\right] \mathrm{d}s, \quad \sigma_\rho := \int_0^T \left|\theta_s\right|^2 \mathrm{d}s$$

and denoting the density of ρ by $f_\rho := F'_\rho$, we have

$$\begin{aligned} F''_\rho(x) = \frac{\mathrm{d}}{\mathrm{d}x}\left(f_\rho(x)\right) &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\left(\frac{1}{\sigma_\rho x}\right)\phi\left(\frac{\log x - \mu_\rho}{\sigma_\rho}\right)\right) \\ &= \frac{-\left(\sigma_\rho x\right)\left(\frac{\log x-\mu_\rho}{\sigma_\rho}\right)\phi\left(\frac{\log x-\mu_\rho}{\sigma_\rho}\right)\left(\frac{1}{\sigma_\rho x}\right) - \sigma_\rho\phi\left(\frac{\log x-\mu_\rho}{\sigma_\rho}\right)}{\sigma_\rho^2 x^2} \\ &= \frac{-\phi\left(\frac{\log x-\mu_\rho}{\sigma_\rho}\right)\left[\left(\frac{\log x-\mu_\rho}{\sigma_\rho}\right)+\sigma_\rho\right]}{\sigma_\rho^2 x^2} \leqslant 0 \end{aligned}$$

where ϕ denotes the standard normal density. Therefore F_ρ is concave and has a non-negative RRA given by

$$-\frac{xF''_\rho(x)}{F'_\rho(x)} = \frac{\phi\left(\frac{\log x-\mu_\rho}{\sigma_\rho}\right)\left[\left(\frac{\log x-\mu_\rho}{\sigma_\rho}\right)+\sigma_\rho\right]}{\sigma_\rho^2 x\left(\frac{1}{\sigma_\rho x}\right)\phi\left(\frac{\log x-\mu_\rho}{\sigma_\rho}\right)} = 1 + \left(\frac{\log x - \mu_\rho}{\sigma_\rho^2}\right)$$

Therefore, the inequality becomes, for all $x > 0$,

$$-\frac{xH''(x)}{H'(x)} \geqslant \left(\frac{\log x - \mu_\rho}{\sigma_\rho^2}\right)$$

Therefore, H, which is by definition the *distorted distribution function* of ρ, should have a sufficiently high RRA. An alternative interpretation is that the distortion function w_+ should not decrease the RRA of F_ρ by more than 1. Finally, [7] derives a condition that ensures the monotonicity of $\frac{G_\rho}{w'_+}$ when w_+ is reverse S-shaped: first define the function

$$g(x) := \frac{xH''(x)}{H'(x)} - \frac{xF''_\rho(x)}{F'_\rho(x)}$$

and by definition of H and the inverse function theorem we have

$$\begin{aligned} w_+(x) = H\left(G_\rho(x)\right) &\implies w'_+(x) = \frac{H'\left(G_\rho(x)\right)}{F'_\rho\left(G_\rho(x)\right)} \\ &\implies w''_+(x) = \frac{H''\left(G_\rho(x)\right)F'_\rho\left(G_\rho(x)\right) - H'\left(G_\rho(x)\right)F''_\rho\left(G_\rho(x)\right)}{\left(F'_\rho\left(G_\rho(x)\right)\right)^3} \\ &\implies w''_+\left(F_\rho(x)\right) = \frac{H''(x)F'_\rho(x) - H'(x)F''_\rho(x)}{\left(F'_\rho(x)\right)^3} = \left(\frac{H'(x)}{x\left(F'_\rho(x)\right)^2}\right)g(x) \end{aligned}$$

Since $H = w_+ \circ F_\rho$ is increasing (as it is a composition of increasing functions), $H'(x) \geqslant 0$ for all $x > 0$. If we assume w_+ has a reverse S-shape, $w''_+(x)$ changes from negative to positive when x goes from 0 to 1, and therefore by the equation above $g(x)$ changes from negative to positive when x goes from 0 to 1, and by the earlier derivatiion we require $g(x) \leqslant 1$ for all $x > 0$. Therefore, if w_+ is reverse S-shaped, monotonicity of $\frac{G_\rho}{w'_+}$ is satisfied if there exists $c > 0$ such that

$$g(x) \leqslant 0 \quad \forall x \in (0, c], \quad 0 \leqslant g(x) \leqslant 1 \quad \forall x \in (c, +\infty)$$

An example of a distortion function satisfying this is constructed in Example 6.1 [7] for interested readers. The assumption of monotonicity of $\frac{G_\rho}{w'_+}$ is imposed in [7], however, this is restrictive, especially if we assume that w_+ is reverse S-shaped. [15] showed that this assumption is not satisfied for well known distortion functions such as the one proposed by Tversky and Kahneman.

Therefore, we consider a truncation similar to that in the optimal stopping under probability distortion problem (refer to Remark 2.15) that does not require this assumption to be satisfied. We first verify that $x \mapsto \left(u'_+\right)^{-1}\left(\frac{\lambda G_\rho(1-x)}{w'_+(1-x)}\right)$ has a "U-shape" which is required for the truncation method described in Remark 2.15 to be applicable. First assume w_+ is reverse S-shaped and concave on $[0, 1-q]$ and convex on $[1-q, 1]$ for some $q \in (0, 1)$, then w''_+ is negative on $[0, 1-q]$ and positive on $[1-q, 1]$. Then, examining the first order derivative computed at the beginning of the monotonicity discussion, $\frac{G_\rho}{w'_+}$ is non-decreasing on $[0, 1-q]$ for any reverse S-shaped w_+ (with inflection point $1-q$). Assume the monotonicity condition fails and $\frac{G_\rho}{w'_+}$ is non-increasing on $[1-q, 1]$. Then $x \mapsto \frac{G_\rho(1-x)}{w'_+(1-x)}$ is non-increasing on $[q, 1]$ and non-decreasing on $[0, q]$. Therefore, $x \mapsto \left(u'_+\right)^{-1}\left(\frac{\lambda G_\rho(1-x)}{w'_+(1-x)}\right)$ is non-increasing on $[0, q]$ and non-decreasing on $[q, 1]$, and indeed has the required "U-shape".

The quantile function after truncation is given by

$$G_{X^+}(x) = \alpha \mathbb{1}_{(1-p,\beta]}(x) + \left(\alpha \vee \left(u'_+\right)^{-1}\left(\frac{\lambda G_\rho(1-x)}{w'_+(1-x)}\right)\right) \mathbb{1}_{(\beta,1)}(x)$$

If $\left(\frac{G_\rho(y)}{w'_+(y)}\right)$ is non-decreasing in y, we set $\alpha := 0$, $\beta := 1-p$. λ is then determined by the following equation:

$$\alpha \int_{1-p}^{\beta} G_\rho(1-x)\mathrm{d}x + \int_\beta^1 G_\rho(1-x)\left(\alpha \vee \left(u'_+\right)^{-1}\left(\frac{\lambda G_\rho(1-x)}{w'_+(1-x)}\right)\right)\mathrm{d}x = x_+$$

The left hand side of the above equation is monotone in λ, hence there is a unique λ satisfying the equation. The objective function in the positive part problem becomes

$$u_+(\alpha)\left[w_+(p) - w_+(1-\beta)\right] + \int_\beta^1 u_+\left(\alpha \vee \left(u'_+\right)^{-1}\left(\frac{\lambda G_\rho(1-x)}{w'_+(1-x)}\right)\right) w'_+(1-x)\mathrm{d}x$$

3.2.2 Negative Part Problem

$$\begin{aligned}
\underset{X^-}{\text{minimize}} \quad & V_-\left(X^-\right) := \int_0^\infty w_-\left(\mathbb{P}\left(u_-\left(X^-\right) > x\right)\right) \mathrm{d}x \\
\text{subject to} \quad & \mathbb{E}_\mathbb{P}\left[\rho X^-\right] = x_+ - x_0, \\
& X^- \geqslant 0 \quad a.s., \\
& X^- = 0 \quad a.s. \quad \text{on} \quad A := \{X \geqslant 0\}, \\
& X^- \ \text{ is upper bounded } \ a.s.
\end{aligned}$$

By writing the objective function in terms of its quantile formluation and the fact that

$$G_{X^-}(x) = 0 \implies u_-\left(G_{X^-}(x)\right) = u_-(0) = 0$$

the negative part problem becomes

$$\begin{aligned}
\underset{G_{X^-}}{\text{minimize}} \quad & \mathbb{E}_\mathbb{P}\left[u_-\left(G_{X^-}(U)\right) w_-'\left(1-U\right)\right] = \int_p^1 u_-\left(G_{X^-}(x)\right) w_-'(1-x) \mathrm{d}x \\
\text{subject to} \quad & \mathbb{E}_\mathbb{P}\left[\rho X^-\right] = x_+ - x_0, \\
& 0 \leqslant G_{X^-}(x) < +\infty \quad \forall x \in [0,1], \\
& G_{X^-}(x) = 0 \quad \forall x \in [0,p]
\end{aligned}$$

If $x_+ - x_0 = 0$, the only feasible (and hence optimal) solution is $(X^-)^* = 0$ almost surely, in other words, $G_{(X^-)^*}(x) = 0$ for all $x \in [0,1]$. Assuming $x_+ - x_0 > 0$, similarly to that in the positive part problem, the following lemma characterises an optimal solution $(X^-)^*$, given in terms of its quantile function and ρ. In particular, $(X^-)^*$ must be comonotonic with ρ.

Lemma 3.4. *An optimal solution $(X^-)^*$ to the above problem with distribution function and quantile function $F_{(X^-)^*}$ and $G_{(X^-)^*}$ respectively must satisfy*

$$\left(X^-\right)^* = F^{-1}_{(X^-)^*}\left(F_\rho\left(\rho\right)\right) = G_{(X^-)^*}\left(F_\rho\left(\rho\right)\right) \quad a.s.$$

where F_ρ is the distribution function of ρ.

Proof. Let $\overline{X}^- := G_{(X^+)^*}\left(F_\rho\left(\rho\right)\right)$. Since $F_\rho\left(\rho\right) \sim \mathrm{U}(0,1)$, $\overline{X}^-$ has the same distribution as $(X^-)^*$, and therefore $F_{\overline{X}^-} \equiv F_{(X^-)^*}$ and $G_{\overline{X}^-} \equiv G_{(X^-)^*}$. Furthermore,

$$\mathbb{E}\left[\rho\overline{X}^-\right] = \mathbb{E}\left[\rho G_{(X^-)^*}\left(F_\rho\left(\rho\right)\right)\right] > 0$$

since ρ is a strictly positive random variable. If $(X^-)^* = \overline{X}^-$ almost surely is not true, then by the uniqueness result in (theorem),

$$\mathbb{E}\left[\rho\overline{X}^-\right] > \mathbb{E}\left[\rho\left(X^-\right)^*\right] = x_+ - x_0$$

Define $\widehat{X}^- := k\overline{X}^-$, where $k := \frac{x_+ - x_0}{\mathbb{E}\left[\rho\overline{X}^-\right]} < 1$. Then $\mathbb{E}\left[\rho\widehat{X}^-\right] = x_+ - x_0$ (i.e. $\widehat{X}^-$ is feasible). We also have $\widehat{X}^- < \overline{X}^-$ almost surely, and therefore $G_{\widehat{X}^-}(x) < G_{\overline{X}^-}(x)$ for all $x \in [0,1]$. Since u_- is an increasing function,

$$\int_{1-p}^{1} u_- \left(G_{\widehat{X}^-}(x)\right) w_-'(1-x)\mathrm{d}x < \int_{1-p}^{1} u_- \left(G_{\overline{X}^-}(x)\right) w_-'(1-x)\mathrm{d}x$$
$$= \int_{1-p}^{1} u_- \left(G_{(X^-)^*}(x)\right) w_-'(1-x)\mathrm{d}x$$

which contradicts the optimality of $(X^-)^*$. □

By the lemma and the fact that $F_\rho(\rho)$ is uniformly distributed on $[0,1]$, and that ρ is equal in distribution to $G_\rho(U)$, where $U \sim \mathrm{U}[0,1]$, the left hand side of the constraint becomes

$$\mathbb{E}_{\mathbb{P}}\left[\rho X^-\right] = \mathbb{E}_{\mathbb{P}}\left[G_\rho(U) G_{X^-}(U)\right] = \int_0^1 G_\rho(x) G_{X^-}(x)\,\mathrm{d}x$$

Therefore, the problem that we will solve is

$$\begin{aligned}
\underset{G_{X^-}}{\text{minimize}} \quad & \mathbb{E}_{\mathbb{P}}\left[u_-\left(G_{X^-}(U)\right) w_-'(1-U)\right] = \int_p^1 u_-\left(G_{X^-}(x)\right) w_-'(1-x)\mathrm{d}x \\
\text{subject to} \quad & \mathbb{E}_{\mathbb{P}}\left[G_\rho(U) G_{X^-}(U)\right] = \int_p^1 G_\rho(x) G_{X^-}(x)\,\mathrm{d}x = x_+ - x_0, \\
& 0 \leqslant G_{X^-}(x) < +\infty \quad \forall x \in [0,1], \\
& G_{X^-}(x) = 0 \quad \forall x \in [0,p]
\end{aligned}$$

The negative part problem is not a convex optimisation problem, because the objective function which we are minimising is concave in $G_{X^-}(x)$. We obtain "corner point" solutions in the following proposition:

Proposition 3.5. *Assume u_- is strictly concave at 0. Then the optimal solution, if it exists, must be in the following form: for $x \in [0,1)$,*

$$G_{X^-}(x) = q(x_+, p)\,\mathbb{1}_{(p,1)}(x), \quad q(x_+, p) := \frac{x_+ - x_0}{\int_p^1 G_\rho(x)\mathrm{d}x}$$

Proof. Assume $x_+ - x_0 > 0$, otherwise the result holds trivially. If G_{X^-} is an optimal solution, then $G_{X^-} \not\equiv 0$. Fix $y \in (0,1)$ such that $G_{X^-}(y) > 0$. Define:

$$k := \frac{\int_0^1 G_\rho(x) G_{X^-}(x)\mathrm{d}x}{\int_0^y G_\rho(x) G_{X^-}(x)\mathrm{d}x + G_{X^-}(y)\int_y^1 G_\rho(x)\mathrm{d}x} \geqslant 1, \quad \overline{G}(x) := \begin{cases} kG_{X^-}(x), & x \in [0,y] \\ kG_{X^-}(y), & x \in (y,1) \end{cases}$$

Clearly $\overline{G}$ is another quantile function and

$$\int_0^1 G_\rho(x)\overline{G}(x)\,\mathrm{d}x = k\left(\int_0^y G_\rho(x) G_{X^-}(x)\,\mathrm{d}x + G_{X^-}(y)\int_y^1 G_\rho(x)\mathrm{d}x\right) = \int_0^1 G_\rho(x) G_{X^-}(x)\mathrm{d}x$$

and therefore $\overline{G}$ is feasible. We claim that $G_{X^-}(x) = G_{X^-}(y)$ for almost every $x \in (y,1)$. To prove this, we first assume to the contrary that this is not the case, which means $G_{X^-}(x) \geqslant G_{X^-}(y)$ for almost every $x \in (y,1)$ since G_{X^-} is a non-decreasing quantile function. This implies that $k > 1$. Define

$$\lambda := 1 - \frac{1}{k} \in (0,1), \quad \widetilde{G}(x) := \begin{cases} 0, & x \in [0,y] \\ \frac{G_{X^-}(x) - G_{X^-}(y)}{\lambda}, & x \in (y,1) \end{cases}$$
$$= \left(\frac{1}{\lambda}\left[G_{X^-}(x) - G_{X^-}(y)\right]\right) \mathbb{1}_{(y,1)}(x)$$

Then G_{X^-} can be written as a convex combination of $\overline{G}$ and $\widetilde{G}$:

$$\begin{aligned}&(1-\lambda)\overline{G}(x) + \lambda\widetilde{G}(x) \\ &= \frac{1}{k}\left(kG_{X^-}(x)\mathbb{1}_{[0,y]}(x) + kG_{X^-}(y)\mathbb{1}_{(y,1)}(x)\right) + \lambda\left(\frac{1}{\lambda}\left[G_{X^-}(x) - G_{X^-}(y)\right]\right)\mathbb{1}_{(y,1)}(x) = G_{X^-}(x)\end{aligned}$$

By concavity of u_-,

$$\begin{aligned}&\int_0^1 u_-\left(G_{X^-}(x)\right) w_-'(1-x)\mathrm{d}x \\ &\geqslant (1-\lambda)\int_0^1 u_-\left(\overline{G}(x)\right) w_-'(1-x)\mathrm{d}x + \lambda\int_0^1 u_-\left(\widetilde{G}(x)\right) w_-'(1-x)\mathrm{d}x \\ &= \int_0^1 \left[(1-\lambda)u_-\left(\overline{G}(x)\right) + \lambda u_-\left(\widetilde{G}(x)\right)\right] w_-'(1-x)\mathrm{d}x\end{aligned}$$

and equality holds only if

$$u_-\left(G_{X^-}(x)\right) = (1-\lambda)u_-\left(\overline{G}(x)\right) + \lambda u_-\left(\widetilde{G}(x)\right)$$

holds for almost every $x \in (0,1)$. However, since G_{X^-} is an optimal solution to a minimisation problem, equality must hold. However, the equality holding for $x \leqslant y$ contradicts the assumption of strict concavity at 0. Therefore, we must have $G_{X^-}(x) = G_{X^-}(y)$ for almost every $x \in (y,1)$. Then, since $G_{X^-}(x) = 0$ for all $x \in [0,p]$ we must have $G_{X^-}(x) = q\mathbb{1}_{(p,1)}(x)$ for all $x \in [0,1]$, some $q \in \mathbb{R}_{>0}$. By the feasibility of G_{X^-},

$$\int_p^1 G_\rho(x)G_{X^-}(x)\,\mathrm{d}x = q\int_p^1 G_\rho(x)\mathrm{d}x = x_+ - x_0 \implies q = q\left(x_+, p\right) = \frac{x_+ - x_0}{\int_p^1 G_\rho(x)\mathrm{d}x}$$

□

Then the objective function in the negative part problem becomes

$$\int_p^1 u_-\left(\frac{x_+ - x_0}{\int_p^1 G_\rho(x)\mathrm{d}x}\right) w_-'(1-x)\mathrm{d}x = u_-\left(\frac{x_+ - x_0}{\int_p^1 G_\rho(x)\mathrm{d}x}\right) w_-\left(1-p\right)$$

3.2.3 Determining Optimal Parameters

Combining the objective functions in the positive and negative part problems, we have the following joint objective function

$$\begin{aligned}\mathcal{J}\left(x_+, p, \alpha, \beta, \lambda\right) &:= u_+(\alpha)\left[w_+(p) - w_+(1-\beta)\right] \\ &+ \int_\beta^1 u_+\left(\alpha \vee \left(u_+'\right)^{-1}\left(\frac{\lambda G_\rho(1-x)}{w_+'(1-x)}\right)\right) w_+'(1-x)\mathrm{d}x - u_-\left(\frac{x_+ - x_0}{\int_p^1 G_\rho(x)\mathrm{d}x}\right) w_-\left(1-p\right)\end{aligned}$$

Define the following function

$$\mathcal{K}\left(p, \alpha, \beta, \lambda\right) := \alpha \int_{1-p}^{\beta} G_\rho(1-x)\mathrm{d}x + \int_\beta^1 G_\rho(1-x)\left(\alpha \vee \left(u_+'\right)^{-1}\left(\frac{\lambda G_\rho(1-x)}{w_+'(1-x)}\right)\right)\mathrm{d}x$$

Then we have the final optimisation problem:

$$\begin{aligned}\underset{x_+, p, \alpha, \beta, \lambda}{\text{maximize}} \quad & \mathcal{J}\left(x_+, p, \alpha, \beta, \lambda\right) \\ \text{subject to} \quad & \mathcal{K}(p, \alpha, \beta, \lambda) = x_+, \\ & x_+, \alpha, \lambda \geqslant 0, \quad 0 \leqslant p \leqslant 1, \quad 1 - p \leqslant \beta \leqslant 1\end{aligned}$$

Once the optimal parameters $x_+^*, p^*, \alpha^*, \beta^*, \lambda^*$ are found, we have the optimal quantile functions in the positive part and negative part problems respectively given by

$$\begin{aligned}G_{(X^+)^*}(x) &= \alpha^* \mathbb{1}_{(1-p^*, \beta^*]}(x) + \left(\alpha^* \vee \left(u_+'\right)^{-1}\left(\frac{\lambda^* G_\rho(1-x)}{w_+'(1-x)}\right)\right) \mathbb{1}_{(\beta^*, 1)}(x) \\ G_{(X^-)^*}(x) &= \left(\frac{x_+ - x_0}{\int_p^1 G_\rho(x)\mathrm{d}x}\right) \mathbb{1}_{(p^*, 1)}(x)\end{aligned}$$

and the corresponding optimal random variables $(X^+)^*$ and $(X^-)^*$ are given by

$$\begin{aligned}\left(X^+\right)^* &= G_{(X^+)^*}\left(1 - F_\rho\left(\rho\right)\right) \\ &= \alpha^* \mathbb{1}_{(1-p^*, \beta^*]}\left(1 - F_\rho\left(\rho\right)\right) + \left(\alpha^* \vee \left(u_+'\right)^{-1}\left(\frac{\lambda^* G_\rho\left(F_\rho\left(\rho\right)\right)}{w_+'\left(F_\rho\left(\rho\right)\right)}\right)\right) \mathbb{1}_{(\beta^*, 1)}\left(1 - F_\rho\left(\rho\right)\right) \\ &= \alpha^* \mathbb{1}_{[G_\rho(1-\beta^*), G_\rho(p^*))}(\rho) + \left(\alpha^* \vee \left(u_+'\right)^{-1}\left(\frac{\lambda^* \rho}{w_+'(F_\rho\left(\rho\right))}\right)\right) \mathbb{1}_{[0, G_\rho(1-\beta^*))}\left(\rho\right)\end{aligned}$$

$$\begin{aligned}\left(X^-\right)^* = G_{(X^-)^*}\left(F_\rho\left(\rho\right)\right) &= \left(\frac{x_+ - x_0}{\int_p^1 G_\rho(x)\mathrm{d}x}\right) \mathbb{1}_{(p^*, 1)}\left(F_\rho\left(\rho\right)\right) \\ &= \left(\frac{x_+ - x_0}{\int_p^1 G_\rho(x)\mathrm{d}x}\right) \mathbb{1}_{(G_\rho(p^*), \infty)}\left(\rho\right)\end{aligned}$$

and the optimal random variable X^* is

$$\begin{aligned} X^* &= (X^+)^* - (X^-)^* \\ &= \left(\alpha^* \vee \left(u_+'\right)^{-1}\left(\frac{\lambda^*\rho}{w_+'(F_\rho(\rho))}\right)\right)\mathbb{1}_{[0,G_\rho(1-\beta^*))}(\rho) + \alpha^*\mathbb{1}_{[G_\rho(1-\beta^*),G_\rho(p^*))}(\rho) \\ &- \left(\frac{x_+ - x_0}{\int_p^1 G_\rho(x)\mathrm{d}x}\right)\mathbb{1}_{(G_\rho(p^*),\infty)}(\rho) \end{aligned}$$

3.3 Addition of Loss Control

We now present the problem with the addition of loss control as solved in [8], where an investor wants to bound the losses of his terminal wealth position by a constant L. The positive part problem remains the same and the negative part problem becomes

$$\begin{aligned} \underset{X^-}{\text{minimize}} \quad & V_-(X_-) := \int_0^\infty w_-\left(\mathbb{P}\left(u_-\left(X^-\right) > x\right)\right)\mathrm{d}x \\ \text{subject to} \quad & \mathbb{E}_\mathbb{P}\left[\rho X^-\right] = x_+ - x_0, \\ & 0 \leqslant X^- \leqslant L \quad a.s., \\ & X^- = 0 \quad a.s. \quad \text{on} \quad A := \{X \geqslant 0\} \end{aligned}$$

$$\begin{aligned} \underset{G_{X^-}}{\text{minimize}} \quad & \mathbb{E}_\mathbb{P}\left[u_-\left(G_{X^-}(U)\right)w_-'(1-U)\right] = \int_p^1 u_-\left(G_{X^-}(x)\right)w_-'(1-x)\mathrm{d}x \\ \text{subject to} \quad & \mathbb{E}_\mathbb{P}\left[G_\rho(U)G_{X^-}(U)\right] = \int_p^1 G_\rho(x)G_{X^-}(x)\,\mathrm{d}x = x_+ - x_0, \\ & 0 \leqslant G_{X^-}(x) \leqslant L \quad \forall x \in [0,1], \\ & G_{X^-}(x) = 0 \quad \forall x \in [0,p] \end{aligned}$$

The following proposition provides the form of the optimal solution, which consists of three points, $0, q$ and L. In [8], their proof involves a proof by contradiction, first assuming that there are more than three points, and then constructing linear interpolations of quantile function and eventually arriving at a contradiction. We provide an alternative proof which also involves proof by contradiction, but by using a more fundamental approach, and this is adapted from the proof in [10], which proves that there must be at most three distinct points in the optimal quantile function, but in an optimal stopping under probability distortion problem with loss control (where no pricing kernel is involved).

Proposition 3.6. *The optimal solution, if it exists, must be in the following form: for $x \in [0,1)$,*

$$G_{X^-}(x) = q\mathbb{1}_{(p,\gamma)}(x) + L\mathbb{1}_{[\gamma,1)}(x)$$

where

$$q := q\left(x_+, p, \gamma\right) := \frac{x_+ - x_0 - L\int_\gamma^1 G_\rho(x)\mathrm{d}x}{\int_p^\gamma G_\rho(x)\mathrm{d}x} < L$$

Proof. Let G be an optimal solution. We first prove that the optimal solution G is the quantile function of a random variable having a three-point distribution with masses at 0, q and L. Assume to the contrary that the image of G contains more than three distinct elements. Then we can choose $c \in (0,1)$ and $q \in (0, L)$ such that $q \in [G(c), G(c+)]$ such that there exists $x_1 \in (0, c)$ and $x_2 \in (c, 1)$ with $0 < G\left(x_1\right) < q$ and $q < G\left(x_2\right) < L$. Define a convex function $\overline{\eta}_1$ and a concave function $\overline{\eta}_2$ by

$$\overline{\eta}_1(x) = \overline{\eta}_1\left(x; \varepsilon_1\right) := x^{\varepsilon_1}, \quad \overline{\eta}_2(x) = \overline{\eta}_2\left(x; \varepsilon_2\right) := 1 - (1-x)^{\varepsilon_2}$$

with $\varepsilon_1 > 1$ and $\varepsilon_2 > 1$. Note that $\overline{\eta}_1(0) = \overline{\eta}_2(0) = 0$ and $\overline{\eta}_1(1) = \overline{\eta}_2(1) = 1$ and we can define another quantile function $\overline{G}$ via

$$\frac{\overline{G}(x)}{q} := \overline{\eta}_1\left(\frac{G(x)}{q}\right), \quad 0 \leqslant x \leqslant c \qquad \frac{\overline{G}(x) - q}{L - q} := \overline{\eta}_2\left(\frac{G(x) - q}{L - q}\right), \quad c < x < 1$$

$$\Longrightarrow \overline{G}(x) = \begin{cases} q\overline{\eta}_1\left(\frac{G(x)}{q}\right), & 0 \leqslant x \leqslant c \\ q + (L-q)\,\overline{\eta}_2\left(\frac{G(x)-q}{L-q}\right), & c < x < 1 \end{cases}$$

By construction, $\overline{G}(x) \leqslant G(x)$ on $[0, c]$ and $\overline{G}(x) \geqslant G(x)$ on $(c, 1)$. Since $\overline{\eta}_1(x) = x$ and $\overline{\eta}_2(x) = x$ if and only if $x \in \{0, 1\}$, we have that in the first inequality, equality only holds for x such that $G(x) = 0 \iff x = 0$ or $G(x) = q \iff x = c$, and strict inequality holds for $x \in (0, c)$. Similarly for the second inequality, equality only holds for x such that $G(x) = q \iff x = c+$ or $G(x) = L$, and strict inequality holds for all other values of x. Then we can define the quantities

$$\Delta_1\left(\varepsilon_1\right) := \int_0^c G_\rho(x)G(x)\mathrm{d}x - \int_0^c G_\rho(x)\overline{G}(x)\mathrm{d}x > 0$$
$$\Delta_2\left(\varepsilon_2\right) := \int_c^1 G_\rho(x)\overline{G}(x)\mathrm{d}x - \int_c^1 G_\rho(x)G(x)\mathrm{d}x > 0$$

Observe that Δ_1 and Δ_2 are strictly increasing in ε_1 and ε_2 respectively, and

$$\lim_{\varepsilon_1 \downarrow 1} \Delta_1 = \lim_{\varepsilon_2 \downarrow 1} \Delta_2 = 0$$

Therefore, we can choose ε_1^* and ε_2^* such that $\Delta_1\left(\varepsilon_1^*\right) = \Delta_2\left(\varepsilon_2^*\right)$, equivalently

$$\int_0^1 G_\rho(x)\overline{G}\left(x; \varepsilon_1^*; \varepsilon_2^*\right)\mathrm{d}x = \int_0^1 G_\rho(x)G\left(x\right)\mathrm{d}x = x_+ - x_0$$

In other words, $\overline{G}$ is feasible. Now fix $0 < \lambda^* < \min\left\{\frac{1}{\varepsilon_1^*}, \frac{1}{\varepsilon_2^*}\right\} < 1$. Consider another pair of functions $\widetilde{\eta}_1$ and $\widetilde{\eta}_2$ given by

$$\widetilde{\eta}_1(x) := \frac{x - \lambda^*\overline{\eta}_1\left(x; \varepsilon_1^*\right)}{1 - \lambda^*} = x + \left(\frac{\lambda^*}{1 - \lambda^*}\right)\left(x - \overline{\eta}_1\left(x; \varepsilon_1^*\right)\right)$$
$$\widetilde{\eta}_2(x) := \frac{x - \lambda^*\overline{\eta}_2\left(x; \varepsilon_2^*\right)}{1 - \lambda^*} = x + \left(\frac{\lambda^*}{1 - \lambda^*}\right)\left(x - \overline{\eta}_2\left(x; \varepsilon_2^*\right)\right)$$

which also satisfy $\widetilde{\eta}_1(0) = \widetilde{\eta}_2(0) = 0$ and $\widetilde{\eta}_1(1) = \widetilde{\eta}_2(1) = 1$, and $\widetilde{\eta}_1$ (respectively $\widetilde{\eta}_2$) is a strictly increasing concave (respectively convex) function on $[0,1]$. Define another quantile function $\widetilde{G}$ by

$$\widetilde{G}(x) := \begin{cases} q\widetilde{\eta}_1\left(\frac{G(x)}{q}\right), & 0 \leqslant x \leqslant c \\ q + (L-q)\,\widetilde{\eta}_2\left(\frac{G(x)-q}{L-q}\right), & c < x < 1 \end{cases}$$

By construction of $\widetilde{\eta}_1$ and $\widetilde{\eta}_2$, it can be easily verified that $\widetilde{G}$ is feasible:

$$\int_0^1 G_\rho(x)\widetilde{G}\left(x;\varepsilon_1^*;\varepsilon_2^*\right)\mathrm{d}x = \int_0^1 G_\rho(x)\overline{G}\left(x;\varepsilon_1^*;\varepsilon_2^*\right)\mathrm{d}x = \int_0^1 G_\rho(x)G\left(x\right)\mathrm{d}x = x_+ - x_0$$

and G can be written as a convex combination of $\overline{G}$ and $\widetilde{G}$:

$$G(x) = \lambda^*\overline{G}\left(x;\varepsilon_1^*;\varepsilon_2^*\right) + (1-\lambda^*)\,\widetilde{G}\left(x;\varepsilon_1^*;\varepsilon_2^*\right)$$

By concavity of u_-,

$$\begin{aligned}
&\int_0^1 u_-\left(G(x)\right)w_-'(1-x)\mathrm{d}x \\
&\geqslant \lambda^*\int_0^1 u_-\left(\overline{G}(x)\right)w_-'(1-x)\mathrm{d}x + (1-\lambda^*)\int_0^1 u_-\left(\widetilde{G}(x)\right)w_-'(1-x)\mathrm{d}x \\
&= \int_0^1\left[\lambda^* u_-\left(\overline{G}(x)\right) + (1-\lambda^*)\,u_-\left(\widetilde{G}(x)\right)\right]w_-'(1-x)\mathrm{d}x
\end{aligned}$$

Since G is an optimal solution to a minimisation problem, equality must hold, and we must have $G(x) = \overline{G}(x) = \widetilde{G}(x)$ for all $x \in [0,1]$. In particular, by definition of $\overline{G}$, we must have $\frac{G(x)}{q} \in \{0,1\}$ for $x \in [0,c]$ and $\frac{G(x)-q}{L-q} \in \{0,1\}$ for $x \in (c,1)$, equivalently, $G(x) \in \{0,q\}$ for $x \in [0,c]$ and $G(x) \in \{q,L\}$ for $x \in (c,1)$, which contradicts the assumption that the image of G contains more than three distinct elements. Therefore the optimal quantile function must be a three-step step function, taking values $0, L$ and intermediate level $q \in (0,L)$. By feasibility of G, $G(x) = 0$ for all $x \in [0,p]$ and

$$\begin{aligned}
\int_p^1 G_\rho(x)G\left(x\right)\mathrm{d}x = q\int_p^\gamma G_\rho(x)\mathrm{d}x + L\int_\gamma^1 G_\rho(x)\mathrm{d}x = x_+ - x_0 \\
\implies q = q\left(x_+, p, \gamma\right) = \frac{x_+ - x_0 - L\int_\gamma^1 G_\rho(x)\mathrm{d}x}{\int_p^\gamma G_\rho(x)\mathrm{d}x}
\end{aligned}$$

□

Then the objective function becomes

$$\begin{aligned}
&\int_p^\gamma u_-\left(\frac{x_+ - x_0 - L\int_\gamma^1 G_\rho(x)\mathrm{d}x}{\int_p^\gamma G_\rho(x)\mathrm{d}x}\right)w_-'(1-x)\mathrm{d}x + \int_\gamma^1 u_-(L)w_-'(1-x)\mathrm{d}x \\
&= u_-\left(\frac{x_+ - x_0 - L\int_\gamma^1 G_\rho(x)\mathrm{d}x}{\int_p^\gamma G_\rho(x)\mathrm{d}x}\right)\left[w_-\left(1-p\right) - w_-\left(1-\gamma\right)\right] + u_-(L)w_-(1-\gamma)
\end{aligned}$$

3.3.1 Determining Optimal Parameters

Combining the objective functions in the positive and negative part problems, we have the following joint objective function

$$\begin{aligned}\mathcal{J}'(x_+, p, \alpha, \beta, \gamma, \lambda) :=\ & u_+(\alpha)\left[w_+(p) - w_+(1-\beta)\right] \\ &+ \int_\beta^1 u_+\left(\alpha \vee \left(u_+'\right)^{-1}\left(\frac{\lambda G_\rho(1-x)}{w_+'(1-x)}\right)\right) w_+'(1-x)\mathrm{d}x \\ &- u_-\left(\frac{x_+ - x_0 - L\int_\gamma^1 G_\rho(x)\mathrm{d}x}{\int_p^\gamma G_\rho(x)\mathrm{d}x}\right)\left[w_-(1-p) - w_-(1-\gamma)\right] \\ &- u_-(L)w_-(1-\gamma)\end{aligned}$$

Then we have the final optimisation problem:

$$\begin{aligned}\underset{x_+, p, \alpha, \beta, \gamma, \lambda}{\text{maximize}} \quad & \mathcal{J}'(x_+, p, \alpha, \beta, \gamma, \lambda) \\ \text{subject to} \quad & \mathcal{K}(p, \alpha, \beta, \lambda) = x_+, \\ & x_+, \alpha, \lambda \geqslant 0, \quad 0 \leqslant p \leqslant 1, \quad 1 - p \leqslant \beta \leqslant 1\end{aligned}$$

Once the optimal parameters $x_+^*, p^*, \alpha^*, \beta^*, \gamma^*, \lambda^*$ are found,

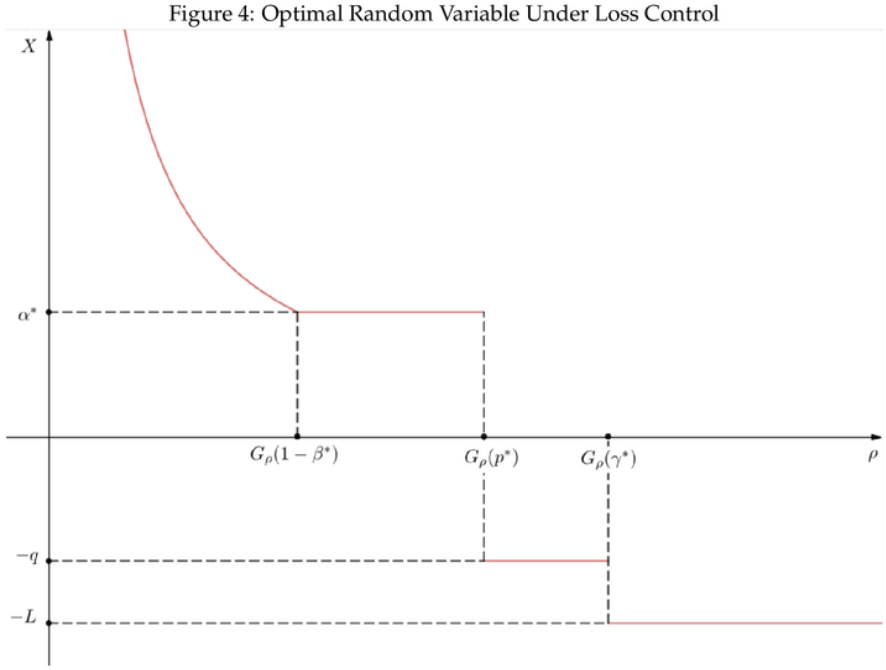
$$\begin{aligned}(X^-)^* &= G_{(X^-)^*}(F_\rho(\rho)) \\ &= \left(\frac{x_+ - x_0 - L\int_\gamma^1 G_\rho(x)\mathrm{d}x}{\int_p^\gamma G_\rho(x)\mathrm{d}x}\right)\mathbb{1}_{(p^*,\gamma^*)}(F_\rho(\rho)) + L\mathbb{1}_{[\gamma^*,1)}(F_\rho(\rho)) \\ &= \left(\frac{x_+ - x_0 - L\int_\gamma^1 G_\rho(x)\mathrm{d}x}{\int_p^\gamma G_\rho(x)\mathrm{d}x}\right)\mathbb{1}_{(G_\rho(p^*),G_\rho(\gamma^*))}(\rho) + L\mathbb{1}_{[G_\rho(\gamma^*),\infty)}(\rho)\end{aligned}$$

and the optimal random variable is

$$\begin{aligned}X^* &= (X^+)^* - (X^-)^* \\ &= \left(\alpha^* \vee \left(u_+'\right)^{-1}\left(\frac{\lambda^*\rho}{w_+'(F_\rho(\rho))}\right)\right)\mathbb{1}_{[0,G_\rho(1-\beta^*))}(\rho) + \alpha^*\mathbb{1}_{[G_\rho(1-\beta^*),G_\rho(p^*))}(\rho) \\ &\quad - \left(\frac{x_+ - x_0 - L\int_\gamma^1 G_\rho(x)\mathrm{d}x}{\int_p^\gamma G_\rho(x)\mathrm{d}x}\right)\mathbb{1}_{(G_\rho(p^*),G_\rho(\gamma^*))}(\rho) - L\mathbb{1}_{[G_\rho(\gamma^*),\infty)}(\rho)\, s\end{aligned}$$

Figure 2 on the following page provides an illustration of the distribution of the optimal random variable X^*, where $q := \left(\frac{x_+ - x_0 - L\int_\gamma^1 G_\rho(x)\mathrm{d}x}{\int_p^\gamma G_\rho(x)\mathrm{d}x}\right)$, and the values X^* takes depends on the values of ρ. The figure shows that in good states of the economy corresponding to $\rho \in [0, G_\rho(p^*)]$, a gambling policy is adopted, while in intermediate states corresponding to $\rho \in (G_\rho(p^*), G_\rho(\gamma^*))$, a constant moderate loss q is accepted, and in bad states corresponding to $[G_\rho(\gamma^*), \infty)$, the constant maximal loss L is accepted.

Figure 4: Optimal Random Variable Under Loss Control



3.4 Discussion

In [7], Jin and Zhou study an explicit example involving a two-piece CRRA utility function and solve the problem (without loss control) explicitly, and discuss the effects of CPT preferences on allocations of risky assets - under certain conditions investors underweigh risky assets in the portfolio compared to the allocation prescribed by a model in the EUT framework; interested readers can find further details in [7].

Compared with the optimal stopping problem, the approach for this problem is more flexible. The behavioural portfolio selection problem can be solved even when coefficients are time-dependent (deterministic or stochastic). However in the optimal stopping problem, only problems involving time-homogeneous processes, whose scale function can be computed in closed form, can be solved explicitly.

4 Optimal Contract in Employee Stock Options

Employee stock options (ESOs) are a form of equity compensation contract issued by companies to their employees and executives, whose contract payoff has a similar structure to a call option on the company's stock [21]. Unlike standard listed or exchange-traded options, employee stock options cannot be sold by employees [21]. These options are exercised when the company's stock rises above the strike price and the holder obtains the company's stock at a discount, upon which the holder may sell the stock in the open market for a profit or retain the stock for a period of time [21]. This type of equity compensation plan is mutually beneficial for employers and employees: employees are incentivized and motivated to be productive and contribute to the company's success (increasing the company's stock), and may offer potential tax savings to employees upon sale of shares in stock [21]. A company may be interested in minimising the cost of such compensation while ensuring that employees are sufficiently incentivized to increasing the company's firm value (instead of working for another company). This chapter aims to address this optimisation problem, assuming that the firm is risk-neutral and that employees have CPT preferences.

The model proposed in [9] is the first paper to incorporate probability weighting (of employees having CPT preferences) into the context of equity compensation. Spalt does not solve for the optimal contract, but instead calibrates the model to experimental data and demonstrates that probability weighting can explain why lower-level employees overvalue stock options relative to the Black-Scholes (1973) benchmark, an observation that is inconsistent with the assumption in EUT framework that employees are uniformly risk-averse. He also shows that the model explains why firms with more volatile stock returns grant more ESOs, a phenomenon that cannot be explained with an EUT framework, since risk-averse employees would demand a higher compensation from a more volatile company.

We will solve for the optimal contract assuming it has the specific structure proposed by Spalt, with some minor modifications. Then we will consider a more general structure and solve it using similar techniques to those in the optimal stopping and behavioural portfolio selection chapter, and analyse the solutions obtained.

4.1 Formulation of Problem

Assume that the firm value, denoted by $(P_t)_{t\geqslant 0}$, is a continuous stochastic process defined on complete filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geqslant 0}, \mathbb{P})$ is given by the following firm value model where the firm value evolves according to a geometric Brownian motion (with constant coefficients), with corresponding strong solution:

$$\frac{\mathrm{d}P_t}{P_t} = \mu \mathrm{d}t + \sigma \mathrm{d}W_t \qquad P_t = P_0 \exp\left(\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W_t\right)$$

where $\mu \in \mathbb{R}$, $\sigma, P_0 > 0$ are constants, $(W_t)_{t\geqslant 0}$ is a standard one-dimensional Brownian motion under physical measure $\mathbb{P}$, and r denotes the constant risk-less interest rate such that the price process $(B_t)_{t\geqslant 0}$ of the bank account evolves according to

$$\mathrm{d}B_t = rB_t\mathrm{d}t, \quad B_0 = 1 \iff B_t = \mathrm{e}^{rt}$$

Under the *risk-neutral measure* $\mathbb{Q}$, we take $(B_t)_{t\geqslant 0}$ as the numeraire, such that the numeraire re-based firm value process $\left(\frac{P_t}{B_t}\right)_{t\geqslant 0} = \left(\mathrm{e}^{-rt}P_t\right)_{t\geqslant 0}$ is a martingale under $\mathbb{Q}$ (having zero drift). Using integration by parts (a special case of Itô's formula), we have the following dynamics:

$$\begin{aligned}\mathrm{d}\left(\mathrm{e}^{-rt}P_t\right) &= -r\mathrm{e}^{-rt}P_t\mathrm{d}t + \mathrm{e}^{-rt}\mathrm{d}P_t = \mathrm{e}^{-rt}P_t\sigma\mathrm{d}W_t + \mathrm{e}^{-rt}P_t(\mu - r)\mathrm{d}t \\ &= \sigma\mathrm{e}^{-rt}P_t\left(\mathrm{d}W_t + \left(\frac{\mu - r}{\sigma}\right)\mathrm{d}t\right) =: \sigma\mathrm{e}^{-rt}P_t\mathrm{d}\widetilde{W}_t\end{aligned}$$

where $\left(\widetilde{W}_t\right)_{t\geqslant 0} := \left(W_t + \left(\frac{\mu-r}{\sigma}\right)t\right)_{t\geqslant 0} := (W_t + \theta t)_{t\geqslant 0}$ is a standard Brownian motion under $\mathbb{Q}$ and $\theta := \dfrac{\mu - r}{\sigma}$ denotes the *market price of risk* or *Sharpe ratio*. By Girsanov's theorem, the Radon-Nikodym derivative of the $\mathbb{Q}$ with respect to $\mathbb{P}$ is given by

$$\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} = \exp\left(-\theta W_T - \frac{1}{2}\theta^2 T\right)$$

and the arbitrage-free price of the contract $\Pi\left(P_T\right)$ written on the firm value P_T is

$$\mathbb{E}_{\mathbb{Q}}\left[\mathrm{e}^{-rT}\Pi\left(P_T\right)\right] = \mathbb{E}_{\mathbb{P}}\left[\mathrm{e}^{-rT}\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\Pi\left(P_T\right)\right] =: \mathbb{E}_{\mathbb{P}}\left[\rho_T\Pi(P_T)\right]$$

where $\mathbb{E}_{\mathbb{Q}}$ and $\mathbb{E}_{\mathbb{P}}$ denote expectations under probability measures $\mathbb{Q}$ and $\mathbb{P}$ respectively, Π denotes the (non-decreasing) contract function, and the random variable $\rho := \rho_T := \mathrm{e}^{-rT}\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}$ is called the *pricing kernel*.

In the model in [9], Π has the following form:

$$\Pi(x) = c(x - K)^+ + \eta$$

for some constant positive multiplier $c > 0$ which represents the number of units of *European call options*, *strike price* $K > 0$ and *base salary* η. Define the random variable $Y : \Omega \to \mathbb{R}$ by

$$Y := c(P_T - K)^+ + \eta - R$$

where R is a non-negative constant that represents a reference level: payoffs exceeding this level are interpreted as "gains", while those falling below the level are "losses". Y represents the difference between the payoff and reference level, with its sign indicating a gain or a loss, and its magnitude indicating the size of the gain or loss. We assume that $R \geqslant \eta$, otherwise Y is almost surely non-negative, which is unrealistic as it indicates that employees are satisfied with the payoff no matter how well or badly the firm performs. Let $u : \mathbb{R} \to \mathbb{R}$ be the utility function of gains/losses of the employees and let us assume that u is S-shaped, i.e. concave on $\mathbb{R}_{\geqslant 0}$ and convex on $\mathbb{R}_{<0}$. Then we may write

$$u\left(Y\right) := u_+\left(Y^+\right) - u_-\left(Y^-\right)$$

where Y^+, Y^- ($\geqslant 0$ almost surely) denote the positive and negative parts of Y (gains and losses) respectively and $u_+ : \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ and $u_- : \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ are concave functions representing the utilities associated with the magnitudes of gains and losses respectively. If we further assume that the probabilities associated with random variable $u_+(Y^+)$ are distorted with the function $w_+ : [0,1] \to [0,1]$ and those associated with $u_-(Y^-)$ are distorted with the function $w_- : [0,1] \to [0,1]$, then the problem that we solve is:

$$\begin{aligned} \underset{c,K,\eta}{\text{minimize}} \quad & \mathbb{E}_{\mathbb{P}}\left[\rho\left(c(P_T-K)^+ + \eta\right)\right] = c\left(P_0\Phi(d_1) - K\mathrm{e}^{-rT}\Phi(d_2)\right) + \eta \mathrm{e}^{-rT} \\ \text{subject to} \quad & \int_0^\infty w_+\left(\mathbb{P}(u_+(Y^+) > x\right)\mathrm{d}x - \int_0^\infty w_-\left(\mathbb{P}(u_-(Y^-) > x\right)\mathrm{d}x \geqslant V, \\ & c \geqslant 0 \end{aligned} \tag{4.1}$$

where

$$d_1 = \frac{\log\left(\frac{P_0}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}, \quad d_2 = d_1 - \sigma\sqrt{T}$$

The financial interpretation of this problem is as follows: the objective function the firm wishes to minimise is its expected *compensation costs*, subjected to a *participation constraint* of employees. The non-negative constant V represents the minimum value function, by employees with CPT preferences, of the deviation of the firm's compensation from their reference level. The constraint $c \geqslant 0$ is imposed due to the fact that employees cannot sell these stock options. This is a *slight modification* of the model in [9], which proposed the following minimisation problem:

$$\begin{aligned} \underset{c,\eta}{\text{minimize}} \quad & \mathbb{E}_{\mathbb{P}}\left[\rho\left(c(P_T-K)^+ + \eta\right)\right] = c\left(P_0\Phi(d_1) - K\mathrm{e}^{-rT}\Phi(d_2)\right) + \eta \mathrm{e}^{-rT} \\ \text{subject to} \quad & \int_0^\infty w_+\left(\mathbb{P}(u_+(Y^+) > x\right)\mathrm{d}x - \int_0^\infty w_-\left(\mathbb{P}(u_-(Y^-) > x\right)\mathrm{d}x \geqslant u\left(\overline{V} - R\right), \\ & c \geqslant 0 \end{aligned}$$

where K is exogenous and fixed at P_0, in the other words, only *at-the-money* call options are considered. According to [9], if K is endogenized (that is, we optimise over K), it may lead to optimal strike prices higher than at-the-money ones due to probability distortion by employees, and K was fixed in [9] to illustrate the effects of probability weighting. In the right hand side of the participation constraint, $\overline{V}$ represents the *outside opportunity* of the employee, and can be interpreted as the certainty equivalent of an ESO contract offered at another firm [9]. Furthermore, [9] assumes that the reference level R has the functional form

$$R = c\vartheta + \eta$$

where $\vartheta \geqslant 0$ is some constant that represents the payoff expectation or aspiration level held by an employee for one call option. For simplicity, we assume that R does not explicitly depend on c and η, and can perhaps be estimated from experimental studies instead, since R is to some extent subjective.

We now solve problem (4.1): we first notice that we may express the constraint in distribution formulation, using the lemma below:

Lemma 4.1. *Let F_{Y^+} and F_{Y^-} denote the cumulative distribution functions of Y^+ and Y^- respectively. Then*

$$F_{Y^+}(x) = \begin{cases} 0, & x < 0 \\ \Phi\left(\frac{\log\left(\frac{K+\frac{x+R-\eta}{c}}{P_0}\right)-(\mu-\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right), & x \geqslant 0 \end{cases}$$

and

$$F_{Y^-}(x) = \begin{cases} 0, & x < 0 \\ 1-\Phi\left(\frac{\log\left(\frac{K+\frac{R-\eta-x}{c}}{P_0}\right)-(\mu-\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right), & 0 \leqslant x < R-\eta \\ 1, & x \geqslant R-\eta \end{cases}$$

Proof. By (equation), $\log(P_T) \sim \mathcal{N}\left(\log(P_0) + \left(\mu - \frac{1}{2}\sigma^2\right)T, \sigma^2 T\right)$ and so the cumulative distribution function of P_T is given by

$$\mathbb{P}(P_T \leqslant x) = \Phi\left(\frac{\log(x) - \left(\log(P_0) + \left(\mu - \frac{1}{2}\sigma^2\right)T\right)}{\sigma\sqrt{T}}\right) = \Phi\left(\frac{\log\left(\frac{x}{P_0}\right) - \left(\mu - \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}\right)$$

Then,

$$\begin{aligned} F_{Y^+}(x) = \mathbb{P}(Y^+ \leqslant x) &= \mathbb{P}\left(\left(c(P_T-K)^+ + \eta - R\right)^+ \leqslant x\right) \\ &= \begin{cases} 0, & x < 0 \\ \mathbb{P}\left((P_T-K)^+ \leqslant \frac{x+R-\eta}{c}\right), & x \geqslant 0 \end{cases} \end{aligned}$$

and

$$\begin{aligned} \mathbb{P}\left((P_T-K)^+ \leqslant \frac{x+R-\eta}{c}\right) &= \begin{cases} 0, & x+R-\eta < 0 \\ \mathbb{P}\left((P_T-K) \leqslant \frac{x+R-\eta}{c}\right), & x+R-\eta \geqslant 0 \end{cases} \\ &= \begin{cases} 0, & x < \eta - R \\ \Phi\left(\frac{\log\left(\frac{K+\frac{x+R-\eta}{c}}{P_0}\right)-(\mu-\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right), & x \geqslant \eta - R \end{cases} \end{aligned}$$

Since $\eta \leqslant R$ (that is, $\eta - R \leqslant 0$) as assumed previously, the inequalities $x < 0$ and $x < \eta - R$ imply $x < 0$. Similarly, the inequalities $x \geqslant 0$ and $x \geqslant \eta - R$ imply $x \geqslant 0$, and therefore we have the required expression for F_{Y^+}. Next,

$$F_{Y^-}(x) = \mathbb{P}\left(Y^- \leqslant x\right) = \begin{cases} 0, & x < 0 \\ \mathbb{P}\left(-Y \leqslant x\right), & x \geqslant 0 \end{cases}$$

and

$$\begin{aligned} \mathbb{P}\left(-Y \leqslant x\right) = \mathbb{P}\left(Y \geqslant -x\right) &= 1 - \mathbb{P}\left(Y < -x\right) \\ &= 1 - \mathbb{P}\left(\left(P_T - K\right)^+ < \frac{R - \eta - x}{c}\right) \\ &= \begin{cases} 1, & x \geqslant R - \eta \\ 1 - \Phi\left(\frac{\log\left(\frac{K + \frac{R-\eta-x}{c}}{P_0}\right) - \left(\mu - \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}\right), & 0 \leqslant x < R - \eta \end{cases} \end{aligned}$$

□

Therefore by the Lemma, the left-hand side of the constraint may be written as:

$$\begin{aligned} &\int_0^\infty w_+\left(\mathbb{P}(u_+\left(Y^+\right) > x\right) \mathrm{d}x - \int_0^\infty w_-\left(\mathbb{P}(u_-\left(Y^-\right) > x\right) \mathrm{d}x \\ &= \int_0^\infty w_+\left(1 - F_{Y^+}(x)\right) u_+^{'}(x) \mathrm{d}x - \int_0^\infty w_-\left(1 - F_{Y^-}(x)\right) u_-^{'}(x) \mathrm{d}x \\ &= \int_{\eta - R}^\infty w_+\left(1 - \Phi\left(\frac{\log\left(\frac{K + \frac{x+R-\eta}{c}}{P_0}\right) - \left(\mu - \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}\right)\right) u_+^{'}(x) \mathrm{d}x \\ &\quad - \int_0^{R-\eta} w_-\left(\Phi\left(\frac{\log\left(\frac{K + \frac{R-\eta-x}{c}}{P_0}\right) - \left(\mu - \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}\right)\right) u_-^{'}(x) \mathrm{d}x \end{aligned}$$

Since the left-hand side of the constraint is now expressed in terms of c, K and η, we can use numerical optimisation to find the optimal parameters c^*, K^* and η^* and hence the optimal European call option style contract $\Pi^*\left(P_T\right) = c^*\left(P_T - K^*\right)^+ + \eta^*$. However, this may not be the optimal function, due to its restriction to a particular structure, similar to that of a European call option. We shall aim to minimise the price of the contract over all possible functions Π, solving a more general optimisation problem.

Define the random variable $X : \Omega \to \mathbb{R}$ by

$$X := \Pi\left(P_T\right) - R \tag{4.2}$$

which, similarly to the random variable Y defined earlier, represents the difference between the payoff and reference level, with its sign indicating a gain or a loss, and its magnitude indicating the size of the gain or loss. The problem can be written as:

$$\begin{aligned} &\underset{\Pi}{\text{minimize}} \quad \mathfrak{J}(\Pi) := \mathbb{E}_{\mathbb{P}}\left[\rho\Pi(P_T)\right] \\ &\text{subject to} \quad \int_0^\infty w_+\left(\mathbb{P}(u_+\left(X^+\right) > x\right) \mathrm{d}x - \int_0^\infty w_-\left(\mathbb{P}(u_-\left(X^-\right) > x\right) \mathrm{d}x \geqslant V \end{aligned}$$

where X^+ and X^- are the positive and negative parts of X. We will consider instead minimising the following objective function involving X, which can be written in terms of the objective function involving Π as follows:

$$\mathbb{E}_{\mathbb{P}}\left[\rho X\right] = \mathbb{E}_{\mathbb{P}}\left[\rho\Pi(P_T)\right] - R\mathbb{E}_{\mathbb{P}}\left[\rho\right] = \mathfrak{J}(\Pi) - R\mathbb{E}_{\mathbb{P}}\left[\mathrm{e}^{\left(-r-\frac{1}{2}\theta^2\right)T+\theta\sqrt{T}Z}\right] = \mathfrak{J}(\Pi) - R\mathrm{e}^{-rT}$$

where $Z \sim \mathcal{N}(0,1)$ and using the fact that ρ is log-normally distributed with

$$\log\rho \sim \mathcal{N}\left(\left(-r-\frac{1}{2}\theta^2\right)T, \theta^2 T\right)$$

Therefore our problem becomes:

$$\begin{aligned} &\underset{X}{\text{minimize}} \quad f_0\left(X\right) := \mathbb{E}_{\mathbb{P}}\left[\rho X\right] \\ &\text{subject to} \quad \int_0^\infty w_+\left(\mathbb{P}(u_+\left(X^+\right) > x\right) \mathrm{d}x - \int_0^\infty w_-\left(\mathbb{P}(u_-\left(X^-\right) > x\right) \mathrm{d}x \geqslant V \end{aligned}$$

It is worth noting that the space of contract functions Π is much larger than the space of random variables X. This is because X is a function of only the terminal firm value P_T subtract a constant R, while the space of contract functions Π includes path-dependent ones, that is, $\Pi\left(\left(P_T\right)_{t\in[0,T]}\right)$ that is a function of the whole history of the firm value in $[0,T]$. However, it will turn out that minimising over X is equivalent to minimising over contract functions Π, and this will be justified subsequently. The objective function may be written as:

$$f_0\left(X\right) = \mathbb{E}_{\mathbb{P}}\left[\rho X\right] = \mathbb{E}_{\mathbb{P}}\left[\rho X^+\right] - \mathbb{E}_{\mathbb{P}}\left[\rho X^-\right] = f_0\left(X^+\right) - f_0\left(X^-\right)$$

Let us assume as before that u is S-shaped, i.e. concave on $\mathbb{R}_{\geqslant 0}$ and convex on $\mathbb{R}_{<0}$. Then we may write

$$u\left(X\right) := u_+\left(X^+\right) - u_-\left(X^-\right)$$

where u_+ and u_- are concave functions as defined earlier. If we further assume as before that the probabilities associated with random variable $u_+\left(X^+\right)$ are distorted with the function w_+ and those associated with $u_-\left(X^-\right)$ are distorted with the function w_-, then we may split the constraint into:

$$\begin{aligned} &\int_0^\infty w_+\left(\mathbb{P}(u_+\left(X^+\right) > x\right) \mathrm{d}x - \int_0^\infty w_-\left(\mathbb{P}(u_-\left(X^-\right) > x\right) \mathrm{d}x \geqslant V \\ \iff &\int_0^\infty w_+\left(\mathbb{P}(u_+\left(X^+\right) > x\right) \mathrm{d}x \geqslant V + s; \quad \int_0^\infty w_-\left(\mathbb{P}(u_-\left(X^-\right) > x\right) \mathrm{d}x = s \end{aligned}$$

for some $s \geqslant 0$. Then similarly to behavioural portfolio selection, we may split the problem into positive part and negative part problems as follows:

Positive Part Problem

$$\begin{aligned} \underset{X^+}{\text{minimize}} \quad & f_0\left(X^+\right) = \mathbb{E}_{\mathbb{P}}\left[\rho X^+\right] \\ \text{subject to} \quad & \int_0^\infty w_+\left(\mathbb{P}(u_+\left(X^+\right) > x\right) \mathrm{d}x \geqslant V + s, \\ & X^+ \geqslant 0 \quad a.s., \\ & X^+ = 0 \quad a.s. \quad \text{on} \quad A^{\complement} := \{X \geqslant 0\}^{\complement} = \{X < 0\} \end{aligned}$$

Now, we may argue that the first constraint in the problem above should be binding:

Proposition 4.2. *The solution* $(X^+)^*$ *to the positive part problem satisfies*

$$\int_0^\infty w_+\left(\mathbb{P}(u_+\left(\left(X^+\right)^*\right) > x\right) \mathrm{d}x = V + s$$

Proof. Assume to the contrary that (X^+) is a solution satisfying

$$\int_0^\infty w_+\left(\mathbb{P}(u_+\left(X^+\right) > x\right) \mathrm{d}x > V + s$$

Then there exists $\varepsilon > 0$ such that $X^+ - \varepsilon \geqslant 0$ almost surely, $X^+ - \varepsilon = 0$ almost surely on $\{X - \varepsilon < 0\}$ and hence on $A^{\complement}$ (since $\{X < 0\} \subset \{X < \varepsilon\}$), and since u_+ and w_+ are increasing functions,

$$\int_0^\infty w_+\left(\mathbb{P}(u_+\left(X^+\right) > x\right) \mathrm{d}x > \int_0^\infty w_+\left(\mathbb{P}(u_+\left(X^+ - \varepsilon\right) > x\right) \mathrm{d}x \geqslant V + s$$

In other words, $\widetilde{X}^+ := X^+ - \varepsilon$ is feasible. However,

$$\mathbb{E}_{\mathbb{P}}\left[\rho \widetilde{X}^+\right] = \mathbb{E}_{\mathbb{P}}\left[\rho X^+\right] - \varepsilon e^{-rT} < \mathbb{E}_{\mathbb{P}}\left[\rho X^+\right]$$

which contradicts the optimality of (X^+). □

Negative Part Problem

$$\begin{aligned} \underset{X^-}{\text{maximize}} \quad & f_0\left(X^-\right) := \mathbb{E}_{\mathbb{P}}\left[\rho X^-\right] \\ \text{subject to} \quad & \int_0^\infty w_-\left(\mathbb{P}(u_-\left(X^-\right) > x\right) \mathrm{d}x = s, \\ & X^- \geqslant 0 \quad a.s., \\ & X^- = 0 \quad a.s. \quad \text{on} \quad A := \{X \geqslant 0\} \end{aligned}$$

The two problems are solved assuming that $s \geqslant 0$ and $A \in \mathcal{F}_T$ are given, and therefore the optimal values (denoted by $v_+(s, A)$ and $v_-(s, A)$ respectively) and corresponding solutions for each problem (denoted by $(X^+)^*(s,A)$ and $(X^-)^*(s,A)$) are given in terms of s and A. We then determine the "optimal" s and A (denoted by s^* and A^*), given by

$$(s^*, A^*) = \underset{(s,A)}{\arg\min}\, v(s,A) := \underset{(s,A)}{\arg\min}\, [v_+(s,A) - v_-(s,A)]$$

Finally, we have

$$X^* := (X^+)^*(s^*, A^*) - (X^-)^*(s^*, A^*)$$

In the next section, we will solve for X^* explicitly. In particular, we will show that ρ can be expressed as a function h of P_T and that X^* can be written as a function of ρ, i.e.

$$X^* = f(\rho) = (f \circ h)(P_T)$$

for some function f to be determined, and we may then deduce that the optimal contract function is of the form

$$\Pi^*(P_T) := R + (f \circ h)(P_T)$$

This justifies the equivalence of minimising over functions Π and minimising over random variables X, and it is not necessary to consider path-dependent contracts, only European style contracts.

4.2 Solving the Problem

We first determine the function h such that $\rho = h(P_T)$:

$$\begin{aligned}
\rho &= e^{-rT}\frac{d\mathbb{Q}}{d\mathbb{P}} \\
&= \exp\left(-\left(\frac{\mu-r}{\sigma}\right)W_T - \frac{1}{2}\left(\frac{\mu-r}{\sigma}\right)^2 T - rT\right) \\
&= \exp\left(-\left(\frac{\mu-r}{\sigma^2}\right)\left(\left(\mu-\frac{1}{2}\sigma^2\right)T + \sigma W_T\right)\right)\exp\left(\left(\frac{(\mu-r)(\mu-\frac{1}{2}\sigma^2)-\frac{1}{2}(\mu-r)^2 - r\sigma^2}{\sigma^2}\right)T\right) \\
&= \left(\frac{P_T}{P_0}\right)^{-\left(\frac{\mu-r}{\sigma^2}\right)}\exp\left(\left(\frac{\mu^2-\frac{1}{2}\mu\sigma^2 - r\mu + \frac{1}{2}r\sigma^2 - \frac{1}{2}\mu^2 + r\mu - \frac{1}{2}r^2 - r\sigma^2}{\sigma^2}\right)T\right) \\
&= \left(\frac{P_T}{P_0}\right)^{-\left(\frac{\theta}{\sigma}\right)}\exp\left(\left(\frac{\mu^2 - r^2 - \mu\sigma^2 - r\sigma^2}{2\sigma^2}\right)T\right) = \left(\frac{P_T}{P_0}\right)^{-\left(\frac{\theta}{\sigma}\right)}\exp\left(\frac{1}{2}\left(\theta^2 - \mu - r\right)T\right)
\end{aligned}$$

Therefore, $\rho = h(P_T)$ where the function $h : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ is defined by

$$h(x) := Cx^{\delta}, \qquad \delta := -\frac{\theta}{\sigma}, \qquad C := P_0^{\left(\frac{\theta}{\sigma}\right)}\exp\left(\frac{1}{2}\left(\theta^2-\mu-r\right)T\right)$$

with inverse $h^{-1} : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ given by

$$h^{-1}(x) = \left(\frac{x}{C}\right)^{\frac{1}{\delta}}$$

where δ and C are constants.

Remark 4.3. We assume that $\mu > r \implies \theta > 0 \implies \delta < 0$. This is a reasonable assumption in this context because if $\mu < r$, it is more worthwhile for an employee to to put his money in a bank account than invest in the firm's stocks, and it is unlikely that the employee would work for the firm. Alternatively, if the firm value process were the price process of a stock instead, it is more worthwhile to short sell the stock if $\mu < r$, however in this context employees cannot short sell a firm's stock or the ESO. Under this assumption, $h(x)$ and $h^{-1}(x)$ are decreasing in x, i.e. ρ is a decreasing function of P_T and vice versa, and this relationship is crucial in deriving the optimal contract function.

By (4.2),

$$X = \Pi(P_T) - R = \left(\Pi \circ h^{-1}\right)(\rho) - R$$

and therefore the random variables X^+ and X^- are functions of ρ:

$$X^+ = \left(\left(\Pi \circ h^{-1}\right)(\rho) - R\right)^+, \quad X^- = \left(\left(\Pi \circ h^{-1}\right)(\rho) - R\right)^-$$

Therefore, by (theorem in appendix), the optimal random variables (denoted by $(X^+)^*$ and $(X^-)^*$ respectively) are of the form

$$(X^+)^* = F^{-1}_{(X^+)^*}\left(1 - F_\rho(\rho)\right) = G_{(X^+)^*}\left(1 - F_\rho(\rho)\right) \tag{4.3}$$

$$(X^-)^* = F^{-1}_{(X^-)^*}\left(F_\rho(\rho)\right) = G_{(X^-)^*}\left(F_\rho(\rho)\right) \tag{4.4}$$

where $F_\rho : (0, \infty) \to [0, 1]$ is the cumulative distribution function of ρ and is given by

$$F_\rho(x) = \Phi\left(\frac{\log x + \left(r + \frac{1}{2}\theta^2\right)T}{\theta\sqrt{T}}\right)$$

for all $x \in (0, \infty)$, where Φ is the cumulative distribution function of a standard normal distribution. $G_\rho := F_\rho^{-1} : [0, 1] \to (0, \infty)$ is the inverse distribution function (or quantile function) of ρ and is given by

$$G_\rho(u) = \exp\left(\theta\sqrt{T}\Phi^{-1}(u) - \left(r + \frac{1}{2}\theta^2\right)T\right)$$

for all $u \in [0, 1]$. $F_{(X^+)^*}$ and $G_{(X^+)^*}$ are respectively the cumulative distribution function and quantile functions of $(X^+)^*$, and analogously for $(X^-)^*$. Now, using the fact that $F_\rho(\rho)$ is uniformly distributed on $[0, 1]$, and that ρ is equal in distribution to $G_\rho(U)$, where $U \sim \mathrm{U}[0, 1]$, the optimal values of the objective functions in the positive and negative part problems are given by

$$\begin{aligned}\min_{X^+} \mathbb{E}_{\mathbb{P}}\left[\rho X^+\right] = \min_{G_{X^+}} \mathbb{E}_{\mathbb{P}}\left[G_\rho(U) G_{X^+}(1 - U)\right] &= \min_{G_{X^+}} \int_0^1 G_\rho(y) G_{X^+}(1 - y)\,\mathrm{d}y \\ &= \min_{G_{X^+}} \int_0^1 G_\rho(1 - x) G_{X^+}(x)\,\mathrm{d}x\end{aligned}$$

$$\max_{X^-} \mathbb{E}_{\mathbb{P}}\left[\rho X^-\right] = \max_{G_{X^-}} \mathbb{E}_{\mathbb{P}}\left[G_\rho(U)G_{X^-}(U)\right] = \max_{G_{X^-}} \int_0^1 G_\rho(x)G_{X^-}(x)\,\mathrm{d}x$$

The constraints can be rewritten in quantile formulation as follows:

$$\int_0^1 u_+\left(G_{X^+}(x)\right) w'_+(1-x)\mathrm{d}x = V+s, \quad \int_0^1 u_-\left(G_{X^-}(x)\right) w'_-(1-x)\mathrm{d}x = s$$

Next, for any random variable Y with quantile function G_Y,

$$Y \geqslant 0 \quad a.s. \iff G_Y(x) \geqslant 0$$

for all $x \in [0,1]$. Define the probability $p := \mathbb{P}(X \geqslant 0)$. Then, $X^+ = 0$ almost surely on $A^{\complement} := \{X \geqslant 0\}^{\complement} = \{X < 0\}$ is equivalent to $X^+(\omega) = 0$ for all $\omega \in \{X < 0\}$, which is equivalent to $G_{X^+}\left[U^+(\omega)\right] = 0$ for all $\omega \in \{X < 0\}$, where $U^+ := F_{X^+}(X^+) \sim \mathrm{U}[0,1]$, and F_{X^+} and G_{X^+} are the cumulative distribution function and quantile function of X^+ respectively. Since

$$\{X < 0\} = \{G_X(U) < 0\} = \{U < F_X(0)\} = \{U < 1-p\}$$

where $U := F_X(X) \sim \mathrm{U}[0,1]$ and F_X and G_X are the cumulative distribution function and quantile function of X respectively,

$$\mathbb{P}(X < 0) = 1-p = \mathbb{P}(U < 1-p) = \ell([0, 1-p])$$

where ℓ denotes the Lebesgue measure on $[0,1]$. Therefore, $G_{X^+}(x) = 0$ for all $x \in [0, 1-p]$. Similarly, $X^- = 0$ almost surely on $A = \{X \geqslant 0\}$ is equivalent to $G_{X^-}(x) = 0$ for all $x \in [0,p]$. Then A^* is related to p^* via $p^* = \mathbb{P}(A^*) = \mathbb{P}(X^* \geqslant 0)$. The positive part problem becomes:

$$\begin{aligned}
\underset{G_{X^+}}{\text{minimize}} \quad & \mathbb{E}_{\mathbb{P}}\left[G_\rho(1-U)G_{X^+}(U)\right] = \int_{1-p}^1 G_\rho(1-x)G_{X^+}(x)\,\mathrm{d}x \\
\text{subject to} \quad & \mathbb{E}_{\mathbb{P}}\left[u_+\left(G_{X^+}(U)\right) w'_+(1-U)\right] = \int_{1-p}^1 u_+\left(G_{X^+}(x)\right) w'_+(1-x)\mathrm{d}x = V+s, \\
& G_{X^+}(x) \geqslant 0 \quad \forall x \in [0,1], \\
& G_{X^+}(x) = 0 \quad \forall x \in [0, 1-p]
\end{aligned}$$

and the negative part problem becomes:

$$\begin{aligned}
\underset{G_{X^-}}{\text{maximize}} \quad & \mathbb{E}_{\mathbb{P}}\left[G_\rho(1-U)G_{X^-}(U)\right] = \int_p^1 G_\rho(x)G_{X^-}(x)\,\mathrm{d}x \\
\text{subject to} \quad & \mathbb{E}_{\mathbb{P}}\left[u_-\left(G_{X^-}(U)\right) w'_-(1-U)\right] = \int_p^1 u_-\left(G_{X^-}(x)\right) w'_-(1-x)\mathrm{d}x = s, \\
& G_{X^-}(x) \geqslant 0 \quad \forall x \in [0,1], \\
& G_{X^-}(x) = 0 \quad \forall x \in [0, p]
\end{aligned}$$

4.2.1 Positive Part Problem

In the positive part problem, we observe that the objective function that we want to minimise is affine (and hence convex) in G_{X^+}. Since u_+ is concave (i.e. $-u_+$ is convex), we may use the

Lagrange multiplier method and solve the following unconstrained convex optimisation problem for a given $\lambda \geqslant 0$:

$$\underset{G_{X^+}}{\text{minimize}} \quad \int_{1-p}^{1} G_\rho(1-x) G_{X^+}(x)\,\mathrm{d}x + \lambda \left(V + s - \int_{1-p}^{1} u_+\left(G_{X^+}(x)\right) w_+'(1-x)\mathrm{d}x \right)$$

By monotonicity of expectation, it is sufficient to minimise the integrand, therefore we may solve the following unconstrained convex optimisation problem for each x:

$$\underset{G_{X^+}}{\text{minimize}} \quad \mathcal{L}\left(G_{X^+}(x), \lambda\right) := G_\rho(1-x) G_{X^+}(x) - \lambda \left(u_+\left(G_{X^+}(x)\right) w_+'(1-x) \right)$$

Since we have a convex optimisation problem, the following First Order Conditions are both necessary and sufficient:

$$\frac{\partial \mathcal{L}\left(G_{X^+}, \lambda\right)}{\partial G_{X^+}} = G_\rho(1-x) - \lambda u_+'\left(G_{X^+}(x)\right) w_+'(1-x) = 0$$
$$\iff G_{X^+}(x) = \left(u_+'\right)^{-1} \left(\frac{G_\rho(1-x)}{\lambda w_+'(1-x)} \right)$$

We then determine λ by substituting the above expression for G_{X^+} into the original constraint:

$$\int_{1-p}^{1} u_+ \left(\left(u_+'\right)^{-1} \left(\frac{G_\rho(1-x)}{\lambda w_+'(1-x)} \right) \right) w_+'(1-x)\mathrm{d}x = V + s$$

Since u_+ is concave, u_+' as well as $\left(u_+'\right)^{-1}$ are decreasing functions. Therefore, we require $\left(\frac{G_\rho(1-x)}{w_+'(1-x)}\right)$ to be non-increasing in $x \in (0,1]$ (i.e. $\left(\frac{G_\rho(y)}{w_+'(y)}\right)$ should be non-decreasing in y), in order for $G_{X^+}(x)$ to be non-decreasing in $x \in (0,1]$ (to satisfy the property of being a quantile function). Similarly to the optimal stopping under probability distortion and behavioural portfolio selection problems, we introduce a truncation when this condition is not satisfied:

$$G_{X^+}(x) = \alpha \mathbb{1}_{(1-p,\beta]}(x) + \left(\alpha \vee \left(u_+'\right)^{-1} \left(\frac{G_\rho(1-x)}{\lambda w_+'(1-x)} \right) \right) \mathbb{1}_{(\beta,1)}(x)$$

If $\left(\frac{G_\rho(y)}{w_+'(y)}\right)$ is non-decreasing in y, we set $\alpha := 0$, $\beta := 1-p$. λ is then determined by the following equation:

$$u_+(\alpha)\left[w_+(p) - w_+(1-\beta)\right] + \int_{\beta}^{1} u_+ \left(\alpha \vee \left(u_+'\right)^{-1} \left(\frac{G_\rho(1-x)}{\lambda w_+'(1-x)} \right) \right) w_+'(1-x)\mathrm{d}x = V + s$$

The left hand side of the above equation is monotone in λ, hence there is a unique λ satisfying the equation.

Lemma 4.4. *For $-\infty \leqslant a \leqslant b \leqslant \infty$,*

$$\int_a^b G_\rho(x)\mathrm{d}x = \mathrm{e}^{-rT}\left[\Phi\left(\Phi^{-1}(b) - \theta\sqrt{T}\right) - \Phi\left(\Phi^{-1}(a) - \theta\sqrt{T}\right)\right]$$

Proof.

$$\int_a^b G_\rho(x)\mathrm{d}x = \mathrm{e}^{-\left(r+\frac{1}{2}\theta^2\right)T}\int_a^b \mathrm{e}^{\theta\sqrt{T}\Phi^{-1}(x)}\mathrm{d}x = \mathrm{e}^{-\left(r+\frac{1}{2}\theta^2\right)T}\int_{\Phi^{-1}(a)}^{\Phi^{-1}(b)} \mathrm{e}^{\theta\sqrt{T}y}\phi(y)\mathrm{d}y$$
$$= \mathrm{e}^{-\left(r+\frac{1}{2}\theta^2\right)T}\frac{1}{\sqrt{2\pi}}\int_{\Phi^{-1}(a)}^{\Phi^{-1}(b)} \mathrm{e}^{-\frac{1}{2}\left(y^2-2\theta\sqrt{T}y\right)}\mathrm{d}y$$
$$= \mathrm{e}^{-\left(r+\frac{1}{2}\theta^2\right)T}\mathrm{e}^{\frac{1}{2}\theta^2 T}\frac{1}{\sqrt{2\pi}}\int_{\Phi^{-1}(a)}^{\Phi^{-1}(b)} \mathrm{e}^{-\frac{1}{2}\left(y-\theta\sqrt{T}\right)^2}\mathrm{d}y$$
$$= \mathrm{e}^{-rT}\left[\Phi\left(\Phi^{-1}(b)-\theta\sqrt{T}\right)-\Phi\left(\Phi^{-1}(a)-\theta\sqrt{T}\right)\right]$$

□

Remark 4.5. In particular, for $a, b \in \mathbb{R}$,

$$\int_a^\infty G_\rho(x)\mathrm{d}x = \mathrm{e}^{-rT}\left[1-\Phi\left(\Phi^{-1}(a)-\theta\sqrt{T}\right)\right], \quad \int_{-\infty}^b G_\rho(x)\mathrm{d}x = \mathrm{e}^{-rT}\left[\Phi\left(\Phi^{-1}(b)-\theta\sqrt{T}\right)\right]$$

The integral can also be written as the following expectation:

$$\int_a^b G_\rho(x)\mathrm{d}x = \int_{G_\rho(a)}^{G_\rho(b)} y f_\rho(y)\mathrm{d}y = \mathbb{E}_{\mathbb{P}}\left[\rho \mathbb{1}_{\{G_\rho(a)\leqslant\rho\leqslant G_\rho(b)\}}\right]$$

where $f_\rho := F_\rho'$ is the probability density function of ρ. In particular,

$$\int_a^\infty G_\rho(x)\mathrm{d}x = \mathbb{E}_{\mathbb{P}}\left[\rho \mathbb{1}_{\{\rho\geqslant G_\rho(a)\}}\right], \quad \int_{-\infty}^b G_\rho(x)\mathrm{d}x = \mathbb{E}_{\mathbb{P}}\left[\rho \mathbb{1}_{\{\rho\leqslant G_\rho(b)\}}\right]$$

Therefore by the Lemma, the objective function of the positive part problem becomes:

$$\alpha\int_{1-p}^{\beta} G_\rho(x)\mathrm{d}x + \int_\beta^1 \left(\alpha \vee \left(u_+'\right)^{-1}\left(\frac{G_\rho(1-x)}{\lambda w_+'(1-x)}\right)\right) G_\rho(x)\mathrm{d}x$$
$$= \alpha\mathrm{e}^{-rT}\left[\Phi\left(\Phi^{-1}(\beta)-\theta\sqrt{T}\right)-\Phi\left(\Phi^{-1}(1-p)-\theta\sqrt{T}\right)\right]$$
$$+\left(\alpha\mathrm{e}^{-rT}\left[1-\Phi\left(\Phi^{-1}(\beta)-\theta\sqrt{T}\right)\right]\vee\int_\beta^1\left(\left(u_+'\right)^{-1}\left(\frac{G_\rho(1-x)}{\lambda w_+'(1-x)}\right)\right)G_\rho(x)\mathrm{d}x\right)$$

Remark 4.6. The actual constraint in the positive part problem is

$$\int_{1-p}^1 u_+\left(G_{X^+}(x)\right) w_+'(1-x)\mathrm{d}x \geqslant V+s \iff V+s-\int_{1-p}^1 u_+\left(G_{X^+}(x)\right) w_+'(1-x)\mathrm{d}x \leqslant 0$$

where the latter inequality is in standard form (that is, $f\left[G_{X^+}(x)\right] \leqslant 0$ where f is convex and therefore the original positive part problem is a convex optimisation problem (and therefore strong duality holds). Clearly $\lambda = 0$ is not admissible (otherwise we obtain $G_\rho(1-x) = 0$, which is a contradiction) and therefore by complementary slackness conditions (for further explanations see [5] equality should hold, and this corroborates with Proposition 4.2.

4.2.2 Negative Part Problem

The negative part problem is not a convex optimisation problem, because the left hand side of the equality constraint is not affine in $G_{X^-}(x)$. Having assumed that the affine functional in $G_{X^+}(x)$

that we were minimising was convex, we will also assume that the affine functional in $G_{X^-}(x)$ in this negative part problem is convex. We obtain "corner point" solutions in the following proposition:

Proposition 4.7. *The optimal solution, if it exists, must be in the following form: for $x \in [0,1)$,*

$$G_{X^-}(x) = q(s,p)\,\mathbb{1}_{(p,1)}(x), \quad q(s,p) := (u_-)^{-1}\left(\frac{s}{\int_p^1 w_-'(1-x)\mathrm{d}x}\right) = (u_-)^{-1}\left(\frac{s}{w_-(1-p)}\right)$$

Proof. Assume $s > 0$, otherwise the result holds trivially. If G_{X^-} is an optimal solution, then $G_{X^-} \not\equiv 0$. Fix $y \in (0,1)$ such that $G_{X^-}(y) > 0$. Define:

$$k := \frac{\int_0^1 G_\rho(x)G_{X^-}(x)\mathrm{d}x}{\int_0^y G_\rho(x)G_{X^-}(x)\mathrm{d}x + G_{X^-}(y)\int_y^1 G_\rho(x)\mathrm{d}x} \geqslant 1, \quad \overline{G}(x) := \begin{cases} kG_{X^-}(x), & x \in [0,y] \\ kG_{X^-}(y), & x \in (y,1) \end{cases}$$

Clearly $\overline{G}$ is another quantile function and

$$\int_0^1 G_\rho(x)\overline{G}(x)\,\mathrm{d}x = k\left(\int_0^y G_\rho(x)G_{X^-}(x)\,\mathrm{d}x + G_{X^-}(y)\int_y^1 G_\rho(x)\mathrm{d}x\right) = \int_0^1 G_\rho(x)G_{X^-}(x)\mathrm{d}x$$

and therefore $\overline{G}$ is also optimal. We claim that $G_{X^-}(x) = G_{X^-}(y)$ for almost every $x \in (y,1)$. To prove this, we first assume to the contrary that this is not the case, which means $G_{X^-}(x) \geqslant G_{X^-}(y)$ for almost every $x \in (y,1)$ since G_{X^-} is a non-decreasing quantile function. This implies that $k > 1$. Define

$$\lambda := 1 - \frac{1}{k} \in (0,1), \quad \widetilde{G}(x) := \begin{cases} 0, & x \in [0,y] \\ \frac{G_{X^-}(x) - G_{X^-}(y)}{\lambda}, & x \in (y,1) \end{cases}$$
$$= \left(\frac{1}{\lambda}\left[G_{X^-}(x) - G_{X^-}(y)\right]\right)\mathbb{1}_{(y,1)}(x)$$

Then G_{X^-} can be written as a convex combination of $\overline{G}$ and $\widetilde{G}$:

$$(1-\lambda)\overline{G}(x) + \lambda\widetilde{G}(x)$$
$$= \frac{1}{k}\left(kG_{X^-}(x)\mathbb{1}_{[0,y]}(x) + kG_{X^-}(y)\mathbb{1}_{(y,1)}(x)\right) + \lambda\left(\frac{1}{\lambda}\left[G_{X^-}(x) - G_{X^-}(y)\right]\right)\mathbb{1}_{(y,1)}(x) = G_{X^-}(x)$$

and

$$\int_0^1 G_\rho(x)G_{X^-}(x)\,\mathrm{d}x = \int_0^1 G_\rho(x)\left((1-\lambda)\overline{G}(x) + \lambda\widetilde{G}(x)\right)\mathrm{d}x$$
$$= (1-\lambda)\int_0^1 G_\rho(x)\overline{G}(x)\,\mathrm{d}x + \lambda\int_0^1 G_\rho(x)\widetilde{G}(x)\,\mathrm{d}x$$

Therefore, since $\overline{G}$ is optimal, $\widetilde{G}$ must be optimal as well. By concavity of u_-,

$$u_-(G_{X^-}(x)) \geqslant (1-\lambda)u_-\left(\overline{G}(x)\right) + \lambda u_-\left(\widetilde{G}(x)\right)$$

and the inequality is strict for some values of x and therefore

$$\begin{aligned}&\int_0^1 u_-\left(G_{X^-}(x)\right) w_-'(1-x)\mathrm{d}x \\ &> \int_0^1 \left[(1-\lambda)u_-\left(\overline{G}(x)\right) + \lambda u_-\left(\widetilde{G}(x)\right)\right] w_-'(1-x)\mathrm{d}x \\ &= (1-\lambda)\int_0^1 u_-\left(\overline{G}(x)\right) w_-'(1-x)\mathrm{d}x + \lambda\int_0^1 u_-\left(\widetilde{G}(x)\right) w_-'(1-x)\mathrm{d}x = (1-\lambda)s + \lambda s = s\end{aligned}$$

where the penultimate equality follows from the feasibility of $\overline{G}$ and $\widetilde{G}$, but this contradicts the feasibility of G_{X^-} (assumed to be an optimal solution). Therefore, we must have $G_{X^-}(x) = G_{X^-}(y)$ for almost every $x \in (y, 1)$. Then, since $G_{X^-}(x) = 0$ for all $x \in [0, p]$ we must have $G_{X^-}(x) = q\mathbb{1}_{(p,1)}(x)$ for all $x \in [0, 1]$, some $q \in \mathbb{R}_{>0}$. By the feasibility of G_{X^-},

$$\int_p^1 u_-\left(G_{X^-}(x)\right) w_-'(1-x)\mathrm{d}x = u_-(q)\int_p^1 w_-'(1-x)\mathrm{d}x = s$$

$$\implies q = q\left(s, p\right) := \left(u_-\right)^{-1}\left(\frac{s}{\int_p^1 w_-'(1-x)\mathrm{d}x}\right) = \left(u_-\right)^{-1}\left(\frac{s}{w_-\left(1-p\right)}\right)$$

□

By the Lemma, the objective function in the negative part problem becomes

$$q(s,p)\int_p^1 G_\rho(x)\mathrm{d}x = q(s,p)\mathrm{e}^{-rT}\left(1 - \Phi\left(\Phi^{-1}(p) - \theta\sqrt{T}\right)\right)$$

4.2.3 Determining Optimal Parameters and Optimal Contract

Combining the objective functions in the positive and negative part problems, we have the following objective function:

$$\begin{aligned}\mathcal{J}\left(s,p,\alpha,\beta,\lambda\right) = {}&\alpha\mathrm{e}^{-rT}\left[\Phi\left(\Phi^{-1}(\beta) - \theta\sqrt{T}\right) - \Phi\left(\Phi^{-1}(1-p) - \theta\sqrt{T}\right)\right] \\ &+ \left(\alpha\mathrm{e}^{-rT}\left[1 - \Phi\left(\Phi^{-1}(\beta) - \theta\sqrt{T}\right)\right] \vee \int_\beta^1 \left(\left(u_+'\right)^{-1}\left(\frac{G_\rho(1-x)}{\lambda w_+'(1-x)}\right)\right) G_\rho(x)\mathrm{d}x\right) \\ &- q(s,p)\mathrm{e}^{-rT}\left(1 - \Phi\left(\Phi^{-1}(p) - \theta\sqrt{T}\right)\right)\end{aligned}$$

Define the following function:

$$\begin{aligned}\mathcal{K}(p,\alpha,\beta,\lambda) :={}& u_+(\alpha)\left[w_+(p) - w_+(1-\beta)\right] \\ &+ \int_\beta^1 u_+\left(\alpha \vee \left(u_+'\right)^{-1}\left(\frac{G_\rho(1-x)}{\lambda w_+'(1-x)}\right)\right) w_+'(1-x)\mathrm{d}x\end{aligned}$$

Then we have the final optimisation problem:

$$\begin{aligned}\underset{s,\,p,\,\alpha,\,\beta,\,\lambda}{\text{minimize}} \quad & \mathcal{J}\left(s,p,\alpha,\beta,\lambda\right) \\ \text{subject to} \quad & \mathcal{K}(p,\alpha,\beta,\lambda) = V + s, \\ & q\left(s,p\right) < R, \\ & s,\alpha,\lambda \geqslant 0, \quad 0 \leqslant p \leqslant 1, \quad 1-p \leqslant \beta \leqslant 1\end{aligned}$$

Once the optimal parameters $s^*, p^*, \alpha^*, \beta^*, \lambda^*$ are found, we have the optimal quantile functions in the positive part and negative part problems respectively given by

$$G_{(X^+)^*}(x) = \alpha^* \mathbb{1}_{(1-p^*,\beta^*]}(x) + \left(\alpha^* \vee \left(u_+'\right)^{-1}\left(\frac{G_\rho(1-x)}{\lambda^* w_+'(1-x)}\right)\right)\mathbb{1}_{(\beta^*,1)}(x)$$
$$G_{(X^-)^*}(x) = (u_-)^{-1}\left(\frac{s^*}{w_-(1-p^*)}\right)\mathbb{1}_{(p^*,1)}(x)$$

and the corresponding optimal random variables $(X^+)^*$ and $(X^-)^*$ are given by

$$\begin{aligned}
(X^+)^* &= G_{(X^+)^*}\left(1 - F_\rho(\rho)\right) \\
&= \alpha^* \mathbb{1}_{(1-p^*,\beta^*]}\left(1 - F_\rho(\rho)\right) + \left(\alpha^* \vee \left(u_+'\right)^{-1}\left(\frac{G_\rho\left(F_\rho(\rho)\right)}{\lambda^* w_+'\left(F_\rho(\rho)\right)}\right)\right)\mathbb{1}_{(\beta^*,1)}\left(1 - F_\rho(\rho)\right) \\
&= \alpha^* \mathbb{1}_{[G_\rho(1-\beta^*),G_\rho(p^*))}(\rho) + \left(\alpha^* \vee \left(u_+'\right)^{-1}\left(\frac{\rho}{\lambda^* w_+'(F_\rho(\rho))}\right)\right)\mathbb{1}_{[0,G_\rho(1-\beta^*))}(\rho) \\
&= \alpha^* \mathbb{1}_{(h^{-1}(G_\rho(p^*)),h^{-1}(G_\rho(1-\beta^*))]}(P_T) \\
&\quad + \left(\alpha^* \vee \left(u_+'\right)^{-1}\left(\frac{h(P_T)}{\lambda^* w_+'(F_\rho(h(P_T)))}\right)\right)\mathbb{1}_{(h^{-1}(G_\rho(1-\beta^*)),\infty)}(P_T)
\end{aligned}$$

$$\begin{aligned}
(X^-)^* = G_{(X^-)^*}\left(F_\rho(\rho)\right) &= (u_-)^{-1}\left(\frac{s^*}{w_-(1-p^*)}\right)\mathbb{1}_{(p^*,1)}\left(F_\rho(\rho)\right) \\
&= (u_-)^{-1}\left(\frac{s^*}{w_-(1-p^*)}\right)\mathbb{1}_{(G_\rho(p^*),\infty)}(\rho) \\
&= (u_-)^{-1}\left(\frac{s^*}{w_-(1-p^*)}\right)\mathbb{1}_{(0,h^{-1}(G_\rho(p^*)))}(P_T)
\end{aligned}$$

where we have used the fact that $h^{-1}(x)$ is decreasing in x and, since $\delta < 0$,

$$h^{-1}(0) = \lim_{x\downarrow 0}\left(\frac{x}{C}\right)^{\frac{1}{\delta}} = \infty, \quad h^{-1}(\infty) = \lim_{x\uparrow\infty}\left(\frac{x}{C}\right)^{\frac{1}{\delta}} = 0$$

Therefore, the optimal contract is given by

$$\begin{aligned}
\Pi^*(P_T) &= R + (X^+)^* - (X^-)^* \\
&= R + \alpha^* \mathbb{1}_{(h^{-1}(G_\rho(p^*)),h^{-1}(G_\rho(1-\beta^*))]}(P_T) \\
&\quad + \left(\alpha^* \vee \left(u_+'\right)^{-1}\left(\frac{h(P_T)}{\lambda^* w_+'(F_\rho(h(P_T)))}\right)\right)\mathbb{1}_{(h^{-1}(G_\rho(1-\beta^*)),\infty)}(P_T) \\
&\quad - (u_-)^{-1}\left(\frac{s^*}{w_-(1-p^*)}\right)\mathbb{1}_{(0,h^{-1}(G_\rho(p^*)))}(P_T) \\
&= \left(R - (u_-)^{-1}\left(\frac{s^*}{w_-(1-p^*)}\right)\right)\mathbb{1}_{(0,h^{-1}(G_\rho(p^*)))}(P_T) \\
&\quad + (R + \alpha^*)\,\mathbb{1}_{(h^{-1}(G_\rho(p^*)),h^{-1}(G_\rho(1-\beta^*))]}(P_T) \\
&\quad + \left((R+\alpha^*) \vee \left(R + \left(u_+'\right)^{-1}\left(\frac{h(P_T)}{\lambda^* w_+'(F_\rho(h(P_T)))}\right)\right)\right)\mathbb{1}_{(h^{-1}(G_\rho(1-\beta^*)),\infty)}(P_T)
\end{aligned}$$

4.3 Addition of Loss Control

Similarly to behavioural portfolio selection, a firm may want to bound its compensation costs by a constant $L < R$. The positive part problem is unchanged, and the negative part problem becomes

$$\begin{aligned} \underset{G_{X^-}}{\text{maximize}} \quad & \int_p^1 G_\rho(x) G_{X^-}(x)\,\mathrm{d}x \\ \text{subject to} \quad & \int_p^1 u_-\left(G_{X^-}(x)\right) w_-'(1-x)\mathrm{d}x = s, \\ & 0 \leqslant G_{X^-}(x) \leqslant L \quad \forall x \in [0,1], \\ & G_{X^-}(x) = 0 \quad \forall x \in [0,p] \end{aligned}$$

The proof below for the solution to this loss control problem has the same idea as the proof for behavioural portfolio selection, but the contradiction is slightly different.

Proposition 4.8. *The optimal solution, if it exists, must be in the following form: for $x \in [0,1)$,*

$$G_{X^-}(x) = q\,(s,p,\gamma)\,\mathbb{1}_{(p,\gamma)}(x) + L\mathbb{1}_{[\gamma,1)}(x)$$

where

$$q\,(s,p,\gamma) := (u_-)^{-1}\left(\frac{s - u_-(L)\int_\gamma^1 w_-'(1-x)\mathrm{d}x}{\int_p^\gamma w_-'(1-x)\mathrm{d}x}\right) = (u_-)^{-1}\left(\frac{s - u_-(L)w_-(1-\gamma)}{w_-(1-p) - w_-(1-\gamma)}\right) < L$$

Proof. Let G be an optimal solution. We first prove that the optimal solution G is the quantile function of a random variable having a three-point distribution with masses at 0, q and L. Assume to the contrary that the image of G contains more than three distinct elements. Then we can choose $c \in (0,1)$ and $q \in (0,L)$ such that $q \in [G(c), G(c+)]$ such that there exists $x_1 \in (0,c)$ and $x_2 \in (c,1)$ with $0 < G(x_1) < q$ and $q < G(x_2) < L$. Define a convex function $\overline{\eta}_1$ and a concave function $\overline{\eta}_2$ by

$$\overline{\eta}_1(x) = \overline{\eta}_1(x;\varepsilon_1) := x^{\varepsilon_1}, \quad \overline{\eta}_2(x) = \overline{\eta}_2(x;\varepsilon_2) := 1-(1-x)^{\varepsilon_2}$$

with $\varepsilon_1 > 1$ and $\varepsilon_2 > 1$. Note that $\overline{\eta}_1(0) = \overline{\eta}_2(0) = 0$ and $\overline{\eta}_1(1) = \overline{\eta}_2(1) = 1$ and we can define another quantile function $\overline{G}$ via

$$\frac{\overline{G}(x)}{q} := \overline{\eta}_1\left(\frac{G(x)}{q}\right), \quad 0 \leqslant x \leqslant c \qquad \frac{\overline{G}(x)-q}{L-q} := \overline{\eta}_2\left(\frac{G(x)-q}{L-q}\right), \quad c < x < 1$$

$$\implies \overline{G}(x) = \begin{cases} q\overline{\eta}_1\left(\frac{G(x)}{q}\right), & 0 \leqslant x \leqslant c \\ q + (L-q)\,\overline{\eta}_2\left(\frac{G(x)-q}{L-q}\right), & c < x < 1 \end{cases}$$

By construction, $\overline{G}(x) \leqslant G(x)$ on $[0,c]$ and $\overline{G}(x) \geqslant G(x)$ on $(c,1)$. Since u_- is an increasing function, we also have $u_-\left(\overline{G}(x)\right) \leqslant u_-\left(G(x)\right)$ on $[0,c]$ and $u_-\left(\overline{G}(x)\right) \geqslant u_-\left(G(x)\right)$ on $(c,1)$.

Since $\overline{\eta}_1(x) = x$ and $\overline{\eta}_2(x) = x$ if and only if $x \in \{0,1\}$, we have that in the first inequality, equality only holds for x such that $G(x) = 0 \iff x = 0$ or $G(x) = q \iff x = c$, and strict inequality holds for $x \in (0,c)$. Similarly for the second inequality, equality only holds for x such that $G(x) = q \iff x = c+$ or $G(x) = L$, and strict inequality holds for all other values of x. Then we can define the quantities

$$\Delta_1(\varepsilon_1) := \int_0^c u_-(G(x))\, w'_-(1-x)\mathrm{d}x - \int_0^c u_-\left(\overline{G}(x)\right) w'_-(1-x)\mathrm{d}x > 0$$
$$\Delta_2(\varepsilon_2) := \int_c^1 u_-\left(\overline{G}(x)\right) w'_-(1-x)\mathrm{d}x - \int_c^1 u_-(G(x))\, w'_-(1-x)\mathrm{d}x > 0$$

Observe that Δ_1 and Δ_2 are strictly increasing in ε_1 and ε_2 respectively, and

$$\lim_{\varepsilon_1 \downarrow 1} \Delta_1 = \lim_{\varepsilon_2 \downarrow 1} \Delta_2 = 0$$

Therefore, we can choose ε_1^* and ε_2^* such that $\Delta_1(\varepsilon_1^*) = \Delta_2(\varepsilon_2^*)$, equivalently

$$\int_0^1 u_-\left(\overline{G}(x;\varepsilon_1^*;\varepsilon_2^*)\right) w'_-(1-x)\mathrm{d}x = \int_0^1 u_-(G(x))\, w'_-(1-x)\mathrm{d}x = s$$

In other words, $\overline{G}$ is feasible. Now fix $0 < \lambda^* < \min\left\{\frac{1}{\varepsilon_1^*}, \frac{1}{\varepsilon_2^*}\right\} < 1$. Consider another pair of functions $\widetilde{\eta}_1$ and $\widetilde{\eta}_2$ given by

$$\widetilde{\eta}_1(x) := \frac{x - \lambda^* \overline{\eta}_1(x;\varepsilon_1^*)}{1-\lambda^*} = x + \left(\frac{\lambda^*}{1-\lambda^*}\right)\left(x - \overline{\eta}_1(x;\varepsilon_1^*)\right)$$
$$\widetilde{\eta}_2(x) := \frac{x - \lambda^* \overline{\eta}_2(x;\varepsilon_2^*)}{1-\lambda^*} = x + \left(\frac{\lambda^*}{1-\lambda^*}\right)\left(x - \overline{\eta}_2(x;\varepsilon_2^*)\right)$$

which also satisfy $\widetilde{\eta}_1(0) = \widetilde{\eta}_2(0) = 0$ and $\widetilde{\eta}_1(1) = \widetilde{\eta}_2(1) = 1$, and $\widetilde{\eta}_1$ (respectively $\widetilde{\eta}_2$) is a strictly increasing concave (respectively convex) function on $[0,1]$. Define another quantile function $\widetilde{G}$ by

$$\widetilde{G}(x) := \begin{cases} q\widetilde{\eta}_1\left(\frac{G(x)}{q}\right), & 0 \leqslant x \leqslant c \\ q + (L-q)\,\widetilde{\eta}_2\left(\frac{G(x)-q}{L-q}\right), & c < x < 1 \end{cases}$$

By construction of $\widetilde{\eta}_1$ and $\widetilde{\eta}_2$, it can be easily verified that $\widetilde{G}$ is feasible:

$$\begin{aligned}\int_0^1 u_-\left(\widetilde{G}(x;\varepsilon_1^*;\varepsilon_2^*)\right) w'_-(1-x)\mathrm{d}x &= \int_0^1 u_-\left(\overline{G}(x;\varepsilon_1^*;\varepsilon_2^*)\right) w'_-(1-x)\mathrm{d}x \\ &= \int_0^1 u_-(G(x))\, w'_-(1-x)\mathrm{d}x = s\end{aligned}$$

and G can be written as a convex combination of $\overline{G}$ and $\widetilde{G}$:

$$G(x) = \lambda^* \overline{G}(x;\varepsilon_1^*;\varepsilon_2^*) + (1-\lambda^*)\,\widetilde{G}(x;\varepsilon_1^*;\varepsilon_2^*)$$

and therefore we have

$$\begin{aligned}\int_0^1 G_\rho(x) G(x)\,\mathrm{d}x &= \int_0^1 G_\rho(x)\left(\lambda^* \overline{G}(x;\varepsilon_1^*;\varepsilon_2^*) + (1-\lambda^*)\,\widetilde{G}(x;\varepsilon_1^*;\varepsilon_2^*)\right)\mathrm{d}x \\ &= \lambda^* \int_0^1 G_\rho(x)\overline{G}(x;\varepsilon_1^*;\varepsilon_2^*)\,\mathrm{d}x + (1-\lambda^*)\int_0^1 G_\rho(x)\widetilde{G}(x;\varepsilon_1^*;\varepsilon_2^*)\,\mathrm{d}x\end{aligned}$$

We must have $G(x) = \overline{G}(x) = \widetilde{G}(x)$ for all $x \in [0,1]$, otherwise G is not feasible, which contradicts the assumption of G being an optimal solution: if $G(x) \neq \overline{G}(x)$ or $G(x) \neq \overline{G}(x)$ for some $x \in [0,1]$, by concavity of u_- and feasibility of $\overline{G}$ and $\widetilde{G}$ we have

$$\int_0^1 u_-\left(G\left(x\right)\right) w_-'(1-x)\mathrm{d}x = \int_0^1 u_-\left(\lambda^*\overline{G}\left(x\right) + (1-\lambda^*)\,\widetilde{G}\left(x\right)\right) w_-'(1-x)\mathrm{d}x$$
$$> \lambda^* \int_0^1 u_-\left(\overline{G}\left(x\right)\right) w_-'(1-x)\mathrm{d}x + (1-\lambda^*)\int_0^1 u_-\left(\widetilde{G}\left(x\right)\right) w_-'(1-x)\mathrm{d}x = \lambda^* s + (1-\lambda^*)\, s = s$$

In particular, by definition of $\overline{G}$, we must have $\frac{G(x)}{q} \in \{0,1\}$ for $x \in [0,c]$ and $\frac{G(x)-q}{L-q} \in \{0,1\}$ for $x \in (c,1)$, equivalently, $G(x) \in \{0,q\}$ for $x \in [0,c]$ and $G(x) \in \{q,L\}$ for $x \in (c,1)$, which contradicts the assumption that the image of G contains more than three distinct elements. Therefore the optimal quantile function must be a three-step step function, taking values 0, L and intermediate level $q \in (0,L)$. By feasibility of G, $G(x) = 0$ for all $x \in [0,p]$ and

$$\int_p^1 u_-\left(G\left(x\right)\right) w_-'(1-x)\mathrm{d}x = u_-(q)\int_p^{\gamma} w_-'(1-x)\mathrm{d}x + u_-(L)\int_{\gamma}^1 w_-'(1-x)\mathrm{d}x = s$$

$$\Longrightarrow\ q = q\left(s,p,\gamma\right) = \left(u_-\right)^{-1}\left(\frac{s - u_-(L)\int_{\gamma}^1 w_-'(1-x)\mathrm{d}x}{\int_p^{\gamma} w_-'(1-x)\mathrm{d}x}\right)$$
$$= \left(u_-\right)^{-1}\left(\frac{s - u_-(L) w_-\left(1-\gamma\right)}{w_-\left(1-p\right) - w_-\left(1-\gamma\right)}\right)$$

□

The objective function in the negative part problem becomes

$$q\left(s,p,\gamma\right)\int_p^{\gamma} G_\rho(x)\mathrm{d}x + L\int_{\gamma}^1 G_\rho(x)\mathrm{d}x$$
$$= q\left(s,p,\gamma\right)\mathrm{e}^{-rT}\left[\Phi\left(\Phi^{-1}(\gamma) - \theta\sqrt{T}\right) - \Phi\left(\Phi^{-1}(p) - \theta\sqrt{T}\right)\right] + L\mathrm{e}^{-rT}\left(1 - \Phi\left(\Phi^{-1}(\gamma) - \theta\sqrt{T}\right)\right)$$

4.3.1 Determining Optimal Parameters and Optimal Contract

Combining the objective functions in the positive and negative part problems, we have the following objective function:

$$\begin{aligned}
\mathcal{J}'\left(s,p,\alpha,\beta,\gamma,\lambda\right) = {} & \alpha\mathrm{e}^{-rT}\left[\Phi\left(\Phi^{-1}(\beta) - \theta\sqrt{T}\right) - \Phi\left(\Phi^{-1}(1-p) - \theta\sqrt{T}\right)\right] \\
& + \left(\alpha\mathrm{e}^{-rT}\left[1 - \Phi\left(\Phi^{-1}(\beta) - \theta\sqrt{T}\right)\right] \vee \int_{\beta}^1 \left(\left(u_+'\right)^{-1}\left(\frac{G_\rho(1-x)}{\lambda w_+'(1-x)}\right)\right) G_\rho(x)\mathrm{d}x\right) \\
& - q\left(s,p,\gamma\right)\mathrm{e}^{-rT}\left[\Phi\left(\Phi^{-1}(\gamma) - \theta\sqrt{T}\right) - \Phi\left(\Phi^{-1}(p) - \theta\sqrt{T}\right)\right] \\
& - L\mathrm{e}^{-rT}\left(1 - \Phi\left(\Phi^{-1}\left(\gamma\right) - \theta\sqrt{T}\right)\right)
\end{aligned}$$

Then we have the final optimisation problem:

$$\begin{aligned} \underset{s,p,\alpha,\beta,\gamma,\lambda}{\text{minimize}} \quad & \mathcal{J}'(s,p,\alpha,\beta,\gamma,\lambda) \\ \text{subject to} \quad & \mathcal{K}(p,\alpha,\beta,\lambda) = V + s, \\ & q(s,p,\gamma) < L, \\ & s,\alpha,\lambda \geqslant 0, \quad 0 \leqslant p \leqslant 1, \quad 1-p \leqslant \beta \leqslant 1, \quad p < \gamma \leqslant 1 \end{aligned}$$

Once the optimal parameters $s^*, p^*, \alpha^*, \beta^*, \gamma^*, \lambda^*$ are found,

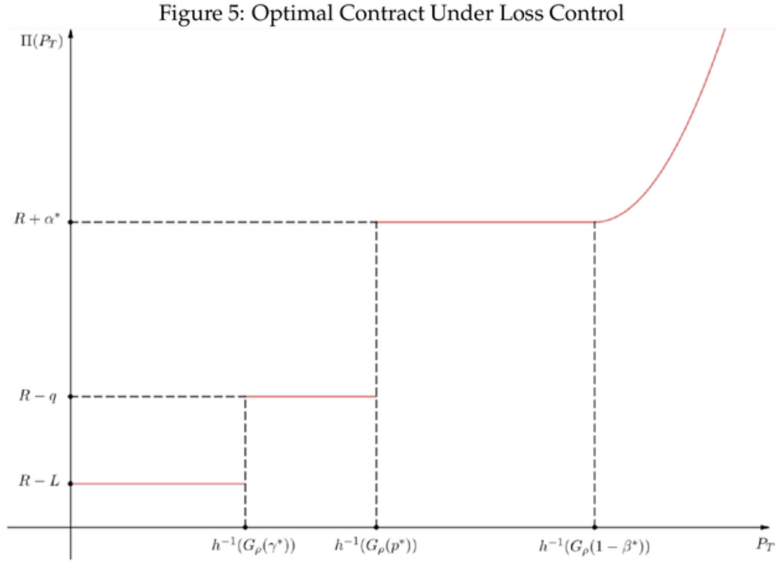
$$\begin{aligned} (X^-)^* &= G_{(X^-)^*}(F_\rho(\rho)) \\ &= (u_-)^{-1}\left(\frac{s^* - u_-(L)w_-(1-\gamma^*)}{w_-(1-p^*) - w_-(1-\gamma^*)}\right)\mathbb{1}_{(p^*,\gamma^*)}(F_\rho(\rho)) + L\mathbb{1}_{[\gamma^*,1)}(F_\rho(\rho)) \\ &= (u_-)^{-1}\left(\frac{s^* - u_-(L)w_-(1-\gamma^*)}{w_-(1-p^*) - w_-(1-\gamma^*)}\right)\mathbb{1}_{(h^{-1}(G_\rho(\gamma^*)),h^{-1}(G_\rho(p^*)))}(P_T) \\ &\quad + L\mathbb{1}_{[0,h^{-1}(G_\rho(\gamma^*)))}(P_T) \end{aligned}$$

Therefore, the optimal contract is given by

$$\begin{aligned} \Pi^*(P_T) &= (R-L)\,\mathbb{1}_{[0,h^{-1}(G_\rho(\gamma^*)))}(P_T) \\ &\quad + \left(R - (u_-)^{-1}\left(\frac{s^* - u_-(L)w_-(1-\gamma^*)}{w_-(1-p^*) - w_-(1-\gamma^*)}\right)\right)\mathbb{1}_{(h^{-1}(G_\rho(\gamma^*)),h^{-1}(G_\rho(p^*)))}(P_T) \\ &\quad + (R+\alpha^*)\,\mathbb{1}_{(h^{-1}(G_\rho(p^*)),h^{-1}(G_\rho(1-\beta^*))]}(P_T) \\ &\quad + \left((R+\alpha^*) \vee \left(R + \left(u'_+\right)^{-1}\left(\frac{h(P_T)}{\lambda^* w'_+(F_\rho(h(P_T)))}\right)\right)\right)\mathbb{1}_{(h^{-1}(G_\rho(1-\beta^*)),\infty)}(P_T) \end{aligned}$$

and this is depicted in Figure 3 below.

Figure 5: Optimal Contract Under Loss Control



4.4 Optimality of Call Option

Having determined the form of the optimal contract in ESOs, we would like to determine conditions in the CPT framework under which the non-constant portion of the contract is linear, resembling the payoff of a call option. In other words, we derive conditions such that the call option is optimal. In cases where the call option is not optimal, we will use the static replication method by Carr and Madan [12] to show that the payoff could be replicated by a portfolio of call options.

Suppose we have, for $0 < \kappa < 1$, the following utility function of employees evaluated on gains:

$$u_+(x) := x^\kappa \implies \left(u_+^{'}\right)^{-1}(x) = \left(\frac{x}{\kappa}\right)^{\frac{1}{\kappa-1}}$$

where κ is the risk-aversion parameter (lower values of κ correspond to higher risk aversion) and we want to find w_+ such that the non-constant portion of the contract function is given by a *power function* which includes the linear case, and the convexity or concavity of the function can be easily inferred. Precisely, we want to find w_+ such that

$$\left(u_+^{'}\right)^{-1}\left(\frac{h\left(P_T\right)}{\lambda^* w_+^{'}\left(F_\rho\left(h\left(P_T\right)\right)\right)}\right) = \left[\frac{CP_T^\delta}{\kappa\lambda^* w_+^{'}\left(F_\rho\left(CP_T^\delta\right)\right)}\right]^{\frac{1}{\kappa-1}} = AP_T^b$$

for some constants $A, b > 0$, which ensure that the function is strictly increasing. Since F_ρ is strictly increasing (it can be checked that its first order derivative is strictly positive for all values in $\mathbb{R}_{>0}$), its inverse G_ρ is unique, and hence the final equality is only possible if we define w_+ by

$$w_+^{'}(x) := d\left(G_\rho(x)\right)^m = d\exp\left(m\theta\sqrt{T}\Phi^{-1}(x) - m\left(r + \frac{1}{2}\theta^2\right)T\right)$$

for some $d \in \mathbb{R}_{>0}$ and $m \in \mathbb{R} \setminus \{1\}$ to be determined. We exclude $m = 1$ as this would reduce the contract function to a constant. This definition of w_+ such that it depends on G_ρ, the quantile function of ρ is intuitive, as the distortion of probabilities by employees is linked with the performance of the firm. $d \neq 0$ ensures that the function is defined (not dividing by zero) and $d > 0$ ensures that $w_+^{'}(x) > 0$ for all $x \in [0, 1]$ and therefore w_+ is strictly increasing.

Tversky and Kahneman (1992) [3] stipulates that $w_+^{'}(0) = w_+^{'}(1) = +\infty$, which indicates the observation that the most significant distortions are on very small and very large probabilities. For $m > 0$, our definition of w_+ satisfies $w_+^{'}(1) = +\infty$, but $w_+^{'}(0) = 0 < +\infty$, since

$$G_\rho(0) = \exp\left(\theta\sqrt{T}\Phi^{-1}(0) - \left(r + \frac{1}{2}\theta^2\right)T\right) = 0$$

However, for $m < 0$, we have the opposite: $w_+^{'}(1) = 0$ and $w_+^{'}(0) = +\infty$. Nevertheless, this issue will be subsequently resolved.

Next, we can show that w_+ is either convex or concave, depending on the sign of m, by computing its second order derivative:

$$w_+''(x) = dm\left(G_\rho(x)\right)^{m-1}\frac{\mathrm{d}}{\mathrm{d}x}\left(G_\rho(x)\right) = dm\left(G_\rho(x)\right)^{m-1}\left(G_\rho(x)\right)\left(\theta\sqrt{T}\frac{\mathrm{d}}{\mathrm{d}x}\left(\Phi^{-1}(x)\right)\right)$$
$$= \frac{dm\theta\sqrt{T}\left(G_\rho(x)\right)^m}{\phi\left(\Phi^{-1}(x)\right)}$$

Therefore, since $d > 0$, $w_+''(x) \leqslant 0$ for all $x \in [0,1]$ when $m \leqslant 0$ and $w_+''(x) \geqslant 0$ for all $x \in [0,1]$ when $m \geqslant 0$. In other words, w_+ is concave for $m \leqslant 0$ and convex for $m \geqslant 0$. However, Tversky and Kahneman (1992) [3] also proposed that w_+ should have a reverse S-shape. Nonetheless, this issue will be subsequently resolved.

With our definition of w_+, the non-constant portion of the contract function is given by

$$\left[\frac{CP_T^\delta}{\kappa\lambda^* w_+'\left(F_\rho\left(CP_T^\delta\right)\right)}\right]^{\frac{1}{\kappa-1}} = \left(d\kappa\lambda^* C^{1-m}\right)^{\frac{1}{1-\kappa}} P_T^{\frac{\delta(1-m)}{\kappa-1}} =: AP_T^b$$

where

$$A := \left(d\kappa\lambda^* C^{\frac{1}{\delta}}\right)^{\frac{1}{1-\kappa}}, \quad b := \frac{\delta(1-m)}{\kappa-1} = \frac{\delta(m-1)}{1-\kappa}$$

In order for $b > 0$, we require $m < 1$, since $\delta < 0$ and $1-\kappa > 0$. Furthermore,

$$\frac{G_\rho(x)}{w_+'(x)} = \frac{G_\rho(x)}{\left(G_\rho(x)\right)^m} = \left(G_\rho(x)\right)^{1-m}$$

which is an increasing function of x, since G_ρ is an increasing function and $m < 1$, and therefore no truncation is required for this choice of w_+. The optimal contract function (under loss control) would then given by:

$$\begin{aligned}
\Pi^*\left(P_T\right) &= (R-L)\,\mathbb{1}_{[0,h^{-1}(G_\rho(\gamma^*)))}\left(P_T\right) \\
&\quad + \left(R - (u_-)^{-1}\left(\frac{s^* - u_-(L)w_-\left(1-\gamma^*\right)}{w_-\left(1-p^*\right) - w_-\left(1-\gamma^*\right)}\right)\right)\mathbb{1}_{(h^{-1}(G_\rho(\gamma^*)),h^{-1}(G_\rho(p^*)))}\left(P_T\right) \\
&\quad + \left(\left(R + \left(u_+'\right)^{-1}\left(\frac{h\left(P_T\right)}{\lambda^* w_+'(F_\rho\left(h\left(P_T\right)\right))}\right)\right)\right)\mathbb{1}_{(h^{-1}(G_\rho(p^*)),\infty)}\left(P_T\right) \\
&= (R-L)\,\mathbb{1}_{[0,h^{-1}(G_\rho(\gamma^*)))}\left(P_T\right) \\
&\quad + \left(R - (u_-)^{-1}\left(\frac{s^* - u_-(L)w_-\left(1-\gamma^*\right)}{w_-\left(1-p^*\right) - w_-\left(1-\gamma^*\right)}\right)\right)\mathbb{1}_{(h^{-1}(G_\rho(\gamma^*)),h^{-1}(G_\rho(p^*)))}\left(P_T\right) \\
&\quad + \left(R + AP_T^b\right)\mathbb{1}_{(h^{-1}(G_\rho(p^*)),\infty)}\left(P_T\right)
\end{aligned}$$

We can find w_+ through integration:

$$\begin{aligned} w_+(x) = d\int_0^x \left(G_\rho(y)\right)^m \mathrm{d}y + c &= d\mathrm{e}^{-m\left(r+\frac{1}{2}\theta^2\right)T}\int_0^x \mathrm{e}^{m\theta\sqrt{T}\Phi^{-1}(y)}\mathrm{d}y + c \\ &= d\mathrm{e}^{-m\left(r+\frac{1}{2}\theta^2\right)T}\int_{-\infty}^{\Phi^{-1}(x)} \mathrm{e}^{m\theta\sqrt{T}z}\phi(z)\mathrm{d}z + c \\ &= d\mathrm{e}^{-m\left(r+\frac{1}{2}\theta^2\right)T}\mathrm{e}^{\frac{1}{2}m^2\theta^2T}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\Phi^{-1}(x)} \mathrm{e}^{-\frac{1}{2}\left(z-m\theta\sqrt{T}\right)^2}\mathrm{d}z + c \\ &= d\mathrm{e}^{\frac{1}{2}m^2\theta^2T-m\left(r+\frac{1}{2}\theta^2\right)T}\Phi\left(\Phi^{-1}(x)-m\theta\sqrt{T}\right) + c \end{aligned}$$

where $c \in \mathbb{R}$ is a constant that ensures $w_+(0) = 0$ and $w_+(1) = 1$. We have:

$$w_+(0) = d\mathrm{e}^{\frac{1}{2}m^2\theta^2T-m\left(r+\frac{1}{2}\theta^2\right)T}\Phi\left(\Phi^{-1}(0)-m\theta\sqrt{T}\right) + c = d\mathrm{e}^{\frac{1}{2}m^2\theta^2T-m\left(r+\frac{1}{2}\theta^2\right)T}\Phi\left(-\infty\right) + c = c$$

which implies $c = 0$. Then,

$$\begin{aligned} w_+(1) = d\mathrm{e}^{\frac{1}{2}m^2\theta^2T-m\left(r+\frac{1}{2}\theta^2\right)T}\Phi\left(\Phi^{-1}(1)-m\theta\sqrt{T}\right) &= d\mathrm{e}^{\frac{1}{2}m^2\theta^2T-m\left(r+\frac{1}{2}\theta^2\right)T}\Phi\left(+\infty\right) \\ &= d\mathrm{e}^{\frac{1}{2}m^2\theta^2T-m\left(r+\frac{1}{2}\theta^2\right)T} = 1 \end{aligned}$$

implies that we must have

$$d = \mathrm{e}^{-\frac{1}{2}m^2\theta^2T+m\left(r+\frac{1}{2}\theta^2\right)T}$$

We now resolve the issue of w_+ not having a reverse S-shape. Since the non-constant portion of the contract function corresponds to higher values of P_T (and hence smaller probabilities), we only require

$$w_+(x) = d\mathrm{e}^{\frac{1}{2}m^2\theta^2T-m\left(r+\frac{1}{2}\theta^2\right)T}\Phi\left(\Phi^{-1}(x)-m\theta\sqrt{T}\right) =: \overline{d}\Phi\left(\Phi^{-1}(x)-m\theta\sqrt{T}\right) \tag{4.5}$$

for $x \in [0,\xi]$ for some $\xi \in (0,1)$, where $\overline{d} := d\mathrm{e}^{\frac{1}{2}m^2\theta^2T-m\left(r+\frac{1}{2}\theta^2\right)T}$. In particular, we no longer require $w_+(1) = 1$, and hence do not require $d = \mathrm{e}^{-\frac{1}{2}m^2\theta^2T+m\left(r+\frac{1}{2}\theta^2\right)T}$. To be precise, equation (4.5) should hold for probabilities less than or equal to

$$\varphi := \mathbb{P}\left(P_T \geqslant h^{-1}\left(G_\rho\left(p^*\right)\right)\right) = 1 - \Phi\left(\frac{\log\left(\frac{h^{-1}\left(G_\rho(p^*)\right)}{P_0}\right) - \left(\mu - \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}\right)$$

In particular, we only require w_+ to be differentiable on $[0,\xi]$. In order for w_+ to have a reverse S-shape, it should be concave on $[0,\xi]$. Therefore, we now only consider the case $m < 0$ (corresponding to concave w_+ and excluding the case where there is no probability distortion), which also guarantees $m < 1$ (required for the contract function to be increasing), and construct a convex function on $[\xi,1]$ to obtain our reverse S-shaped function. To be precise, we take ξ as our inflection point and construct a convex function $\widehat{w}_+$ on $[\xi,1]$ such that

$$\widehat{w}_+\left(\xi\right) = w_+\left(\xi\right) = \overline{d}\Phi\left(\Phi^{-1}\left(\xi\right)-m\theta\sqrt{T}\right), \quad \widehat{w}_+(1) = 1 \tag{4.6}$$

Empirical studies have shown that the inflection point is approximately 0.4 [10]. Therefore, we may set the inflection point as $\xi := \max\{0.4, \varphi\}$. Next, define for $x \in [\xi, 1]$,

$$\widehat{w}_+(x) := 1 - \nu + \nu\left[\Phi\left(\Phi^{-1}(x) + \zeta\right)\right]$$

where $\nu, \zeta \in \mathbb{R}$ are chosen such that $\widehat{w}_+$ is increasing, convex and satisfies equation (4.6). Indeed, $\widehat{w}_+(1) = 1$ for any $\nu, \zeta \in \mathbb{R}$. The first order derivative is

$$\begin{aligned}\widehat{w}'_+(x) = \nu\left[\frac{\phi\left(\Phi^{-1}(x) + \zeta\right)}{\phi\left(\Phi^{-1}(x)\right)}\right] &= \nu\exp\left[\frac{1}{2}\left(\left(\Phi^{-1}(x)\right)^2 - \left(\Phi^{-1}(x) + \zeta\right)^2\right)\right] \\ &= \nu\exp\left(-\zeta\Phi^{-1}(x) - \frac{1}{2}\zeta^2\right)\end{aligned}$$

In order for $\widehat{w}_+$ to be increasing, we require $\widehat{w}'_+ \geqslant 0$, which is satisfied if $\nu \geqslant 0$. However, we exclude $\nu = 0$ as this reduces to $\widehat{w}'_+(x) = 1$ for all $x \in [\xi, 1]$, the case of no probability distortion for probabilities in $[\xi, 1]$. Since Φ^{-1} is an increasing function, $\widehat{w}'_+$ is increasing ($\widehat{w}_+$ is convex) if $\zeta < 0$. Then,

$$\begin{aligned}\frac{G_\rho(x)}{\widehat{w}'_+(x)} &= \frac{\exp\left(\theta\sqrt{T}\Phi^{-1}(x) - \left(r + \frac{1}{2}\theta^2\right)T\right)}{\nu\exp\left(-\zeta\Phi^{-1}(x) - \frac{1}{2}\zeta^2\right)} \\ &= \frac{1}{\nu}\exp\left(\left(\theta\sqrt{T} + \zeta\right)\Phi^{-1}(x) + \frac{1}{2}\zeta^2 - \left(r + \frac{1}{2}\theta^2\right)T\right)\end{aligned}$$

which is non-decreasing in x only if $\theta\sqrt{T} + \zeta \geqslant 0$, that is, if $-\theta\sqrt{T} \leqslant \zeta < 0$. Therefore, truncation is not required if $-\theta\sqrt{T} \leqslant \zeta < 0$, and is required otherwise. Finally, we determine the values $\nu > 0, \zeta < 0$ (which may not be unique) such that

$$\begin{aligned}1 - \nu + \nu\left[\Phi\left(\Phi^{-1}(\xi) + \zeta\right)\right] &= \overline{d}\Phi\left(\Phi^{-1}(\xi) - m\theta\sqrt{T}\right) \\ \Longleftrightarrow \zeta &= \Phi^{-1}\left[\frac{1}{\nu}\left(\overline{d}\Phi\left(\Phi^{-1}(\xi) - m\theta\sqrt{T}\right) + \nu - 1\right)\right] - \Phi^{-1}(\xi)\end{aligned}$$

In other words, we choose $\nu > 0$ such that the right-hand side of the above equation is negative, and this choice of ν determines the value of ζ. Alternatively, we may choose ζ, which determines whether truncation is necessary and the choice of ζ determines the value of ν:

$$\nu = \frac{\overline{d}\Phi\left(\Phi^{-1}(\xi) - m\theta\sqrt{T}\right) - 1}{\Phi\left(\Phi^{-1}(\xi) + \zeta\right) - 1}$$

We have therefore constructed the following reverse S-shaped probability distortion function:

$$w_+(x) := \begin{cases}\overline{d}\Phi\left(\Phi^{-1}(x) - m\theta\sqrt{T}\right), & x \in [0, \xi] \\ 1 - \nu + \nu\left[\Phi\left(\Phi^{-1}(x) + \zeta\right)\right], & x \in [\xi, 1]\end{cases}$$

Furthermore, since $m < 0$, we have $w'_+(0) = \infty$ and since $\zeta < 0$,

$$\widehat{w}'_+(1) = \nu\exp\left(-\zeta\Phi^{-1}(1) - \frac{1}{2}\zeta^2\right) = \infty$$

and therefore the two issues previously identified have now been resolved. Our construction of this reverse S-shaped probability distortion function is similar to the smooth-pasting of two one-parameter weighting functions, implemented in [15]. We have two free parameters, m and ν (or ζ, whichever is chosen first), and we shall discuss the implications of the choice of these parameters. Firstly, the choice of m determines the shape of the optimal contract function. We can determine the value of m such that the non-constant portion of the contract function is linear in P_T:

$$b = \frac{\delta(m-1)}{1-\kappa} = 1 \iff m = 1 + \frac{1-\kappa}{\delta} = 1 - \frac{\sigma^2(1-\kappa)}{\mu - r}$$

where $\delta < 0$ and $\kappa \in (0,1)$ are given parameters that can be estimated from data or experimental studies, and δ is determined by the performance of the firm while κ is determined by the degree of risk aversion of the employees. However, we require $m < 0$, and therefore we require

$$1 - \frac{\sigma^2(1-\kappa)}{\mu - r} < 0 \implies \frac{\mu - r}{\sigma^2} < 1 - \kappa$$

If the above holds, the non-constant payoff in the contract function is equal to R plus A units of a European call option with strike $h^{-1}\left(G_\rho\left(p^*\right)\right)$. We can also consider values of m such that the function is convex or concave in P_T, determined by examining the second order derivative of the power function:

$$\frac{\mathrm{d}^2}{\mathrm{d}P_T^2}\left[AP_T^b\right] = Ab(b-1)P_T^{b-2}$$

which is negative when $b \leqslant 1$ (concave) and positive when $b \geqslant 1$ (convex). The function is convex when

$$b = \frac{\delta(m-1)}{1-\kappa} \geqslant 1 \iff m \geqslant 1 + \frac{1-\kappa}{\delta} = 1 - \frac{\sigma^2(1-\kappa)}{\mu - r}$$

Since $m < 0$, we must have

$$1 - \frac{\sigma^2(1-\kappa)}{\mu - r} \leqslant m < 0 \implies \frac{\mu - r}{\sigma^2} < 1 - \kappa$$

which corresponds to the same inequalities involving δ and κ for the linear function, however several values of m (greater than or equal to $1 + \frac{1-\kappa}{\delta}$) are allowed compared to only one value of m (equal to $1 + \frac{1-\kappa}{\delta}$) for the linear function. Since we expect employees to be fairly risk averse (corresponding to lower values of κ), we expect that the inequality above usually holds. The function is concave when

$$b = \frac{\delta(m-1)}{1-\kappa} \leqslant 1 \iff m \leqslant 1 + \frac{1-\kappa}{\delta} = 1 - \frac{\sigma^2(1-\kappa)}{\mu - r}$$

Since $m < 0$, there are two possible cases:

$$m \leqslant 1 - \frac{\sigma^2(1-\kappa)}{\mu - r} < 0 \implies \frac{\mu - r}{\sigma^2} < 1 - \kappa$$

or

$$m < 0 \leqslant 1 - \frac{\sigma^2 (1-\kappa)}{\mu - r} \implies \frac{\mu - r}{\sigma^2} \geqslant 1 - \kappa$$

Since our contract function is a power function, it is twice continuously differentiable, and according to Carr-Madan static replication (which has been used to price variance swaps in [13]), the payoff can be replicated by unique portfolio consisting of initial positions in unit discount bonds, shares in stock and out-of-the-money European call and put options of different strikes K, formalised in the theorem below [12].

Theorem 4.9 (Carr-Madan Static Replication). *Let $f \in C^2$ be a twice continuously differentiable function, and let $(S_t)_{t\in[0,T]}$ be the price process of a stock. Then*

$$f(S_T) = [f(S_0) - f'(S_0)S_0] + f'(S_0)S_T + \int_0^{S_0} f''(K)(K - S_T)^+ \,\mathrm{d}K + \int_{S_0}^{\infty} f''(K)(S_T - K)^+ \,\mathrm{d}K$$

In other words, the terminal payoff $f(S_T)$ can be replicated by a position in $[f(S_0) - f'(S_0)S_0]$ unit discount bonds, $f'(S_0)$ shares of stock, and $f''(K)\mathrm{d}K$ out-of-the-money European call and put options of all strikes K.

Proof. By the fundamental theorem of calculus, for any fixed F, letting $S := S_T$,

$$\begin{aligned} f(S) &= f(F) + \mathbb{1}_{\{S>F\}} \int_F^S f'(u)\mathrm{d}u - \mathbb{1}_{\{S<F\}} \int_S^F f'(u)\mathrm{d}u \\ &= f(F) + \mathbb{1}_{\{S>F\}} \int_F^S \left[f'(F) + \int_F^u f''(v)\mathrm{d}v \right] \mathrm{d}u - \mathbb{1}_{\{S<F\}} \int_S^F \left[f'(F) - \int_u^F f''(v)\mathrm{d}v \right] \mathrm{d}u \end{aligned}$$

Applying Fubini's theorem and the fact that $f'(F)$ does not depend on u, we obtain

$$\begin{aligned} f(S) &= f(F) + f'(F)(S-F) + \mathbb{1}_{\{S>F\}} \int_F^S \int_v^S f''(v)\mathrm{d}u\mathrm{d}v + \mathbb{1}_{\{S<F\}} \int_S^F \int_S^v f''(v)\mathrm{d}u\mathrm{d}v \\ &= f(F) + f'(F)(S-F) + \mathbb{1}_{\{S>F\}} \int_F^S f''(v)(S-v)\mathrm{d}v + \mathbb{1}_{\{S<F\}} \int_S^F f''(v)(v-S)\mathrm{d}v \\ &= f(F) + f'(F)(S-F) + \int_F^{\infty} f''(v)(S-v)^+\mathrm{d}v + \int_0^F f''(v)(v-S)^+\mathrm{d}v \\ &= [f(F) - f'(F)F] + f'(F)S + \int_0^F f''(v)(v-S)^+\mathrm{d}v + \int_F^{\infty} f''(v)(S-v)^+\mathrm{d}v \end{aligned}$$

Setting $F := S_0$, we obtain the required equation. □

When the contract function is concave, its second derivative is negative and the payoff must be replicated by taking short positions in the call and put options. However, in our context, employees only take long positions in the employee stock options, and hence we will now only consider convex contract functions, in which the payoff is replicated by taking long positions in the call and put options. We also have that since our contract function is increasing, the first derivative is positive which corresponds to a long position in shares of firm value. We have

$$\begin{aligned} AP_T^b &= \left[AP_0^b - AbP_0^{b-1}P_0\right] + \left(AbP_0^{b-1}\right)P_T \\ &+ \int_0^{P_0} Ab(b-1)K^{b-2}\left(K-P_T\right)^+ \mathrm{d}K + \int_{P_0}^{\infty} Ab(b-1)K^{b-2}\left(P_T-K\right)^+ \mathrm{d}K \\ &= (1-b)AP_0^b + \left(AbP_0^{b-1}\right)P_T \\ &+ Ab(b-1)\left(\int_0^{P_0} K^{b-2}\left(K-P_T\right)^+ \mathrm{d}K + \int_{P_0}^{\infty} K^{b-2}\left(P_T-K\right)^+ \mathrm{d}K\right) \end{aligned}$$

If $P_0 < h^{-1}\left(G_\rho\left(p^*\right)\right)$, then

$$\begin{aligned} \left(K-P_T\right)^+ &= K, \quad P_T \in [0, P_0] \\ \left(P_T-K\right)^+ &= 0, \quad P_T \in \left[P_0, h^{-1}\left(G_\rho\left(p^*\right)\right)\right] \end{aligned}$$

Therefore,

$$\begin{aligned} &\left(R + AP_T^b\right)\mathbb{1}_{\left(h^{-1}\left(G_\rho(p^*)\right),\infty\right)}\left(P_T\right) \\ &= R + (1-b)AP_0^b + \left(AbP_0^{b-1}\right)P_T + Ab(b-1)\left(\int_0^{P_0} K^{b-1}\mathrm{d}K + \int_{h^{-1}\left(G_\rho(p^*)\right)}^{\infty} K^{b-2}\left(P_T-K\right)^+ \mathrm{d}K\right) \\ &= R + (1-b)AP_0^b + \left(AbP_0^{b-1}\right)P_T + Ab(b-1)\left(\left(\frac{P_0^b}{b}\right) + \int_{h^{-1}\left(G_\rho(p^*)\right)}^{\infty} K^{b-2}\left(P_T-K\right)^+ \mathrm{d}K\right) \\ &= R + \left(AbP_0^{b-1}\right)P_T + Ab(b-1)\int_{h^{-1}\left(G_\rho(p^*)\right)}^{\infty} K^{b-2}\left(P_T-K\right)^+ \mathrm{d}K \end{aligned}$$

and therefore the payoff is only replicated with shares in the firm value and European call options. However, in reality the set of strikes corresponding to the European options in the portfolio is discrete, and therefore numerical integration is required to approximate the above payoff. In [13], continuous replication and several methods of discrete replication (Derman's method, trapezoidal method, Simpson's quadrature and the Leung and Lorig Optimal Quadratic Hedge) are considered. Of these methods, the trapezoidal method (also used in [14]) is most easily adapted to this context: we adopt the following partition of strike prices:

$$\log K_i = \log\left(h^{-1}\left(G_\rho\left(p^*\right)\right)\right) + i\eta$$

for all $i = 0, 1, ..., n$ where $\eta > 0$ is chosen to be sufficiently small in order to minimise the numerical integration error. Then the trapezoidal rule gives the following approximation:

$$\begin{aligned} &R + \left(AbP_0^{b-1}\right)P_T + Ab(b-1)\int_{h^{-1}\left(G_\rho(p^*)\right)}^{\infty} K^{b-2}\left(P_T-K\right)^+ \mathrm{d}K \\ &\approx R + \left(AbP_0^{b-1}\right)P_T + Ab(b-1)\sum_{i=1}^{n}\left(\frac{\Delta K_i}{2}\right)\left[\left(K_i\right)^{b-2}\left(P_T-K_i\right)^+ + \left(K_{i-1}\right)^{b-2}\left(P_T-K_{i-1}\right)^+\right] \end{aligned}$$

where $\Delta K_i := K_i - K_{i-1}$. If we further assume that the strikes are equidistributed (as assumed in [13]), that is, $\Delta K_i = h > 0$ for all i (a reasonable assumption if $\eta > 0$ is small enough), then the approximation becomes

$$\begin{aligned}
&\approx R+\left(AbP_0^{b-1}\right)P_T\\
&\quad+\left(\frac{Ab(b-1)h}{2}\right)\left(\left(K_0\right)^{b-2}\left(P_T-K_0\right)^{+}+\sum_{i=1}^{n-1}2\left(K_i\right)^{b-2}\left(P_T-K_i\right)^{+}+\left(K_n\right)^{b-2}\left(P_T-K_n\right)^{+}\right)\\
&=:R+\left(AbP_0^{b-1}\right)P_T+\sum_{i=0}^{n}w_i\left(P_T-K_i\right)^{+}
\end{aligned}$$

where $(w_i)_{i=0,1,\ldots,n}$ are the replication weights of the European call options with strikes $(K_i)_{i=0,1,\ldots,n}$ given by

$$w_i:=\begin{cases}\frac{1}{2}Ab(b-1)h\left(K_i\right)^{b-2}, & i=0,n\\ Ab(b-1)h\left(K_i\right)^{b-2}, & i=1,\ldots n-1\end{cases}$$

If $P_0\geqslant h^{-1}\left(G_\rho\left(p^*\right)\right)$, then

$$\left(K-P_T\right)^{+}=K,\quad P_T\in\left[0,h^{-1}\left(G_\rho\left(p^*\right)\right)\right]$$

and

$$\begin{aligned}
&\left(R+AP_T^b\right)\mathbb{1}_{\left(h^{-1}\left(G_\rho(p^*)\right),\infty\right)}\left(P_T\right)\\
&=R+(1-b)AP_0^b+\left(AbP_0^{b-1}\right)P_T\\
&\quad+Ab(b-1)\left(\int_0^{h^{-1}\left(G_\rho(p^*)\right)}K^{b-1}\mathrm{d}K+\int_{h^{-1}\left(G_\rho(p^*)\right)}^{P_0}K^{b-2}\left(K-P_T\right)^{+}\mathrm{d}K+\int_{P_0}^{\infty}K^{b-2}\left(P_T-K\right)^{+}\mathrm{d}K\right)\\
&=R+(1-b)A\left(P_0^b-\left(h^{-1}\left(G_\rho\left(p^*\right)\right)\right)^b\right)+\left(AbP_0^{b-1}\right)P_T\\
&\quad+Ab(b-1)\left(\int_{h^{-1}\left(G_\rho(p^*)\right)}^{P_0}K^{b-2}\left(K-P_T\right)^{+}\mathrm{d}K+\int_{P_0}^{\infty}K^{b-2}\left(P_T-K\right)^{+}\mathrm{d}K\right)
\end{aligned}$$

and the payoff is replicated by a short position in bonds and long positions in shares in firm value, European call and put options, and numerical integration can be carried out by the trapezoidal method similarly to the previous case.

Finally, due to the flexibility in choice of m for the convex function, m and ν may be chosen to calibrate the function w_+ we constructed to the well-known reverse S-shaped probability distortion function proposed by Tversky and Kahneman (1992):

$$w_{\pm}^{TK}(x):=\frac{x^{\delta_\pm}}{\left(x^{\delta_\pm}+(1-x)^{\delta_\pm}\right)^{\frac{1}{\delta_\pm}}}$$

for $0.28<\delta_\pm\leqslant 1$. Estimates of the probability weighting parameters $\delta_\pm$ have been rather consistent across experimental and empirical studies. In particular, the studies concluded that $\delta_+\approx 0.7$.

4.5 Summary

To summarise, a similar solution structure is obtained in the optimal ESO contract problem to that in the behavioural portfolio selection, but with an additional step to write the solution in terms of P_T, to recover the optimal contract, and this additional step relies on ρ being written explicitly as a function of P_T. We state theorems stating the optimal contract with and without loss control, and a proposition regarding the optimality of a call option contract:

Theorem 4.10 (Optimal Contract without Loss Control). *The optimal contract without loss control is given by*

$$\begin{aligned}\Pi^*(P_T) &= \left(R - (u_-)^{-1}\left(\frac{s^*}{w_-(1-p^*)}\right)\right)\mathbb{1}_{\left(0,h^{-1}\left(G_\rho(p^*)\right)\right)}(P_T)\\ &+ (R+\alpha^*)\,\mathbb{1}_{\left(h^{-1}\left(G_\rho(p^*)\right),h^{-1}\left(G_\rho(1-\beta^*)\right)\right]}(P_T)\\ &+ \left((R+\alpha^*)\vee\left(R+\left(u_+'\right)^{-1}\left(\frac{h(P_T)}{\lambda^* w_+'(F_\rho(h(P_T)))}\right)\right)\right)\mathbb{1}_{\left(h^{-1}\left(G_\rho(1-\beta^*)\right),\infty\right)}(P_T)\end{aligned}$$

Theorem 4.11 (Optimal Contract with Loss Control). *The optimal contract with loss control is given by*

$$\begin{aligned}\Pi^*(P_T) &= (R-L)\,\mathbb{1}_{\left[0,h^{-1}\left(G_\rho(\gamma^*)\right)\right)}(P_T)\\ &+ \left(R-(u_-)^{-1}\left(\frac{s^*-u_-(L)w_-(1-\gamma^*)}{w_-(1-p^*)-w_-(1-\gamma^*)}\right)\right)\mathbb{1}_{\left(h^{-1}\left(G_\rho(\gamma^*)\right),h^{-1}\left(G_\rho(p^*)\right)\right)}(P_T)\\ &+ (R+\alpha^*)\,\mathbb{1}_{\left(h^{-1}\left(G_\rho(p^*)\right),h^{-1}\left(G_\rho(1-\beta^*)\right)\right]}(P_T)\\ &+ \left((R+\alpha^*)\vee\left(R+\left(u_+'\right)^{-1}\left(\frac{h(P_T)}{\lambda^* w_+'(F_\rho(h(P_T)))}\right)\right)\right)\mathbb{1}_{\left(h^{-1}\left(G_\rho(1-\beta^*)\right),\infty\right)}(P_T)\end{aligned}$$

Proposition 4.12 (Optimality of a Call Option Contract). *Suppose u_+ satisfies $u_+(x) := x^\kappa$ and w_+ satisfies*

$$w_+(x) := \begin{cases}\overline{d}\Phi\left(\Phi^{-1}(x) - m\theta\sqrt{T}\right), & x\in[0,\xi]\\ 1-\nu+\nu\left[\Phi\left(\Phi^{-1}(x)+\zeta\right)\right], & x\in[\xi,1]\end{cases}$$

where $\overline{d} := de^{\frac{1}{2}m^2\theta^2T - m\left(r+\frac{1}{2}\theta^2\right)T}$ and d is a constant, and ν, ζ satisfy

$$1-\nu+\nu\left[\Phi\left(\Phi^{-1}(\xi)+\zeta\right)\right] = \overline{d}\Phi\left(\Phi^{-1}(\xi)-m\theta\sqrt{T}\right),\quad -\theta\sqrt{T}\leqslant\zeta<0$$

and

$$m = 1+\frac{1-\kappa}{\delta} = 1-\frac{\sigma^2(1-\kappa)}{\mu-r}$$

Then a call option ESO contract is optimal, and without loss control is given by

$$\begin{aligned}\Pi^*(P_T) &= \left(R-(u_-)^{-1}\left(\frac{s^*}{w_-(1-p^*)}\right)\right)\mathbb{1}_{\left(0,h^{-1}\left(G_\rho(p^*)\right)\right)}(P_T)\\ &+ (R+AP_T)\,\mathbb{1}_{\left(h^{-1}\left(G_\rho(1-\beta^*)\right),\infty\right)}(P_T)\end{aligned}$$

where $A := \left(d\kappa\lambda^* C^{\frac{1}{\delta}}\right)^{\frac{1}{1-\kappa}}$. *When there is loss control, the optimal call option ESO contract is given by*

$$\begin{aligned}\Pi^*(P_T) &= (R-L)\,\mathbb{1}_{[0,h^{-1}(G_\rho(\gamma^*)))}(P_T)\\ &+ \left(R-(u_-)^{-1}\left(\frac{s^*-u_-(L)w_-(1-\gamma^*)}{w_-(1-p^*)-w_-(1-\gamma^*)}\right)\right)\mathbb{1}_{(h^{-1}(G_\rho(\gamma^*)),h^{-1}(G_\rho(p^*)))}(P_T)\\ &+ (R+AP_T)\,\mathbb{1}_{(h^{-1}(G_\rho(1-\beta^*)),\infty)}(P_T)\end{aligned}$$

Remark 4.13. Proposition 4.12 states that the parameters m, ν and ζ must satisfy those conditions in order for the call option contract to be optimal. However, these choices of parameters may not calibrate as well to Tversky and Kahneman's probability distortion function $w_\pm^{TK}$. Parameters that calibrate better to $w_\pm^{TK}$ may give rise to an optimal contract involving a portfolio of call and/or put options, using Carr-Madan replication.

5 Conclusion and Further Research

Two stochastic optimisation problems under probability distortion, namely optimal stopping and behavioural portfolio selection, were reviewed, and a solution for a new such problem was developed: determining the optimal contract for employee stock options. In optimal stopping under probability distortion, several variants of the underlying process were also considered. The behavioural portfolio selection problem is more closely related to the employee stock option problem, as both involve a pricing kernel. Even though the former is a maximisation problem while the latter is a minimisation problem, their solution structure is similar, with the latter requiring an additional step to express the solution in terms of the terminal firm value. Further research in the optimal stopping problem may include developing new methods to address the limitations of the current method as discussed in Section 2.4. For the employee stock option problem, numerical studies could be carried out to verify the optimality of the general optimal contract and compare it with the contract in Spalt's model, and to investigate the calibration of the constructed probability distortion function in Section 4.4 to that of Tversky and Kahneman's.

Even though we have considered a more general contract to the one considered by Spalt in [9], it is still a rather simplified model, as it assumes that the ESO can only be exercised at the maturity date T. In reality, as discussed in [16] ESOs are American style options, meaning employees have the right to exercise the option at any time before a maturity date, and ESOs could be terminated early due to employment shock. In [16], Leung and Wan investigate the valuation of ESOs taking into consideration job termination risk, and show through applications of variational inequalities that the increase of such a risk leads to exercise of the ESO being voluntarily accelerated, which in turn leads to reduced costs to the firm. However, Leung and Wan do not incorporate probability distortion into their analysis, and this remains an open problem and an avenue for further research.

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