Imperial College London

Department of Mathematics

# Robust option pricing: the uncertain volatility model

*Supervisor:* Dr. Pietro SIORPAES
*Author:* Hicham EL JERRARI (CID: 01792391)

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# Declaration

The work contained in this thesis is my own work unless otherwise stated.

Signature and date:

06/09/2020

Hicham EL JERRARI

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**Abstract**

This study presents the uncertain volatility model (UVM) which proposes a new approach for the pricing and hedging of derivatives by considering a band of spot's volatility as input. This model is based on the Black-Scholes-Barenblatt (BSB) equation, which we will solve using two methods: trinomial model and finite-difference.

We have priced different portfolios: call, put, straddle, and bull spread to highlight certain properties of the UVM model, namely: the sub-additivity of the payoff, the comparison between Black-Scholes (BS) model and UVM, or the impact of the change in the volatility band on the portfolio price.

We also looked at a variant of the UVM model: the Lagrangian UVM which relies on option hedging and we have pointed out numerically that the increase in the number of options in the hedging portfolio narrows the price range, thus giving more precise results for the price of the derivatives.

**Keywords:** Uncertain volatility model, Black-Scholes model, Black-Scholes-Barenblatt equation, Finite difference, Trinomial model, Numerical implementation.

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# Introduction

Derivatives are contracts whose value is linked to the associated underlying asset. These products experienced a significant increase in the volumes traded from the 1980s [14]. Derivatives are divided into four main classes, namely: options, swaps, forwards and exotic products: more complex financial instruments that are traded on OTC markets. Derivatives are mainly used by individuals as well as by financial institutions for a variety of purposes ranging from hedging to speculation including 'synthetic exposure' [30]. One of the important issues related to these products is the determination of their "fair value" prices which depends on the evolution of the value of the underlying asset. There are different methods and manners for the pricing of derivatives namely: **analytical solution** like the closed price given by [33] for the volatility swap or **numerical approach** like the Monte-Carlo simulation used by [25], [27] and [24] to price American options, or finite difference method used by [5] paper.

The "fair value" of options can be determined in a simplified context, where markets are considered as free of arbitrage by using the binomial model [12] or the Black-Scholes model. In this context, the option price is the expectation of the discounted payoff under the risk-neutral measure which is unique only in a complete market (every position can be hedged or is replicable), as underlined in the articles [16] and [15]. The Black-Scholes model published [13] is a simplified model widely used in finance which gives an analytical price formula, considering that the volatility of the underlying is constant. However, in the real world, volatility is not constant. More complex models have appeared to correct volatility, based on stochastic volatility such as: Hull White model [19], Stein Stein model [31], the Heston model [17] and Scott model [28]. Another model has taken a different approach like the uncertain volatility model (UVM) introduced by [2], which considers that volatility belong to a certain interval given as input thanks to options prices available in the market. The latter model will specify a price range delimited by a upper price corresponding to the best scenario, in the case of a long position in the derivative, and a lower price representative of the worst scenario of the volatility path.

This study on the uncertain volatility model (UVM) is divided into five parts: In section 1, we present a benchmark of the main volatility models used. In section 2 we will introduce Black-Scholes model and price some vanilla options using this latter model. This part can be seen as an analysis of the different results of the Black-Scholes model (BS) that we will compare with the UVM model. In section 3, we will present the UVM and Lagrangian UVM models, as well as the various properties of those models. In section 4, we will discuss the implementation of the UVM model using the trinominal tree and see the difference with the classic BS model. Finally in section 5, we will introduce and implement the UVM method by adopting the finite difference method on the Black-Scholes-Barenblatt (BSB) equation.

# 1 Literature review on the main volatility models

## 1.1 Black-Scholes (BS) model

In this section, we will examine the existing modeling of asset volatility, namely the Black-Scholes model with constant volatility and the more complex models of stochastic volatility.

We consider a filtered probability space $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{Q}\right)$ where $\mathcal{F}_t$ is a filtration, and $\mathbb{Q}$ the risk-neutral measure.

The Black-Scholes equation (BS) model considers that the volatility of the spot is constant and the returns follow a log-normal distribution, as specified in [6]. Indeed, the underlying asset follows the equation below of GBM (Geometric Brownian motion):

$$dS_t = S_t\left[rdt + \sigma dZ_t\right], \quad 0 \leq t \leq T$$

With: $S_t$: the risky asset price at time $t$, $r$: risk-free rate, $\sigma$: volatility of the spot S, $Z_t$: Brownian motion, and $T$: the maturity.

As quoted in [6]:

$$ln(\frac{S_t}{S_0}) \sim N(r - \frac{\sigma^2}{2}, \sigma^2 T)$$

Returns follow a log-normal distribution, which facilitates the calculation of the price of options, which makes this model very tractable, easy to implement and to calibrate. However, considering constant volatility does not reflect the characteristics of the options market: like the heavy tails and asymmetry of returns or the smile of volatility, as specified in [21] and [10]. Indeed, on the options market we notice that volatility increases when the option is in the money or out of the money, this is called the smile of volatility, this stylised fact is not reflected with the BS model.

## 1.2 Heston Model

The most popular stochastic volatility model is the Heston model because it satisfies a number of characteristics of the options market and it has an explicit formula for the pricing options which gives more accurate prices than those offered by the BS model as showed in the article [11]. Heston model uses a stochastic volatility model where volatility follows a Cox-Ingersoll-Ross (CIR) process [17]:

$$\begin{aligned} dS_t &= rS_t dt + \sqrt{V_t} S_t dW_t^S \\ dV_t &= k\left(\theta - V_t\right) dt + \sigma\sqrt{V_t} dW_t^V \\ dW_t^S dW_t^V &= \rho dt \end{aligned}$$

Where: $V_t$ volatility of the spot S, $\sigma$ the volatility of the volatility (or volatility of $\sqrt{V_t}$), $\theta$: long-term reversion, $k$: the speed of mean reversion, $W_t^S$ and $W_t^V$ Brownian motions associated with respectively spot and volatility, and $\rho$: the correlation between spot and volatility of the risky asset.

$k$, $\theta$ and $\sigma$ are three non-negative constants. We also add the following condition which implies that the volatility $V_t$ is positive [17]:

$$2k\theta > \sigma^2$$

We can also show that $V_t$ follows a non-central chi-square distribution [8], which makes the computations for the price (for example) more complicated and less tractable than for the BS model.

One important property of the Heston model is that the volatility follows a mean-reverting process: the expected value of the volatility $V_t$ converges when t goes to infinity with a variance which does not diverge, as specified in [8]. This mean reversion property is important because it is consistent with market observations: volatility will move towards its average value [22].

Formula of the expectation and the variance of the volatility given below are extracted from [8]:

$$E[V_t] = V_0 e^{-kt} + \theta\left(1 - e^{-kt}\right)$$
$$\mathrm{VAR}(V_t) = V_0 \frac{\sigma^2}{k}\left(e^{-kt} - e^{-2kt}\right) + \theta \frac{\sigma^2}{2k}\left(1 - e^{-kt}\right)^2$$

We note that when t goes to infinity, the expected value of $V_t$ goes to $\theta$, the long-term mean reversion, and the variance of $V_t$ converges to $\frac{\theta\sigma^2}{2k}$. We note that if the mean reversion speed k goes to infinity and that the other variables are constant, the variance converges towards 0: the volatility $V_t$ converges quickly towards the long-term average. Now, if the volatility of the volatility $\sigma$ takes large values, the variance also increases and therefore convergence to the long-term average will occur later in time.

To determine the constants of the model, namely: the long-term reversion $\theta$, the speed of mean reversion $k$ and the volatility of the volatility $\sigma$, we perform the calibration which consists in finding these constants so that the price of the vanilla options given by this model is as close as possible to the market, as mentioned in [8]. [7] presents two calibration methods using historical data or the implied method which uses volatility surface.

The Heston model gives an analytical formula for the price of options, it is more consistent with the market than the BS model. Indeed, this model reproduces the smile of volatility and the returns are modeled by a fat-tailed distribution, as specified in [22]. In addition, the correlation between the stock S and volatility plays an important role in obtaining a more realistic model, as mentioned in [17]. However, its implementation and calibration are more complicated than the BS model.

## 1.3 Others Stochastic volatility model

There are also other stochastic volatility models such as: the Stein-Stein model or the White Hull model which we will briefly present. In the Stein-Stein model, volatility follows the

Ornstein-Uhlenbeck [31] process:

$$\begin{aligned} dS_t &= rS_t dt + \sigma_t S_t dW_t^S \\ d\sigma_t &= k\left(\theta - \sigma_t\right) dt + \tilde{\sigma} dW_t^\sigma \end{aligned}$$

$\sigma_t$: volatility of the spot S, $\tilde{\sigma}$: the volatility of the volatility, $\theta$: long-term reversion, $k$: the speed of mean reversion, $W_t^\sigma$ and $W_t^S$ are independent Brownian motions.

This model is interesting because the stochastic equation of volatility is linear and we can obtain an analytical solution for volatility. However, it is better to have a simpler and faster numerical solution, as specified in [31]. In this model, volatility follows a normal distribution, which makes the model very tractable than the Heston model (non-central chi-square distribution for volatility),[8].

Like Heston model, Stein-Stein model is a mean reverting process. When time goes to infinity the expected volatility converges towards $\theta$ with a finite variance, [8].

The disadvantage of this model is that volatility can take negative values, because volatility follows a normal distribution which can of course take non-positive values with a non-zero probability. To solve this problem, we can think of a model where volatility follows a normal logarithmic distribution, this model is the Hull-White, [18].

$$\begin{aligned} dS_t &= rS_t dt + \sqrt{V_t} S_t dW_t^S \\ dV_t &= \mu V_t dt + \tilde{\sigma} V_t dW_t^V \\ dW_t^S dW_t^V &= \rho dt \end{aligned}$$

Where: $\sigma_t$ volatility of the spot S, $\tilde{\sigma}$ the volatility of the volatility (or volatility of $\sqrt{V_t}$), $\mu$: positive constant, $W_t^S$ and $W_t^V$ Brownian motions associated with respectively spot and volatility, and $\rho \geq 0$: the correlation between spot and volatility of the risky asset.

These different volatility models are the main models that allow us to represent the underlying asset's volatility, in order to price financial products. Some models give explicit formulas for the price like the BS model, others require approximations and a numerical calculation to estimate the price. The stochastic volatility models can give analytical formulas which remain quite difficult to derive, that's why we use a numerical method such as: the finite-difference method or the Monte Carlo method used by [29] for the pricing of options.

# 2 Pricing options with Black-Scholes model

In this section, we will present the basics of the BS model and the price of vanilla options with this same model. This study will allow us to introduce the BS model in order to compare it with the UVM in the following sections.

We consider a filtered probability space $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}\right)$ where $\mathcal{F}_t$ is a filtration. The BS model is based on a number of assumptions that we will highlight below, [6]:

- We consider that underlying asset does not pay any dividend.
- No transaction fees or taxes are included in the model.
- Short selling is allowed and we can buy a share even if it is small: share are divisible.
- Interest rate $r$ and volatility will be considered constant.
- The underlying asset follows a GBM.

Let S be the underlying asset, the risky asset that follows the process below:

$$dS_t = S_t\left[rdt + \sigma dZ_t\right], \quad 0 \leq t \leq T$$

Where $r$ and $\sigma$ are two non negative constants, respectively the risk-free rate and the volatility. $Z_t$ a the Brownian motion under risk neutral measure.

Using Ito formula on $ln(S_t)$ we can write the stock as:

$$dln(S_t) = \frac{1}{S_t}dS_t - \frac{1}{2}\frac{1}{S_t^2}(dS_t)^2 \tag{2.1}$$

$$= rdt + \sigma^2 dZ_t - \frac{1}{2}\sigma^2 \tag{2.2}$$

$$= (r - \frac{\sigma^2}{2})dt + \sigma^2 dZ_t \tag{2.3}$$

$$S_t = S_0 e^{(r-\frac{\sigma^2}{2})t+\sigma^2 Z_t} \tag{2.4}$$

Let $B_t$ be the bond, a riskless asset that follows the equation below:

$$dB_t = B_t rdt$$

With $B_0 = 1$, and $r \geq 0$ the interest rate. The solution of the equation is:

$$B_t = e^{rt}$$

## 2.1 Black-Scholes PDE

We consider that the market is arbitrage free, $W_t$ is the value of a portfolio at time t, and $(\phi_t^B, \phi_t^S)$ are the trading strategy defined by:

$$\begin{cases} \phi_t^S = \frac{\partial W}{\partial S}(t, S_t) \\ \phi_t^B = \frac{W_t - \phi_t^S S_t}{B_t} \end{cases}$$

we note that:

$$\phi_t^S S_t + \phi_t^B B_t = \phi_t^S S_t + \left(\frac{W_t - \phi_t^S S_t}{B_t}\right) B_t \tag{2.5}$$

$$= W_t \tag{2.6}$$

With this trading strategy, we can write a Partial Differential Equation (PDE) for $W_t$. On one hand we have:

$$dW_t = \phi_t^S dS_t + \phi_t^B dB_t \tag{2.7}$$

$$= \frac{\partial W}{\partial S}(t, S_t)\, dS_t + \frac{1}{B_t}\left(W_t - \phi_t^S S_t\right) dB_t \tag{2.8}$$

$$= \frac{\partial W}{\partial S}(t, S_t)\, S_t\left[rdt + \sigma dZ_t\right] + \frac{1}{B_t}\left(W_t - \phi_t^S S_t\right) rB_t dt \tag{2.9}$$

$$= \frac{\partial W}{\partial S}(t, S_t)\, S_t\left[rdt + \sigma dZ_t\right] + \left(W_t - \frac{\partial W}{\partial S} S_t\right) rdt \tag{2.10}$$

On the other hand we use ito formula on $W_t$ and we get:

$$\begin{aligned} dW_t &= \frac{\partial W}{\partial t}(t, S_t)\, dt + \frac{\partial W}{\partial S}(t, S_t)\, dS_t + \frac{1}{2}\frac{\partial^2 W}{\partial S^2}(t, S_t)\, dS_t^2 \\ &= \left[\frac{\partial W}{\partial S}(t, S_t)\, rS_t + \frac{\partial W}{\partial t}(t, S_t) + \frac{1}{2}\frac{\partial^2 W}{\partial S^2}(t, S_t)\, \sigma^2 S_t^2\right] dt + \frac{\partial W}{\partial S}(t, S_t)\, \sigma S_t dZ_t \end{aligned} \tag{2.11}$$

Using the two equations (2.10), (2.11) and that the market is arbitrage free we get the Black-Scholes PDE:

$$\frac{\partial W}{\partial t}(t, S_t) + r\left(\frac{\partial W}{\partial S}(t, S_t)\, S_t - W(t, S_t)\right) + \frac{1}{2}\frac{\partial^2 W}{\partial S^2}(t, S_t)\, \sigma^2 S_t^2 = 0 \tag{2.12}$$

This last equation is the Black-Scholes PDE, as showed in [30] and [8].

## 2.2 The price of Black-Scholes for Vanilla options

In this section, we will present some options that we will use in this study, specifying their Black-Scholes prices and some of the sensitivities used.

Using the Feynman-Kac theorem, we write the portfolio price as the expectation of the option payoff under the risk-neutral measure.
We can write the PDE as follows:

$$\frac{\partial W}{\partial t}(t, S_t) - rW(t, S_t) + AW(t, S_t) = 0 \tag{2.13}$$

With:

$$AW(t,S_t) = rS_t\frac{\partial W}{\partial S}(t,S_t) + \frac{1}{2}\sigma^2 S_t^2\frac{\partial^2 W}{\partial S^2}(t,S_t)$$

We consider an option with payoff $f(S_T)$, so the price of portfolio at time $T$ is equal:

$$W(T,S_T) = f(S_T)$$

$W(t,S_t)$ can be written as:

$$W(t,S_t) = \mathbb{E}_t^Q[e^{-r(T-t)}f(X_T)] \tag{2.14}$$

Where $Q$ is the risk neutral measure, $\mathbb{E}_t^Q$ the conditional expectation on filtration $\mathcal{F}_t$ and $S_\tau$ follows the SDE below:

$$dS_\tau = rS_\tau d\tau + \sigma S_\tau dZ_\tau^Q \tag{2.15}$$

### 2.2.1 Calls prices

If for example the portfolio is composed of a call with strike $K$ and maturity $T$, the payoff is written as:

$$f(S_T) = (S_T - K)^+$$

.

The price of the call at time t is :

$$\begin{aligned} W(t,S_t) &= \mathbb{E}_t^Q[e^{-r(T-t)}(S_T-K)^+] &(2.16)\\ &= e^{-r(T-t)}\mathbb{E}_t^Q[(S_T-K)^+] &(2.17)\\ &= e^{-r(T-t)}\mathbb{E}_t^Q[(S_t\, e^{(r-\frac{\sigma^2}{2})(T-t)+\sigma\sqrt{T-t}\,X} - K)\mathbb{1}_{S_T>K}] &(2.18)\end{aligned}$$

Where $X \sim \mathcal{N}(0,1)$.

$$S_T > K \iff X < \frac{len(\frac{S_t}{K}) + (r-\frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}} = d_2$$

We can proof this last result by using that $S_T = S_t\, e^{(r-\frac{\sigma^2}{2})(T-t)+\sigma\sqrt{T-t}\,X}$ and the fact that $X \sim \mathcal{N}(0,1)$ is equivalent to $-X \sim \mathcal{N}(0,1)$.

Using this result we compute the price of the call $W(t,S_t)$:

$$\begin{aligned} W(t,S_t) &= e^{-r(T-t)}S_t e^{(r-\frac{\sigma^2}{2})(T-t)}\mathbb{E}_t^Q[e^{\sigma\sqrt{T-t}\,X}\mathbb{1}_{X<d_1}] - Ke^{-r(T-t)}\mathbb{E}_t^Q[\mathbb{1}_{X<d_1}] &(2.19)\\ &= e^{-r(T-t)}S_t e^{(r-\frac{\sigma^2}{2})(T-t)}\int_{-\infty}^{d_1} e^{\sigma\sqrt{T-t}\,x}\cdot\frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}dx - Ke^{-r(T-t)}P(X<d_1) &(2.20)\\ &= S_tN(d_1) - Ke^{-r(T-t)}N(d_2) &(2.21)\end{aligned}$$

Where $d_1 = d_2 + \sigma\sqrt{T-t}$ and $N$ is the cumulative distribution function (cdf) of the standard normal distribution.

So the price of the call using Black-Scholes :

$$\boxed{W(t,S_t)_{call} = S_t N(d_1) - Ke^{-r(T-t)} N(d_2)}$$

This proof is more detailed in [30] and [8].

We can compute the "Greeks" or the sensitivities such us:

- The delta: the first derivative of the price with respect to the spot value $S$:

$$\Delta_{call} = \frac{\partial W}{\partial S} = N(d_1)$$

  We notice that delta is between -1 and 1 because cdf of normal distribution belongs to this interval

- The gamma: the second derivative of the price with respect to the spot value $S$, thus it represent the price's convexity with respect to the underlying asset:

$$\gamma_{call} = \frac{\partial^2 W}{\partial S^2} = \frac{N'(d_1)}{S\sigma\sqrt{T-t}}$$

  Gamma of the call is positive.

### 2.2.2 Puts prices

The payoff for the put is $f(S_T) = (K - S_T)^+$. We repeat the same calculations as for the call and we get a price for the put at time t:

$$\boxed{W(t,S_t)_{put} = Ke^{-r(T-t)} N(-d_2) - S_t N(-d_1)}$$

- The delta:

$$\Delta_{put} = \frac{\partial W}{\partial S} = N(d_1) - 1$$

  Delta of the put varies between -1 and 0.

- The gamma:

$$\gamma_{put} = \frac{\partial^2 W}{\partial S^2} = \frac{N'(d_1)}{S\sigma\sqrt{T-t}}$$

  We notice that the gamma of the put is exactly equal to the gamma of the call.

### 2.2.3 Straddle and Bull spread Options prices

PDE of Black-Scholes is linear, the price of a portfolio composed of a linear combination of calls and puts: $\sum_{i=1} \lambda_i \times Option_i$, is only equal to the combination of the price of these options: $\sum_{i=1} \lambda_i \times C_i$ where :

- $Option_i$ is a $call_i$ or a $put_i$
- $C_i$ is the price of $Option_i$
- If $\lambda_i > 0$ is a long position on the option and $\lambda_i < 0$ is a short position.

The straddle is composed of a long call and a long put of the same strike $K$. The price of the straddle is equal to the sum of the call and put prices.

$$W(t, S_t)_{straddle} = W(t, S_t)_{call} + W(t, S_t)_{put} \tag{2.22}$$

Concerning the Greek calculations, we show that the sensitivities of the straddle is equal to the sensitivities of the options composing the portfolio by operator linearity of $\frac{\partial^2 .}{\partial S^2}$ for the gamma, and the linearity of $\frac{\partial .}{\partial S}$ for the delta

$$\gamma_{straddle} = \frac{\partial^2 W(t, S_t)_{call}}{\partial S^2} + \frac{\partial^2 W(t, S_t)_{put}}{\partial S^2} \tag{2.23}$$

$$= \gamma_{call} + \gamma_{put} \tag{2.24}$$

$$= 2\frac{N'(d_1)}{S\sigma\sqrt{T-t}} \tag{2.25}$$

The Bull spread is an option consisting of a long call of strike $K_1$ and a short call of strike $K_2$ with $K_1 < K_2$. Using the linearity of Black-Scholes PDE, we get that the price of the Bull spread:

$$W(t, S_t)_{Bullspread} = W(t, S_t)_{call1} - W(t, S_t)_{call2} \tag{2.26}$$

The gamma of the bull spread using Black-Scholes (BS) is equal to the gamma of call1 minus the gamma of call2.

$$\gamma_{Bullspread} = \frac{\partial^2 W(t, S_t)_{call1}}{\partial S^2} - \frac{\partial^2 W(t, S_t)_{call2}}{\partial S^2} \tag{2.27}$$

$$= \gamma_{call1} - \gamma_{call2} \tag{2.28}$$

## 2.3 Model Critics

The author of this paper [23] compares the Black-Scholes model with the reality of the market and highlights a number of limitations linked to this model. Indeed, one limit of the model is that asset return does not follow a normal distribution, which can be seen on empirical data. However, the returns have a peaked distribution with heavy tails, as specified by the paper [4], which shows the impact that this can have on the risk assessment.

The weak point of the Black-scholes model is that it considers the volatility of the asset to be constant, while empirical studies show that the volatility depends on the strike and the maturity of the options, as pointed out in the article [23]. Indeed, if we consider that time to maturity is the same for the options and we focus on the evolution of volatility according to the strike, we obtain the classic figure of the volatility smile. Moreover, if we add to this evolution of volatility, the maturity parameter we get a three dimensional figure called surface volatility, [23]. Thus, we see that the volatility is not constant on the empirical data.

There are also other limits which we will only mention without going into details, namely: the interest rate is not constant over time as considered by the model, and the transaction cost need to be included in the BS model as specified in [9].

The papers [23] also states that this model is widely used and widespread in finance for its simplicity and give a price close to the market price, despite the limitations that this model may have.

# 3 The uncertain volatility model

## 3.1 Model presentation

Stock's volatility is an important variable, difficult to predict, has an uncertain nature, and is crucial in determining the price of derivatives. In [2] and [3] papers, the authors present the UVM model: we consider that the evolution of volatility will remain in a band determined from the prices of vanilla options in order to price more elaborate options called exotic options.

In this model we consider Stock S as the risky asset, its price follows the dynamics below:

$$\mathrm{d}S_{\mathrm{t}} = S_{\mathrm{t}}\left(\sigma_t \mathrm{d}Z_t + \mu_t \mathrm{d}t\right), \qquad 0 \leq t \leq T \tag{3.1}$$

where:

- $Z_t$ a Brownian motion, $\mu_t \geq 0$ is called the drift. Under the risk neutral measure the drift is equal to the interest rate $r$.
- $\sigma_t$ is the volatility that is bounded between two constants $\sigma_{min}$ and $\sigma_{max}$.

$$\sigma_{\min} \leqslant \sigma_t \leqslant \sigma_{\max} \tag{3.2}$$

These bounds are inputs of the model and can be determined by choosing an implied volatility band where this volatility of the options available in the market falls within this interval. Therefore, determining a volatility interval for implied volatility: $\sigma_{implied}$ is equivalent to finding a interval of value for the asset's volatility $\sigma_t$, as mentioned in [3].

Assuming that the stocks $S$ pays no dividends and that there is no transaction costs in trading, we consider a portfolio composed of: an option of value $W(S,t)$ and a short position $\Delta$ on the asset S, [32].

$$\Pi = W - \Delta S$$

Using Ito formula :

$$d\Pi = \frac{\partial W}{\partial t}dt + \frac{\partial W}{\partial S}dS + \frac{1}{2}\frac{\partial^2 W}{\partial S^2}dSdS - \Delta dS$$

Using also that: $dSdS = \sigma_t^2 S^2 dt$ we get:

$$d\Pi = \left(\frac{\partial W}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 W}{\partial S^2}\right)dt + \left(\frac{\partial W}{\partial S} - \Delta\right)dS$$

Choosing that $\Delta = \partial W / \partial S$ will eliminate the term associated with $dS$.

$$d\Pi = \left(\frac{\partial W}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 W}{\partial S^2}\right)dt$$

The worst-case scenario occurs for volatility which will verify that the change in the value of the portfolio over a small interval dt is equal to:

$$\min_{\sigma^- < \sigma < \sigma^+} (d\Pi) = r\Pi dt$$

We note that $\min_{\sigma^- < \sigma < \sigma^+}(d\Pi)$ depend on the sign of $\frac{\partial^2 W}{\partial S^2}$. For example if the gamma is postive the minimum will be reached in $\sigma_{min}$, otherwise the minimum will be $\sigma_{max}$.

The worst-case scenario $W^-$ is solution of the following equation:

$$\frac{\partial W^-}{\partial t} + \frac{1}{2}S^2 \min_{\sigma^- < \sigma < \sigma^+} \left(\sigma^2 \frac{\partial^2 W^-}{\partial S^2}\right) - \left(W^- - S\frac{\partial W^-}{\partial S}\right) = 0$$

$$\frac{\partial W^-}{\partial t} + \frac{1}{2}\sigma^2 \left[\frac{\partial^2 W^-}{\partial S^2}\right] S^2 \frac{\partial^2 W^-}{\partial S^2} + rS\frac{\partial W^-}{\partial S} - rW^- = 0$$

Where:

$$\sigma\left[\frac{\partial^2 W^-}{\partial S^2}\right] = \begin{cases} \sigma_{max} & \text{if } \frac{\partial^2 W^-}{\partial S^2} < 0 \\ \sigma_{min} & \text{if } \frac{\partial^2 W^-}{\partial S^2} > 0 \end{cases}$$

The best-case scenario $W^+$ of the portfolio can be determined by solving the following equation, called Black Scholes Barenblatt equation (BSB):

$$\frac{\partial W^+}{\partial t} + \frac{1}{2}\sigma \left[\frac{\partial^2 W^+}{\partial S^2}\right]^2 S^2 \frac{\partial^2 W^+}{\partial S^2} + rS\frac{\partial W^+}{\partial S} - rW^+ = 0$$

with

$$\sigma\left[\frac{\partial^2 W^+}{\partial S^2}\right] = \begin{cases} \sigma_{max} & \text{if } \frac{\partial^2 W^+}{\partial S^2} > 0 \\ \sigma_{min} & \text{if } \frac{\partial^2 W^+}{\partial S^2} < 0 \end{cases}$$

The best-case value for a long position in the derivative is equivalent to the worst-case scenario for a short position, as highlighted in the paper [32]. Therefore, for a short or long position, the price of a option is between the two extreme values associated with the best and the worst case, almost surely. In general, we are more interested in the worst case scenario because it gives us information about how risky the portfolio is and we can hedge against this scenario. Therefore, the UVM gives a price band for a portfolio of derivatives delimited by $W^-$ and $W^+$.

Lets now see if we are short position on a option with different cash-flows at dates $t_1$,...$t_N$:

$$F_1(S_{t_1}), F_2(S_{t_2}), \ldots, F_N(S_{t_N})$$

The UVM problem can be represented by the following BSB equation [2]:

$$\begin{cases} \forall t < t_N, & \frac{\partial W(S,t)}{\partial t} + r\left(S\frac{\partial W(S,t)}{\partial S} - W(S,t)\right) + \frac{1}{2}S^2\sigma^2\left[\frac{\partial^2 W(S,t)}{\partial S^2}\right] \cdot \frac{\partial^2 W(S,t)}{\partial S^2} = \sum_{t_k>t}^{N-1} F_k(S) \cdot \delta\left(t - t_k\right) \\ W\left(S, t_N\right) = F_N(S) \end{cases} \tag{3.3}$$

where: $W^+$ and $W^-$ are the upper and lower bound of the derivative price. $W$ in the BSB equation can be either $W^+$ or $W^-$.

The equation (3.3) is a generalization of the Black-Scholes differential equation, called the Black-Scholes-Barenblatt equation (BSB). In the case where $\sigma_{min} = \sigma_{max}$, we find the classic Black-Scholes equation. We note that the BSB is similar to BS provided that the volatility is constant. In the BSB equation, the volatility is determined by the convexity of $W$, and when the volatility is $\sigma_t^* = \sigma\left[\frac{\partial^2 W(S_t,t)}{\partial S^2}\right]$, the value given by the BSB equation corresponds to the extreme value: either the upper values or the lower values, depending on the position on the option, this result will be proved in the next section.

There is also another special case, when the second derivative of W relative to the Stock S (the gamma) has a constant sign (either positive or negative), the volatility in this case will be equal to a constant. This implies that the BSB equation simplifies to the BS equation with a volatility of $\sigma_{max}$ if gamma is positive and $\sigma_{min}$ if gamma is negative (for the higher price $W^+$). We also note that the BSB equation is nonlinear unlike the BS equation, and therefore its resolution can only be done numerically.

The solution of the BSB equation can be formulated as below, [3]:

$$W^+\left(S_t, t\right) = \underset{P}{\text{Sup}}\ \mathbf{E}_t^P\left[\sum_{j=1}^{N} \mathrm{e}^{-r(t_j-t)} F_j\left(S_{t_j}\right)\right]$$

$$W^-\left(S_t, t\right) = \underset{P}{\text{Inf}}\ \mathbf{E}_t^P\left[\sum_{j=1}^{N} \mathrm{e}^{-r(t_j-t)} F_j\left(S_{t_j}\right)\right]$$

Where:

- $P$ is the set of all the measures on the paths which verify the dynamic equation of $S_t$ (3.1) and that the volatility belongs to the band (3.2).
- $\mathbf{E}^p$ represents the conditional expectation under the set $P$ and under the condition of all the information up to time $t$.

## 3.2 Hedging with underlying asset S and Bond B

In this section we will discuss the hedging of a short position in derivative instrument with payoff $\phi = \sum_{j=1}^{N} e^{-r(t_j-t)} F_j\left(S_{t_j}\right)$. If we consider that the volatility $\sigma_t$ satisfy:

$$\sigma\left[\frac{\partial^2 W^+}{\partial S^2}\right] = \begin{cases} \sigma_{\max} & \text{if } \frac{\partial^2 W^+}{\partial S^2} \geqslant 0 \\ \sigma_{\min} & \text{if } \frac{\partial^2 W^+}{\partial S^2} < 0 \end{cases}$$

To hedge this portfolio with payoff $\phi$, we consider the following strategy:

- Buy $\phi_S = \frac{\partial W^+(S_\tau,\tau)}{\partial S}$ units of stock S.
- Position of $\phi_B = W^+(S_t,t) - S_t\frac{\partial W^+(S_t,t)}{\partial S}$ units of bank account.

At each time $t_k$, we will need to maintain a fix position of $\phi_S$ for the stock S and $\phi_B$ of cash.

This strategy is self financing: this means that the variations in the value of the portfolio is only due to price changes of B and S, without any withdrawal of cash or addition of it. This strategy replicates exactly the derivative. We can see it by computing the portfolio value at each time $t$: $\phi_t \times S_t + \phi_B = W^+(S_t,t)$

We assume now that the volatility $\sigma_t$ varies in the interval formed by $\sigma_{min}$, and $\sigma_{max}$. This last self-financing strategy gives us a super-replicating strategy: after having paid the different cash flows $F_j(S_{t_j})$, we find ourselves having a positive cash value. Similarly we can hedge long derivative security, considering $W^-$ instead of $W^+$.

Now let's mathematically prove these results [2], we recall the BSB equation for a derivative with payoff $\phi = \sum_{j=1}^{N} e^{-r(t_j-t)}F_j\left(S_{t_j}\right)$ and assuming that $r=0$.

$$\begin{cases} \forall t < t_N, & \frac{\partial W^+(S,t)}{\partial t} + \frac{1}{2}S^2\sigma^2\left[\frac{\partial^2 W^+(S,t)}{\partial S^2}\right]\cdot\frac{\partial^2 W^+(S,t)}{\partial S^2} = \sum_{t_k>t}^{N-1} F_k(S)\cdot\delta\left(t-t_k\right) \\ W^+(S,t_N) = F_N(S) \end{cases}$$

We consider $V$ as the value of the global portfolio composed of: a short position on the payoff $\phi$ and of the hedging portfolio. The value of the overall portfolio at $t_N$ is:

$$V_{t_N} = W^+(S_t,t) + \int_t^{t_N}\Delta_\tau \mathrm{d}S_\tau - \sum_{t_k>t}^{N} F_k\left(S_{t_k}\right)$$

Using Ito formula for $W^+(S_{t_N},t_N)$, we get:

$$W^+(S_{t_N},t_N) = W^+(S_t,t) + \int_t^{t_N}\frac{\partial W^+(S_\tau,\tau)}{\partial\tau}\mathrm{d}\tau + \int_t^{t_N}\Delta_\tau\mathrm{d}S_\tau + \int_t^{t_N}\frac{1}{2}\sigma_\tau^2 S_\tau^2\frac{\partial^2 W^+(S_r,\tau)}{\partial S^2}\mathrm{d}\tau \tag{3.4}$$

$$= W^+(S_t,t) + \int_t^{t_N}\Delta_\tau\mathrm{d}S_\tau + \int_t^{t_N}\left[\frac{\partial W^+(S_\tau,\tau)}{\partial\tau} + \frac{1}{2}\sigma_\tau^2 S_\tau^2\frac{\partial^2 W^+(S_r,\tau)}{\partial S^2}\right]\mathrm{d}\tau \tag{3.5}$$

For $\sigma \in [\sigma_{min},\sigma_{max}]$, we have the following result:

$$\boxed{\sigma_t^2\frac{\partial^2 W^+(S,\tau)}{\partial S^2} \leqslant \sigma^2\left[\frac{\partial^2 W^+(S,\tau)}{\partial S^2}\right]\cdot\frac{\partial^2 W^+(S,\tau)}{\partial S^2}} \tag{3.6}$$

Because for example if : $\frac{\partial^2 W^+(S,\tau)}{\partial S^2} > 0$, then the inequality can be written as :

$$\sigma_t^2 \leqslant \sigma^2 \left[\frac{\partial^2 W^+(S,\tau)}{\partial S^2}\right] = \sigma_{max}^2$$

and if $\frac{\partial^2 W^+(S,\tau)}{\partial S^2} < 0$, then the inequality becomes:

$$\sigma_t^2 \geqslant \sigma^2 \left[\frac{\partial^2 W^+(S,\tau)}{\partial S^2}\right] = \sigma_{min}^2$$

and if the $\frac{\partial^2 W^+(S,\tau)}{\partial S^2} = 0$, the inequality still true, so this proves the inequality (3.6)

Adding the term $\frac{\partial W^+(S,\tau)}{\partial \tau}$ for the both side of the inequality (3.6):

$$\frac{\partial W^+(S,\tau)}{\partial \tau} + \frac{1}{2}\sigma_t^2 S_t^2 \frac{\partial^2 W^+(S,\tau)}{\partial S^2} \leqslant \frac{\partial W^+(S,\tau)}{\partial \tau} + \frac{1}{2}\sigma^2 \left[\frac{\partial^2 W^+(S,\tau)}{\partial S^2}\right] S_t^2 \frac{\partial^2 W^+(S,\tau)}{\partial S^2}$$

We integrate this last inequality between $t$ and $t_N$, we add $W^+(S_t,t) + \int_t^{t_N} \Delta_\tau \mathrm{d}S_\tau$ to both side of the inequality, and we use the equation (3.5) , we get:

$$W^+(S_{t_N}, t_N) \leqslant W^+(S_t,t) + \int_t^{t_N} \Delta_\tau dS_\tau + \int_t^{t_N} \sum_{t_k>t}^{N-1} F_k(S)\delta(t-t_k)$$

$$W^+(S_{t_N}, t_N) \leqslant W^+(S_t,t) + \int_t^{t_N} \Delta_\tau dS_\tau + \sum_{t_k>t}^{N-1} \int_t^{t_N} F_k(S)\delta(t-t_k)$$

$$F_N(S_{t_N}) \leqslant W^+(S_t,t) + \int_t^{t_N} \Delta_\tau \mathrm{d}S_\tau - \sum_{t_k>t}^{N-1} F_k(S_{t_k})$$

$$0 \leqslant W^+(S_t,t) + \int_t^{t_N} \Delta_\tau \mathrm{d}S_\tau - \sum_{t_k>t}^{N} F_k(S_{t_k}) = V_{t_N+0}$$

$$0 \leqslant V(t_N)$$

This last result prove that the self-financing strategy $(\phi_B, \phi_S)$ is a super-replicating strategy. If we consider now that $\sigma_t = \sigma\left[\frac{\partial^2 W^+(S,\tau)}{\partial S^2}\right]$, we find that $V(t_N) = 0$ because inequalities will become equalities. Therefore, this strategy become a replicating strategy. For $W^-(S,\tau)$, we can have a similar reasoning.

## 3.3 Diversification of the portfolio

An important property of the BSB equation is the subadditivity of the payoff. When diversifying the portfolio by adding new derivatives, the price range offered by the diversified portfolio is narrower than that provided by the sum of the individual portfolios. Indeed, this is due to the fact that the volatility, at time t, given by the overall portfolio may be different from the added volatility of the individual portfolios.

Let's consider two payoffs such that:

$$\Phi = \sum_{j=1}^{N} \mathrm{e}^{-r(t_j - t)} F_j \left(S_{t_j}\right)$$

$$\Psi = \sum_{k=1}^{N'} \mathrm{e}^{-r\left(t'_k - t\right)} G_k \left(S_{t'_k}\right)$$

Where $F_1(S_{t_1}), ...F_N(S_{t_N})$ and $G_1(S_{t'_1}), ...G_{N'}(S_{t'_{N'}})$ are the different cash flows generated by those derivative securities respectively at time $t_1, ..., t_N$ and at $t'_1, ..., t'_{N'}$.

As specified in [2] paper, the risk diversification property can be written as:

$$\operatorname*{Sup}_P \mathbf{E}_t^P[\Phi + \Psi] \leqslant \operatorname*{Sup}_P \mathbf{E}_t^P[\Phi] + \operatorname*{Sup}_P \mathbf{E}_t^P[\Psi]$$

$$\operatorname*{Inf}_P \mathbf{E}_t^P[\Phi + \Psi] \geqslant \operatorname*{Inf}_P \mathbf{E}_t^P[\Phi] + \operatorname*{Inf}_P \mathbf{E}_t^P[\Psi]$$

The price range proposed by the UVM method for the sum of the individual portfolios is greater than the price range given by the overall portfolio.

We highlight this sub-additivity when we choose a portfolio made up of options with gammas of different signs. In this case, there is no constant volatility that will give the value of the overall portfolio. On the one hand, if the option prices are convex (for example the call and the put payoff with the same strike K, then the straddle which is their sum is also convex) the BSB equation is reduced to a classical BS equation with constant volatility, and in this particular case there is equality between the price of the overall portfolio and the sum of the individual portfolios. On the other hand, when the options that make up the portfolio have different convexities, the risk of the overall portfolio is reduced and, therefore, the price range will be narrower compared to the sum of the individual portfolios.

## 3.4 The Lagrangian UVM: Hedging with options

In this section, we consider a short position in a derivative with different cash flows $F_j$ at time $t_j$. To hedge this portfolio, we will use $M$ European options with cash flows $G_1 \leq G_2 \leq \ldots \leq G_M$ at maturities $t'_1 \leq t'_2 \leq \ldots \leq t'_M$. The price of these options is noted $C_1, C_2, \ldots, C_M$ and the proportions associated with each option are $\lambda_1, \lambda_2, \ldots, \lambda_M$. Therefore, the price of the hedging portfolio can be written as follows: $\sum_{i=1}^{M} \lambda_i C_i$. The value of the derivative in the worst-case scenario $W^+$ is defined by, as specified in [3]:

$$W^+ \left(S_t, t; \lambda_1, \ldots, \lambda_M\right) = \sup_{P \in \mathbb{P}} \mathbf{E}^P \left\{ \sum_{j=1}^{N} e^{-r(t_j - t)} F_j \left(S_{t_j}\right) - \sum_{i=1}^{M} \lambda_i e^{-r\left(t'_i - t\right)} G_i \left(S_{r_i}\right) \right\} + \sum_{i=1}^{M} \lambda_i C_i$$

Where: $F_1(S_{t_1}), ...F_N(S_{t_N})$ are the different cash flows generated by the portfolio at time $t_1, ..., t_N$.

To find the optimal hedge in the worst-case scenario, we need to find the proportions of options to buy if $\lambda > 0$ or to sell if $\lambda < 0$, to minimize the hedging cost. Therefore, we need to solve the following problem:

$$\inf_{\lambda_1,\lambda_2,\ldots,\lambda_M} W^+\left(S_t, t; \lambda_1, \lambda_2, \ldots \lambda_M\right) \tag{3.7}$$

This problem has a unique solution because the function $(\lambda_1, \ldots, \lambda_M) \mapsto W^+\left(S_t, t; \lambda_1, \lambda_2, \ldots \lambda_M\right)$ is convex, so there exists a minimum for this function.

In the case where the agent has a long position on the derivative, we do the same reasoning and we find that the worst-case value of the portfolio is now $W^-$:

$$W^-\left(S_t, t; \lambda_1, \ldots, \lambda_M\right) = \inf_{P\in\mathbb{P}} \mathbf{E}^P\left\{\sum_{j=1}^{N} e^{-r(t_j-t)} F_j\left(S_{t_j}\right) - \sum_{i=1}^{M} \lambda_i e^{-r\left(t_i'-t\right)} G_i\left(S_{r_i}\right)\right\} + \sum_{i=1}^{M} \lambda_i C_i$$

The optimal hedge can be found now by solving :

$$\sup_{\lambda_1,\lambda_2,\ldots,\lambda_M} W^-\left(S_t, t; \lambda_1, \lambda_2, \ldots \lambda_M\right)$$

Additional conditions, based for example on availability on options can be added to $\lambda_1, ...\lambda_M$ by choosing them in precise intervals.

$$\Lambda_i^- \leq \lambda_i \leq \Lambda_i^+ \quad \forall i \in \{1, ...M\}.$$

We can notice that when the vector $(\lambda_1, ...\lambda_M)$ is equal to zero, then the problem is simplified to the UVM. In this case, we hedge our portfolio only using the underlying asset and the bond.

It can be noted that the Lagrangian UVM can be reduced to solving the UVM, by considering new cash flows for the derivative to be hedged: $\tilde{\tilde{F}}\left(S_{\tau_i}\right)$ instead of $F\left(S_{\tau_i}\right)$ (See the proof below).

$$W\left(S_t, t, \lambda_1, \ldots, \lambda_M\right) = \sup_{P\in\mathbb{P}} \mathbf{E}^P\left\{\sum_{j=1}^{N} e^{-r(t_j-t)} F_j\left(S_{t_j}\right) - \sum_{i=1}^{M} \lambda_i e^{-r\left(t_i'-t\right)} G_i\left(S_{t_i}\right)\right\} + \sum_{i=1}^{M} \lambda_i C_i \tag{3.8}$$

$$= \sup_{P\in\mathbb{P}} \mathbf{E}^P\left\{\sum_{i=1}^{N'} e^{-r(\tau_i-t)}\left(\tilde{F}_i\left(S_{\tau_i}\right) - \lambda_i \tilde{G}_i\left(S_{\tau_i}\right)\right)\right\} + \sum_{i=1}^{M} \lambda_i C_i \tag{3.9}$$

$$= \sup_{P\in\mathbb{P}} \mathbf{E}^P\left\{\sum_{i=1}^{N'} e^{-r(\tau_i-t)} \tilde{\tilde{F}}\left(S_{\tau_i}\right)\right\} + \sum_{i=1}^{M} \lambda_i C_i \tag{3.10}$$

with:

- $\tilde{\tilde{F}}(S_{\tau_i}) = \tilde{F}_i(S_{\tau_i}) - \lambda_i \tilde{G}_i(S_{\tau_i})$
- $\tilde{F}_i = F_i$ if $i \in \{1, .., N\}$ and 0 otherwise
- $\tilde{G}_i = G_i$ if $i \in \{1, .., M\}$ and 0 otherwise
- $N' = max(N, M)$
- $\{\tau_1, ...\tau_{N'}\} = \{t_1, ..., t_N\} U \{t'_1, ..., t'_M\}$

The first equality term (3.10) corresponds exactly to the price determined by UVM methods, to which we add the second term which corresponds to the price of the hedging portfolio, which gives the price range of the derivatives using the UVM Lagrangian.

We can note that the Lagrangian UVM developed here corresponds to a method of static hedging because the hedge ratio $(\lambda_1, ...\lambda_M)$ are constant and not time dependent.

## 3.5 Calibration of the volatility band

The article of [3], highlights the method of calibration of the volatility band that we present in this section.

To find the extreme values of volatility: $\sigma_{min}$ and $\sigma_{max}$, we can use the implied volatility of vanilla options. Specifically, we will plot the implied volatility found from the market price using the Black-Scholes formula for liquid options, and then we will take as the volatility band, an interval that will contain all the implied volatilities of those options. Indeed, considering a range of volatility for the security is equivalent to finding a range of implied volatility.

$$\sigma_{min} \leq \sigma_{impl}(t, T) \leq \sigma_{max}$$

with $T$ the maturity time of an option.

A better calibration would be to consider a time-dependent volatility interval in order to limit an overestimation of volatility and to give a more accurate range for the value of the portfolio.

$$\sqrt{\frac{1}{T-t}\int_t^T \sigma_{\min}^2(s)ds} \leq \sigma_{impl}(t, T) \leq \sqrt{\frac{1}{T-t}\int_t^T \sigma_{\max}^2(s)ds}$$

We can notice that if we take $\sigma_{min}(t) = \sigma_{min}$ and $\sigma_{max}(t) = \sigma_{max}$, we find exactly the same range of volatility as before. Therefore, the implicit volatility of the derivatives used as inputs must be within the range defined by $\sigma_{min}$ and $\sigma_{max}$. We also note that this time dependent calibration will give a smaller volatility range than the constant calibration and therefore a smaller and more accurate price range for the portfolio. Indeed, option price increases with volatility: the upper price $W^+$ will decrease (because the upper value of the volatility range decreases) and the lower price $W^-$ will increase, thus giving a narrower range for the portfolio's price.

# 4 Numerical Implementation using trinomial tree

## 4.1 Trinomial Tree

The non-linearity of the BSB equation implies a numerical solution to determine the price of the portfolio. To approximate the solution to the BSB equation, we use a finite-difference scheme to provide an explicit solution. We first consider that the stock S follows a trinomial model: at the next time, S can reach three levels: up U, middle M or down D. We also assume that the tree is a recombining tree, which implies that $UD = M^2$. We consider $T$ as the maturity time by years, N the trading periods, and the time discretization becomes $\delta = \frac{T}{N}$. $N$ must be large enough so that the time discretization is as small as possible to consider the stock $S$ continuous.[2].

$$U = \mathrm{e}^{\sigma_{\max}\sqrt{\Delta t}+r\Delta t}$$
$$M = \mathrm{e}^{r\Delta t}$$
$$D = \mathrm{e}^{-\sigma_{\max}\sqrt{\Delta t}+r\Delta t}$$

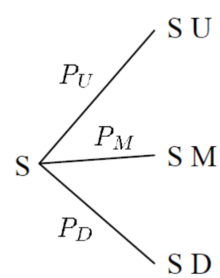


Figure 1: Trinomial tree : Stock S

with

$$P_U(p) = p \cdot \left(1 - \frac{\sigma_{\max}\sqrt{\Delta t}}{2}\right)$$
$$P_M(p) = 1 - 2p$$
$$P_D(p) = p \cdot \left(\frac{1+\sigma_{\max}\sqrt{\Delta t}}{2}\right)$$

Where:

$$\frac{\sigma_{\min}^2}{2\sigma_{\max}^2} \leqslant p \leqslant 1/2$$

## 4.2 Finite-difference scheme

After the introduction of the trinomial model, we will use it to approach the BSB equation. $S_n^j$ denotes the value of the stock S at time n and is located at position j counting from the bottom of the tree, for more precision, see the tree below.

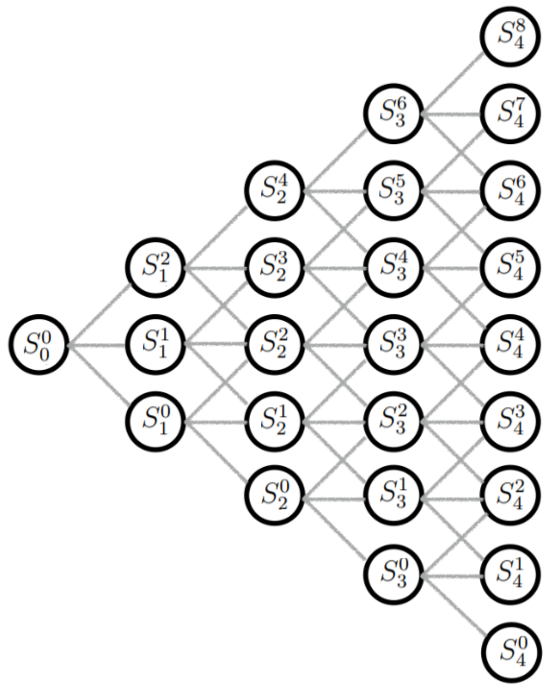


Figure 2: Stock S

Using the trinomial tree presented in the previous subsection, we find the value of $S_n^j$ is, as specified in [2]:

$$S_n^j = S_0^0 \mathrm{e}^{j \cdot \sigma_{\max} \sqrt{\Delta t} + n \cdot r \Delta t}$$

We consider that the boundary of the price of the derivative security is expressed as a function of cash flows as follows:

$$F_n^j = F_n\left(S_n^j\right) \quad \text{and} \quad W_n^{+,j} = \operatorname*{Sup}_P \mathbf{E}_t^P \left[ \sum_{k=j+1}^{N} \mathrm{e}^{-r(t_k - t_n)} F_k\left(S_k\right) \right]$$

$$F_n^j = F_n\left(S_n^j\right) \quad \text{and} \quad W_n^{-,j} = \operatorname*{Inf}_P \mathbf{E}_t^P \left[ \sum_{k=j+1}^{N} \mathrm{e}^{-r(t_k - t_n)} F_k\left(S_k\right) \right]$$

Which is equivalent to the result below, according to the paper of [2]:

$$W_n^{+,j} = F_n^j + \mathrm{e}^{-r\Delta t} \times \mathrm{Sup}_p \left[ P_U(p) W_{n+1}^{+,j+1} + P_M(p) W_{n+1}^{+,j} + P_D(p) W_{n+1}^{+,j+1} \right]$$
$$W_n^{-,j} = F_n^j + \mathrm{e}^{-r\Delta t} \times \mathrm{Inf}_p \left[ P_U(p) W_{n+1}^{-,j+1} + P_M(p) W_{n+1}^{-,j} + P_D(p) W_{n+1}^{-,j+1} \right]$$

With:

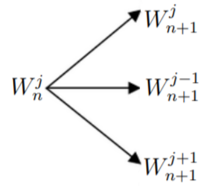


Figure 3: From step $n$ to $n+1$, $W_n^j$ can take three values, namely: $W_{n+1}^{j-1}$, $W_{n+1}^{j}$ and $W_{n+1}^{j+1}$ .

Where $P_U, P_M, P_D$ defined above.

$W_n$ is finally expressed as:

$$W_n^{+,j} = F_n^j + \mathrm{e}^{-r\Delta t} \begin{cases} W_{n+1}^{+,j} + \frac{1}{2} L_{n+1}^{+,j} & \text{if } L_{n+1}^{+,j} \geqslant 0 \\ W_{n+1}^{+,j} + \frac{\sigma_{\min}^2}{2\sigma_{\max}^2} L_{n+1}^{+,j} & \text{if } L_{n+1}^{+,j} < 0 \end{cases}$$

$$W_n^{-,j} = F_n^j + \mathrm{e}^{-r\Delta t} \begin{cases} W_{n+1}^{-j} + \frac{1}{2} L_{n+1}^{-j} & \text{if } L_{n+1}^{-j} < 0 \\ W_{n+1}^{-,j} + \frac{\sigma_{\min}^2}{2\sigma_{\max}^2} L_{n+1}^{-j} & \text{if } L_{n+1}^{-,j} \geqslant 0 \end{cases}$$

Where:

$$L_{n+1}^j = \left(1 - \frac{\sigma_{\max}\sqrt{\Delta t}}{2}\right) W_{n+1}^{j+1} + \left(1 + \frac{\sigma_{\max}\sqrt{\Delta t}}{2}\right) W_{n+1}^{j-1} - 2W_{n+1}^j$$

With $L_{n+1}^j$ and $W_{n+1}^{\cdot}$ can be respectively $L_{n+1}^{-,j}$ or $L_{n+1}^{+,j}$ and $W_{n+1}^{-,\cdot}$ or $W_{n+1}^{+,\cdot}$.

We can determine the price range of the portfolio delimited by $W_0^{+,0}$ and $W_0^{-,0}$ using the backward induction method. Indeed, at each step we can find $W_n^{+,j}$ starting from $W_{n+1}^{+,j-1}$, $W_{n+1}^{+,j}$, $W_{n+1}^{+,j+1}$ and other variables known at time $t_n$. The same reasoning is valid for $W_0^{-,0}$.

## 4.3 UVM Implemetation

For the implementation, we choose to work on the APPLE options (35 calls and 35 puts), taken from [1] for a maturity of $T$ =4 days, interest rate $r = 0.07$, with different strike which varies from 150 to 250 and for a specific day.

The python script corresponding to the UVM implementation using trinomial method is given by the google colab link: `https://colab.research.google.com/drive/1LCcsn4dDTEzoh4_YK7EdLEhH4q7xF-e6?usp=sharing`.
(see the link of the data used at the end of the thesis).

| Trinomial tree | Value |
|---|---|
| $S_0$ | 209.68 |
| T | 4/365 |
| $K_{min}$ | 150 |
| $K_{max}$ | 250 |
| r | 7% |

Table 1: Parameters of Apple's options

#### 4.3.1 Trinomial tree

The first step in the implementation is to represent the stock $S$ as a python list of list as follows:

$[[S_0^0]$

$[S_1^0\ S_1^1\ S_1^2]$

$[S_2^0\ S_2^1\ S_2^2\ S_2^3\ S_2^4]$

$[S_3^0\ S_3^1\ S_3^2\ S_3^3\ S_3^4\ S_3^5\ S_3^6]$

$[S_4^0\ S_4^1\ S_4^2\ S_4^3\ S_4^4\ S_4^5\ S_4^6\ S_4^7\ S_4^8]]$

Let's see an example : for $\sigma_{max} = 0.3$, maturity $T = 15/365$, python list length $N = 3$, interest rate $r = 0.07$ and initial stock $S_0 = 300$:

| Trinomial tree | Value |
|---|---|
| $S_0$ | 300 |
| $\sigma_{max}$ | 0.3 |
| N | 3 |
| r | 7% |

Table 2: Parameters of the trinomial tree

```
[300.0]

[287.79, 300.43, 313.63]

[276.07, 288.2, 300.86, 314.08, 327.89]
```

Figure 4: Output of the python code for the trinomial tree

The result given by the python code is well in the form presented below.

### 4.3.2 Calibration of volatility band

In this part we will find a volatility band for the Apple calls and puts used, an essential step to price the portfolio. As previously specified, to calibrate the volatility band, we will first plot the implied volatility as a function of the strike for Apple options. From the market price of the options, we can find the implied volatility by reversing the Black-Scholes formula.

The analytic formula for European call price is given by

$$C\left(S_t, K, r, \sigma, t, T\right) = S_t \Phi\left(d_1\right) - K e^{-r(T-t)} \Phi\left(d_2\right)$$

where :

$$d_1 = \frac{\log\left(S_t/K\right) + \left(r + \sigma^2/2\right)(T-t)}{\sigma\sqrt{T-t}} \text{ and } d_2 = d_1 - \sigma\sqrt{T-t}.$$

We solve

$$C\left(S_t, K, r, \sigma, t, T\right) = C_{market}$$

$$\implies \sigma_{implied}$$

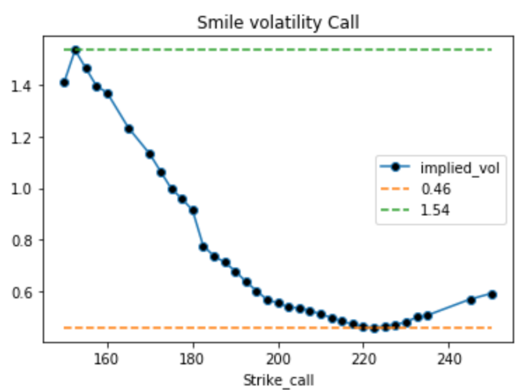


Figure 5: Implied volatility of the call

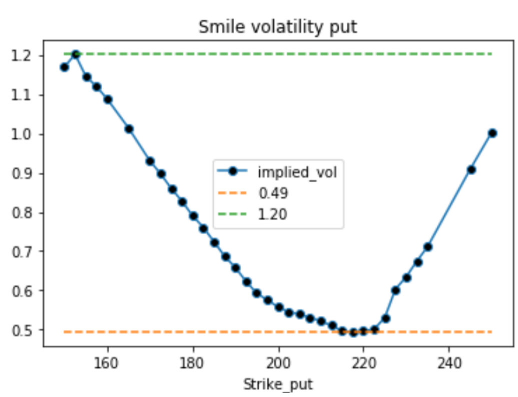


Figure 6: Implied volatility of the put

We also plot the implied volatility of the straddle: long put and long call, as a function of the strike.

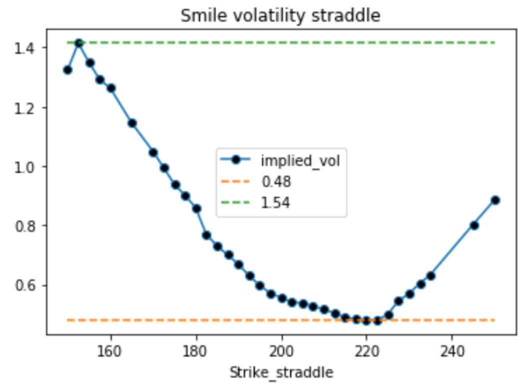


Figure 7: Implied volatility of the straddle

From these plots, we can determine an interval for the volatility band. It is equivalent to choosing a band for the implied volatility which will contains the implied volatilities of the used options.

| Implied volatility | Min | Max |
|---|---|---|
| Call | 0.46 | 1.54 |
| Put | 0.49 | 1.20 |
| Straddle | 0.48 | 1.54 |

Table 3: Extreme values of implied volatility for different vanilla options

We can choose for example $\sigma_{min} = \mathbf{0.1}$ and $\sigma_{max} = \mathbf{1.7}$.

#### 4.3.3 Algorithm presentation

We present the idea of the UVM algorithm which allows to give the extreme prices defining the price interval. To price or hedge an option with the UVM method, we use the backward induction method, we start by determining the payoff at maturity of the option and then we compute $W^j_{T-1}$ , for a call $W^j_T = (S^j_T - K)^+$, $\forall j$ .

$$W^{+,j}_{T-1} = F^j_{T-1} + \mathrm{e}^{-r\Delta t} \begin{cases} W^{+,j}_T + \frac{1}{2} L^{+,j}_T & \text{if } L^{+,j}_T \geqslant 0 \\ W^{+,j}_T + \frac{\sigma^2_{\text{min}}}{2\sigma^2_{\text{max}}} L^{+,j}_T & \text{if } L^{+,j}_T < 0 \end{cases}$$

$$W^{-,j}_{T-1} = F^j_{T-1} + \mathrm{e}^{-r\Delta t} \begin{cases} W^{-,j}_T + \frac{1}{2} L^{-,j}_T & \text{if } L^{-,j}_T < 0 \\ W^{-,j}_T + \frac{\sigma^2_{\text{min}}}{2\sigma^2_{\text{max}}} L^{-,j}_T & \text{if } L^{-,j}_T \geqslant 0 \end{cases}$$

The value of $W^j_{T-1}$ depends on $L^j_T = f(W^j_T, W^{j+1}_T, W^{j-1}_T)$, $F^j_{T-1}$ and $W^j_T$.
Where:

$$f(x, y, z) = \left(1 - \frac{\sigma_{\max}\sqrt{\Delta t}}{2}\right) y + \left(1 + \frac{\sigma_{\max}\sqrt{\Delta t}}{2}\right) z - 2x$$

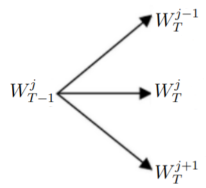


Figure 8: From step $n$ to $n+1$, $W_{T-1}^{j}$ can take three values, namely: $W_T^{j-1}$, $W_T^{j}$ and $W_T^{j+1}$
.

All the variables to compute $W_{T-1}^{j}$ are known, $\forall j$, we repeat the same procedure to compute $W_{T-2}^{\cdot}$ until we reach $W_0^0$, which corresponds to the portfolio price using the UVM method.

#### 4.3.4 Pricing of some vanilla options using UVM

To determine the price of the call using the UVM method, we implemented in Python the *UVM_price_call* function which takes as argument the different parameters of the model namely: extreme volatilities, the maturity T, the level of the tree N, the interest rate r, the initial stock S, the strike K and the *upper_price* parameter which specifies the price to display: the highest price $W^+$ or the lowest price $W^-$.

We have plotted the market price of the options, the extreme prices given by the UVM method, as a function of the strike K. We find that the market price is well within the price range given by the UVM method, almost surely. Indeed, if this condition were not verified, there would be an arbitrage strategy. The drawback of this method is that the price range given by UVM method is wide, especially for the at the money (ATM) calls, as we note in the figures 9 and 10, below for $K = 210$.

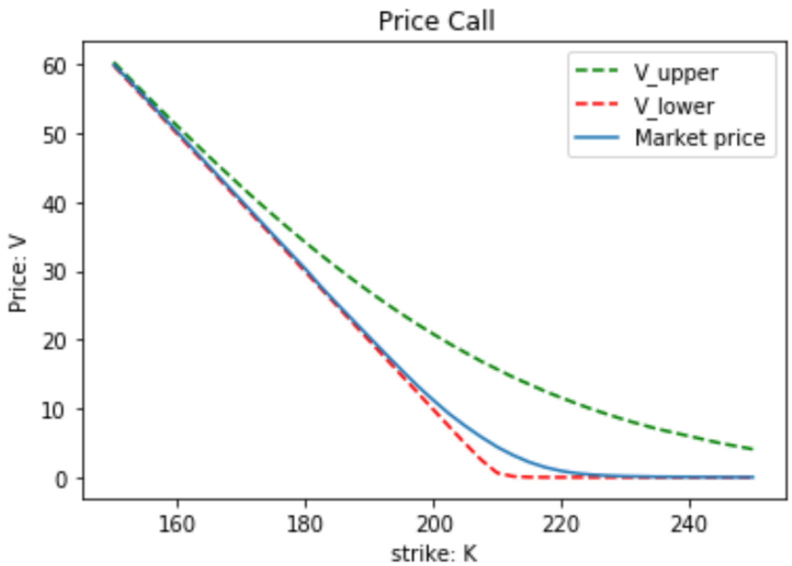


Figure 9: Market and UVM prices of the calls

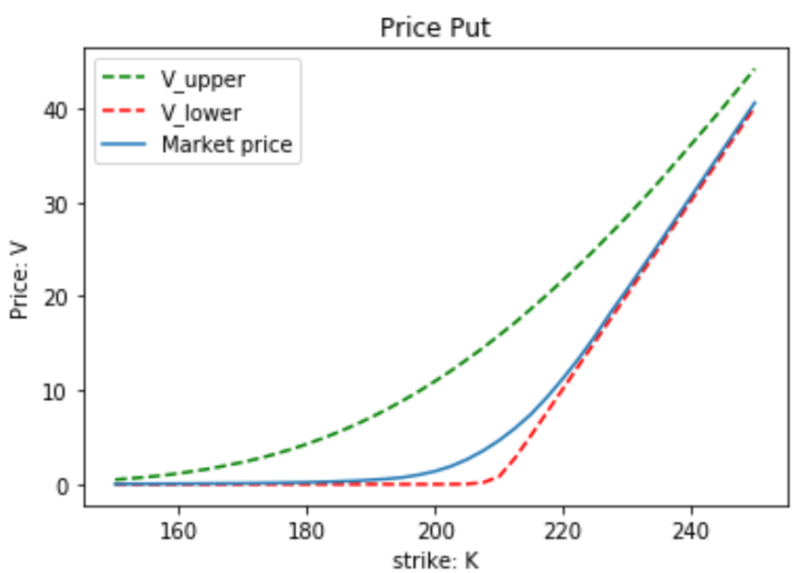


Figure 10: Market and UVM prices of the puts

## 4.4 Comparison with Black and Scholes Prices

In this section, we will compare between the prices given by BS and UVM, for a few options. More precisely, we calculate the upper Black-Scholes price using a constant volatility equal to $\sigma_{max}$ and the lower price using the volatility $\sigma_{min}$.

The plot below shows that the two pricing methods give almost the same price. The UVM price is plotted using a trinomial tree depth $N = 75$.

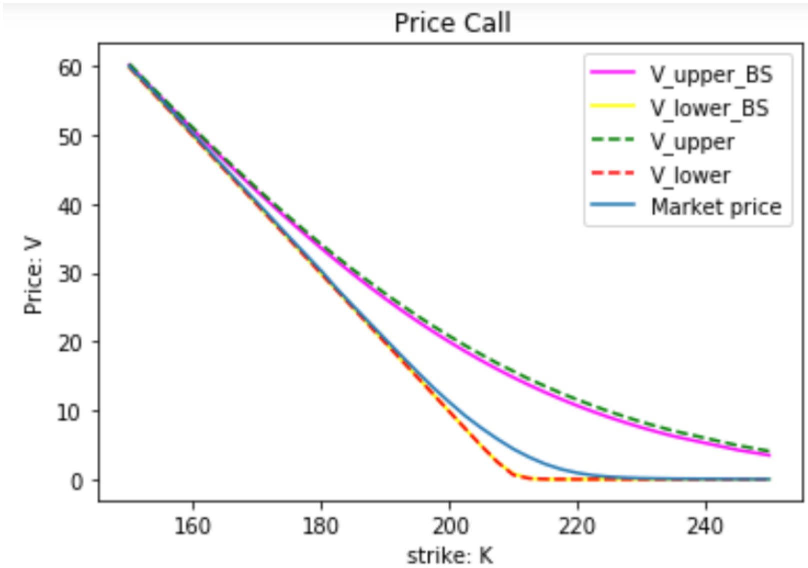


Figure 11: Black-Scholes and UVM prices comparison

To be more precise, we are going to plot the difference between the two upper prices: UVM and BS for Apple calls for different tree depth $N$.

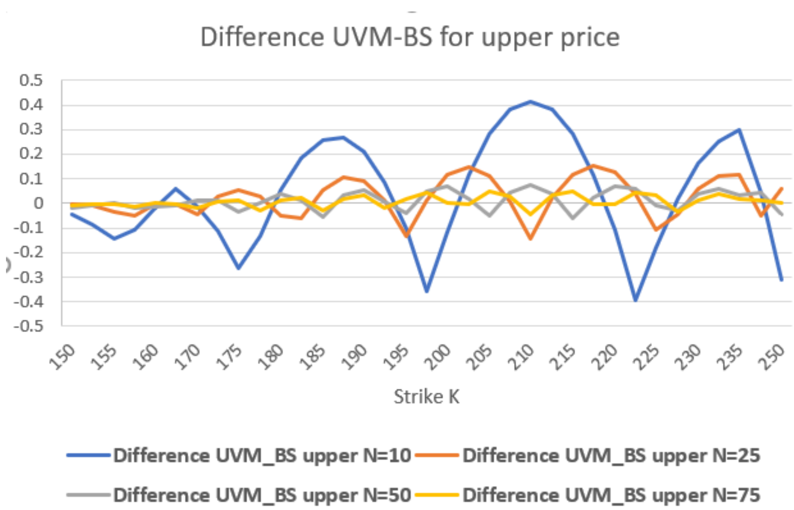


Figure 12: Difference UVM-BS upper price calls

We note that the more N increases the more the difference between the two curves narrows. Indeed, this result is predictable because the gamma (second derivative of the price compared to the stock) is positive, so the price is convex according to the stock. This result is similar for puts and the straddle because their gamma is also positive. The BSB equation for the upper price is:

$$\frac{\partial W(S,t)}{\partial t} + r\left(S\frac{\partial W(S,t)}{\partial S} - W(S,t)\right) + \frac{1}{2}\sigma^2\left[\frac{\partial^2 W(S,t)}{\partial S^2}\right] S^2 \frac{\partial^2 W(S,t)}{\partial S^2} = 0$$

with :

$$\sigma\left[\frac{\partial^2 W^+}{\partial S^2}\right] = \begin{cases} \sigma_{\max} & \text{if } \frac{\partial^2 W^+}{\partial S^2} \geqslant 0 \\ \sigma_{\min} & \text{if } \frac{\partial^2 W^+}{\partial S^2} < 0 \end{cases}$$

we can note that if the gamma is positive, $\sigma^2\left[\frac{\partial^2 W(S,t)}{\partial S^2}\right] = \sigma^2_{max}$

The BSB equation becomes:

$$\frac{\partial W(S,t)}{\partial t} + r\left(S\frac{\partial W(S,t)}{\partial S} - W(S,t)\right) + \frac{1}{2}\sigma^2_{max}S^2\frac{\partial^2 W(S,t)}{\partial S^2} = 0$$

which exactly corresponds to the BS equation (we can see the PDE for BS in section 2).

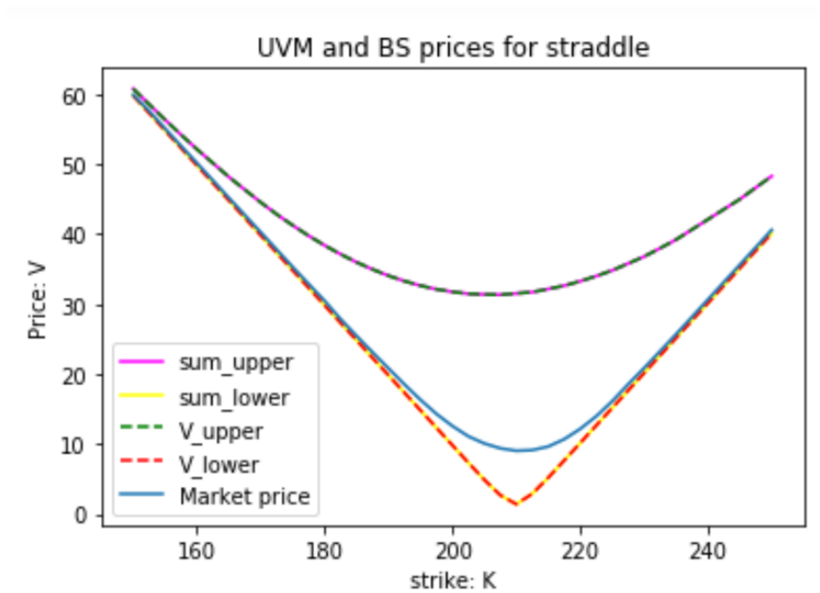


Figure 13: Black-Scholes and UVM prices comparison

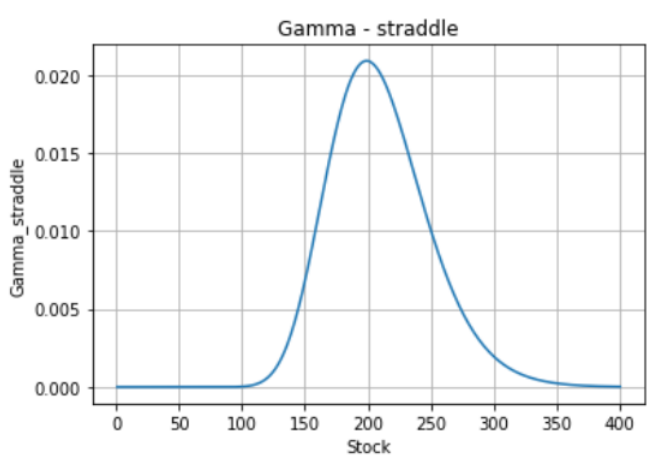


Figure 14: Gamma of the straddle

Now let's take a portfolio where we have a mix of convexity and compare the prices given by UVM and by BS. We consider a portfolio composed of two calls: a first long call with strike K1 and a short second call with strike K2, where $K1 < K2$, this spread option is called: bull spread. We will first plot the gamma of this option to show that its sign is not constant as a function of the Stock S. We took $K1 = 160$ and $K2 = 230$.

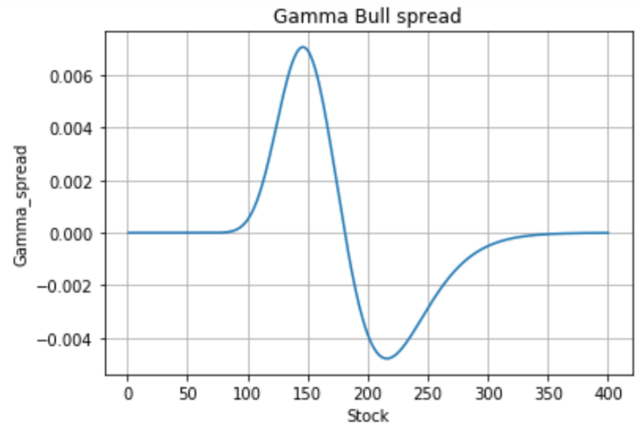


Figure 15: Gamma of the Bull spread

We see that the gamma of the bull spread is positive over the interval $[0, 180]$ then it becomes negative over $[180, 400]$. In this case, the UVM price will be different from the Black-Scholes price because there is no constant volatility that will give exactly the UVM price.

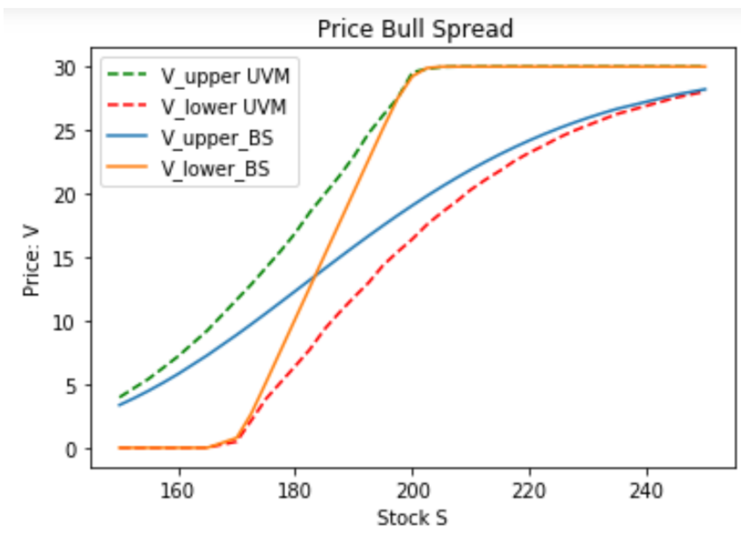


Figure 16: Black-Scholes and UVM prices comparison

We note that the prices given by Black-Scholes (taking the extreme volatilities) are in the price range proposed by UVM. Precisely, to calculate the Black-Scholes upper price for the bull spread, we keep a constant volatility equal to $\sigma_{max}$ and for the lower price we took a volatility of $\sigma_{min}$.

## 4.5 Risk diversification

In this section, we will highlight the property of the sub-additivity of payoffs. We will take the same portfolio composed of Bull spread options (whose payoff can be written as the payoff of long call K1 and short call K2) as the previous section and compare the price of the overall and the individual portfolios. More precisely, we will study the prices for the bull spread with the sum of the prices of the following portfolios: long call with strike K1 and short call with strike K2, using UVM method.

The plot highlights the following inequality.

$$\begin{aligned}&\mathrm{Sup}_P\, \mathrm{E}_t^P[\Phi+\Psi] \leqslant \mathrm{Sup}_P\, \mathrm{E}_t^P[\Phi] + \mathrm{Sup}_P\, \mathrm{E}_t^P[\Psi]\\ &\mathrm{Inf}_P\, \mathrm{E}_t^P[\Phi+\Psi] \geqslant \mathrm{Inf}_P\, \mathrm{E}_t^P[\Phi] + \mathrm{Inf}_P\, \mathrm{E}_t^P[\Psi]\end{aligned}$$

The price range given by the overall portfolio is narrower than the price range suggested by the sum of the individual portfolios. The inequalities above becomes equalities when the two payoffs are identical. Intuitively, we can interpret this result as a consequence of the decrease in volatility risk of the overall portfolio when the payoff has a mixed convexity.

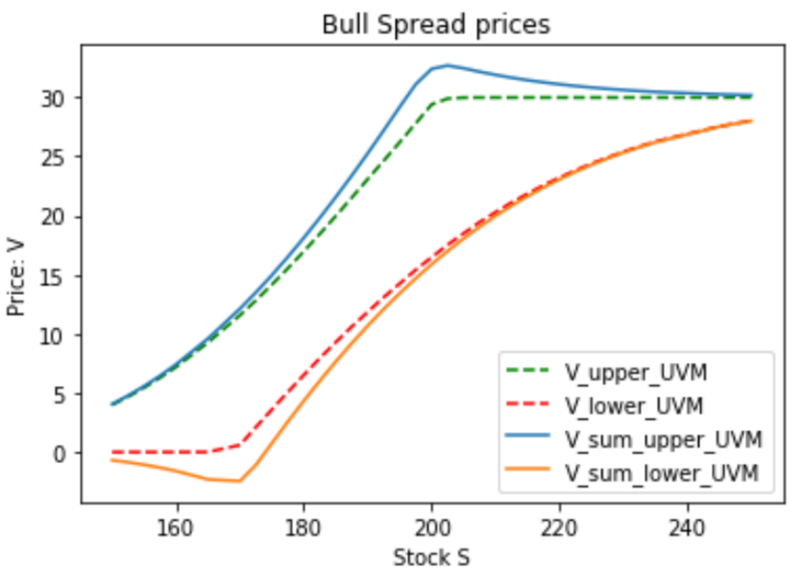


Figure 17: The price envelope determined by UVM methods for the bull spread and the sum of the prices given by BS for the portfolios: long call1 and short call2, taken separately

## 4.6 Change in the volatility band

In this section, we will highlight the impact of changing the volatility band on the price of options using the UVM method.

We first plotted the difference between higher and lower prices for different volatility bands on Apple calls. We chose a wide band where $\sigma_{max} = 1.7$ and $\sigma_{min} = 0.1$ and a smaller band with $\sigma_{max} = 1.5$ and $\sigma_{min} = 0.4$

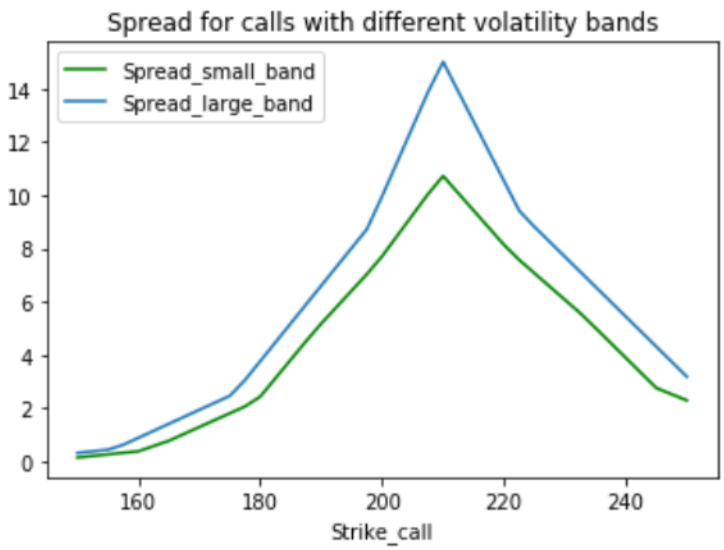


Figure 18: Different bands of volatility and its impact on UVM prices

We note that the thinner the volatility band, the smaller the difference between lower and higher prices. If we consider the extreme case where $\sigma_{min}$ and $\sigma_{max}$ are equal, we find the BS price with constant volatility.

We were then interested in modeling the difference of prices (between upper and lower) as a function of the width of the volatility band for the ATM Apple call with strike $K = 210$. To determine the width of the volatility range, we calculate the difference between $\sigma_{max}$ and $\sigma_{min}$. We kept $\sigma_{min}$ fixed equal to $\sigma_{min} = 0.4$ and we gradually increase $\sigma_{max}$ from 1.5 to 1.8. We note that the larger the volatility range, the higher the spread (and this increase seems linear), which joins the previous remark. (see figure 19 below)

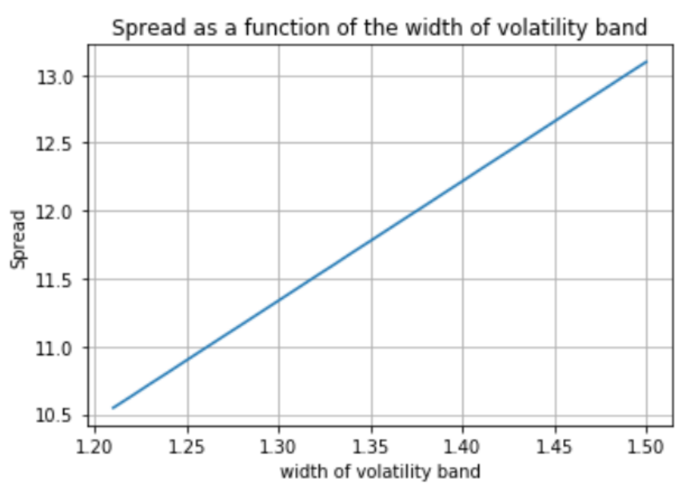


Figure 19: Plot of spread price according to width volatility band

## 4.7 Lagrangian UVM

### 4.7.1 Presentation of the algorithm

To present the Lagrangian UVM algorithm, we will dissociate two cases, first we consider the case where the derivative securities (considered as inputs) have the same maturity $T$ as the portfolio we are studying. Then we study a more general framework where the inputs and the portfolio have different maturities.

- **Portfolio and inputs have the same maturity T**

  In this particular case, we consider for example that we have two calls (with strike $K_1$ and $K_2$) as inputs and a portfolio composed of one call of strike $K$. The Lagragian UVM algorithm will be similar to the UVM algorithm with a different portfolio's payoff:

  $$W_T^j = (S_T^j - K)^+ - \lambda_1(S_T^j - K_{hedg1})^+ - \lambda_2(S_T^j - K_{hedg2})^+$$

  The last step of the algorithm is to find $\lambda_1$ and $\lambda_2$ which minimize $W^+$, maximize $W^-$, and we perform this last step on python thanks to the scipy library.

- **Portfolio and inputs with different maturities**

  For the computation of $W_T^j$ we must compare between the maturities of the different options. We consider for example the case where we have available a portfolio composed of a call of maturity T and two hedging calls of maturity $T1$, $T2$ with $T1 < T2$.

| $W_T^j$ | Value |
|---|---|
| if T2<T | $W_T^j = (S_T^j - K)^+$ |
| if T2>T | $W_T^j = -\lambda_1(S_T^j - K_{hedg1})^+ - \lambda_2(S_T^j - K_{hedg2})^+$ |
| if T2=T | $W_T^j = (S_T^j - K)^+ - \lambda_1(S_T^j - K_{hedg1})^+ - \lambda_2(S_T^j - K_{hedg2})^+$ |

Table 4: Payoff of the portfolio: $W_T^j$

As for the previous algorithm, to compute $W_n^j$ for $n \leq T-1$ and $j \in \{0, 1, ..., 2n+1\}$ we use the backward induction method. Indeed, to compute $W_n^j$ we must first determine $L_{n+1}^j$ as for UVM algorithm for each n and j:

$$L_{n+1}^{\pm,j} = \left(1 - \frac{\sigma_{\max}\sqrt{\Delta t}}{2}\right) W_{n+1}^{\pm,j+1} + \left(1 + \frac{\sigma_{\max}\sqrt{\Delta t}}{2}\right) W_{n+1}^{\pm,j-1} - 2W_{n+1}^{\pm,j}$$

$L_{n+1}^{\pm,j}$ can be computed using $W$ at the step n+1 found by descending induction from the payoff in table 4 (the indices $j, j+1$ and $j-1$ of $W$ are represented in the tree below).

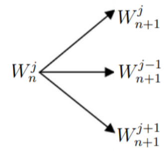


Figure 20: From step $n$ to $n+1$, $W_n^j$ can take three values, namely: $W_{n+1}^{j-1}$, $W_{n+1}^{j}$ and $W_{n+1}^{j+1}$

We will now present the formulas that exists between $W_n^j$ and $L_{n+1}^{\pm,j}$.

We define two lists: the list of the maturities of the portfolio options which we will call *settlement_date* and the list of the maturities of the hedging derivatives: *maturity_hed_opt*. We will assume that these two lists are sorted in ascending order. In addition, we will denote by $t_k$ and $t^{'}_{k'}$ the respective elements of *settlement_date* and *maturity_hed_opt*. $k$ and $k^{'}$ will be at the start of the algorithm equal to the length of their respective lists. When one of the conditions in the table 5 below is met the value of $k$, $k^{'}$ or both will be reduced by one (depending on the variable involved), once we go to the following time step. For example if the first condition of the table below is met, i.e. $t^{'}_{k'} < n \times \Delta T \leq t_k$, when we go to step $n-1$, $k$ becomes $k-1$ and $k^{'}$ will keep the same value.

| $W_n^j$ | Value |
|---|---|
| if $t^{'}_{k'} < n\times\Delta T \leq t_k$ | $W_n^j = (S_n^j - K)^+ + e^{-r\times dt}(W_{n+1}^{j+1} + p \times L_{n+1}^{\pm,j})$ |
| if $t_k < n\times\Delta T \leq t^{'}_{k'}$ | $W_n^j = -\lambda_1(S_n^j - K_{hedg1})^+ - \lambda_2(S_T^j - K_{hedg2})^+ + e^{-r\times dt}(W_{n+1}^{j+1} + p \times L_{n+1}^{\pm,j})$ |
| if $n\times\Delta T = t_k = t^{'}_{k'}$ | $W_n^j = (S_n^j - K)^+ - \lambda_1(S_n^j - K_{hedg1})^+ - \lambda_2(S_n^j - K_{hedg2})^+ + e^{-r\times dt}(W_{n+1}^{j+1} + p\times L_{n+1}^{\pm,j})$ |

Table 5: Value of the portfolio at time n: $W_n^j$

The parameter p in table 5 depends on which extreme price we compute: the upper price or the lower price. For the upper price, the value of p is:

$$p = \begin{cases} 1/2 & \text{if } L_{n+1}^j \geq 0 \\ \sigma_{\min}^2/2\sigma_{\max}^2 & \text{if } L_{n+1}^j < 0 \end{cases}$$

and for the lower price, the system condition is reversed.

We will continue in this way until we determine $W_0^0$.

The last step of the algorithm is to find the value of $\lambda_i$ for $i \in \{1, ...M\}$ that solve the following problem.

$$\inf_{\lambda_1,\lambda_2,\ldots,\lambda_M} W^+(S_t, t; \lambda_1, \lambda_2, \ldots \lambda_M)$$
$$\sup_{\lambda_1,\lambda_2,\ldots,\lambda_M} W^-(S_t, t; \lambda_1, \lambda_2, \ldots \lambda_M)$$

This problem has a unique solution because the function $(\lambda_1, \ldots, \lambda_M) \mapsto W^+(S_t, t; \lambda_1, \lambda_2, \ldots \lambda_M)$ is convex and $(\lambda_1, \ldots, \lambda_M) \mapsto W^-(S_t, t; \lambda_1, \lambda_2, \ldots \lambda_M)$ is concave. [3]

### 4.7.2 Comparison between three and four input derivative instruments

We have a portfolio of calls of maturity T, which we want to hedge using first three then four calls as inputs or hedging derivatives.

We consider now that the hedging derivative instruments to be used are three calls with the same maturity $T$ as that of the Apple calls.

| Inputs calls | strike K | Maturity |
|---|---|---|
| CALL1 | 180 | T=4/365 |
| CALL2 | 210 | T=4/365 |
| CALL3 | 220 | T=4/365 |

Table 6: Parameters of the hedging portfolio's calls

We plot the upper and lower prices using the Lagrangian UVM method as a function of the strike K, by choosing that $\lambda_i$ varies between -1 and 1.

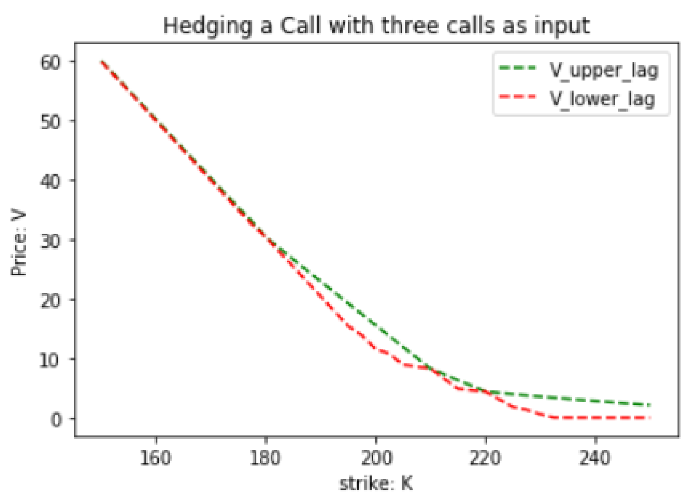


Figure 21: Market and lagragian UVM prices of the calls, using three calls as hedging options

We notice that if $\lambda_i$ is equal to zero for all the hedging options, we find that the range of price given by UVM and Lagragian UVM methods are the same. To check this results graphically, we plot the upper and lower price as a function of the strike using the Lagrangian algorithm UVM with lambda equal to 0. Indeed, we find exactly the same graph as that plotted in the section 4.1.4 for a call.

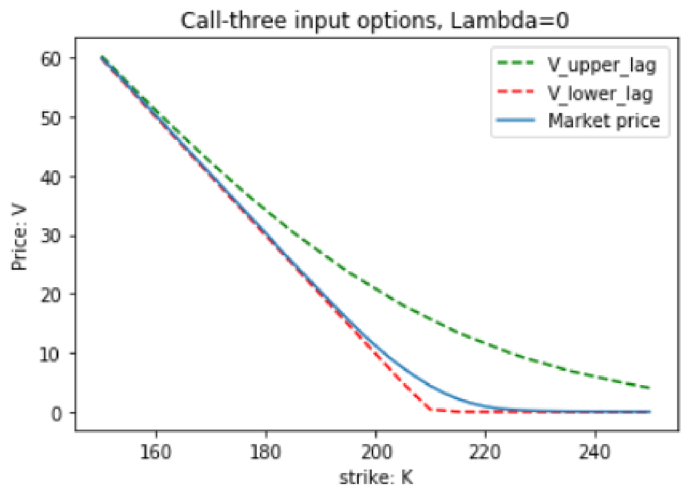


Figure 22: Market and lagragian UVM prices of the calls with vector lambda equal to zero

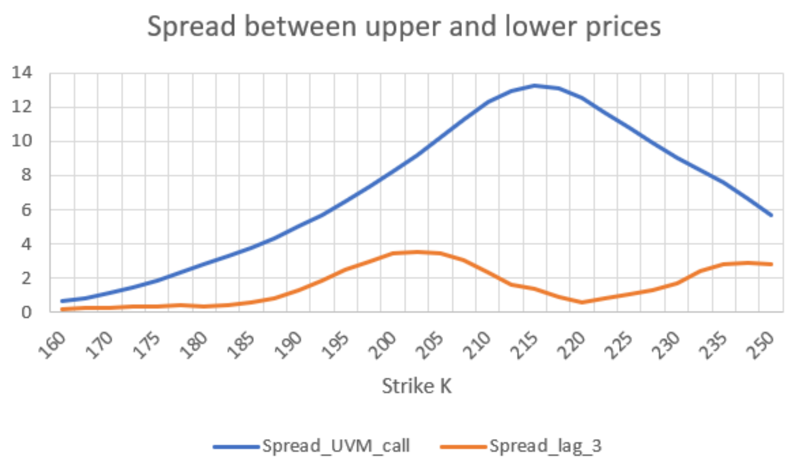


Figure 23: Smoothed curve of the difference between upper and lower prices for UVM method and Lagrangian UVM with three hedging instruments

The Lagrangian method makes it possible to reduce the difference between the extreme prices in comparison with the classic UVM method, as the plot using the two methods underlines.

We notice that the spread between upper and lower prices is strongly reduced with the lambda UVM method. We can also note that the spread is almost equal to zero for the Apple calls with the same strike as the hedging portfolio's calls. For example we see in the figure 21 that the Apple calls with strikes 180, 210 and 220 have a spread of zero.

We keep the same portfolio as before but this time we use four calls as inputs to hedge the portfolio.

| Inputs calls | strike K | Maturity |
|---|---|---|
| CALL1 | 180 | T=4/365 |
| CALL2 | 200 | T=4/365 |
| CALL3 | 210 | T=4/365 |
| CALL4 | 220 | T=4/365 |

Table 7: Parameters of the hedging portfolio's calls

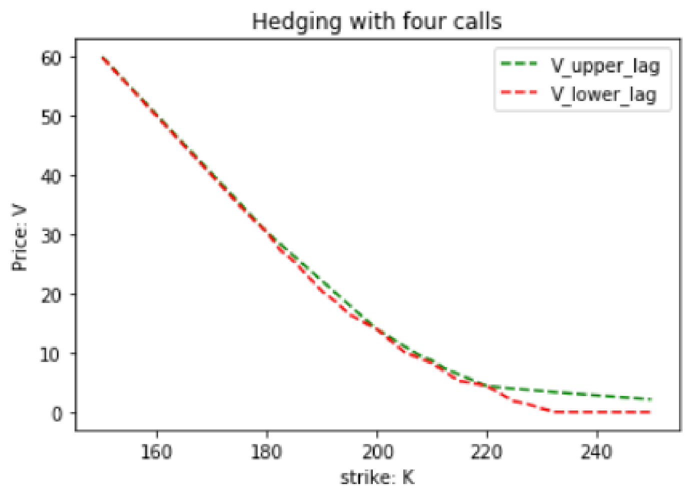


Figure 24: Market and lagragian UVM prices of the calls, using four calls as hedging options

We compare the difference called the spread between the upper and lower price for three and four hedging instruments. We find an important result: the more we increase the number of hedging options the more the difference between the two extreme prices decreases (see figure 25). Indeed, let us consider 4 hedging options with $\lambda_2 = 0$, and keeping the other $\lambda$ equal to the optimal ratio given for the three hedging options (as for the previous example), in this case the price range proposed by 4 hedging options would be equal to that given by 3 inputs, and as we optimize the lambdas the result given by 4 inputs would necessarily be better than that given by a lower number of hedging options.

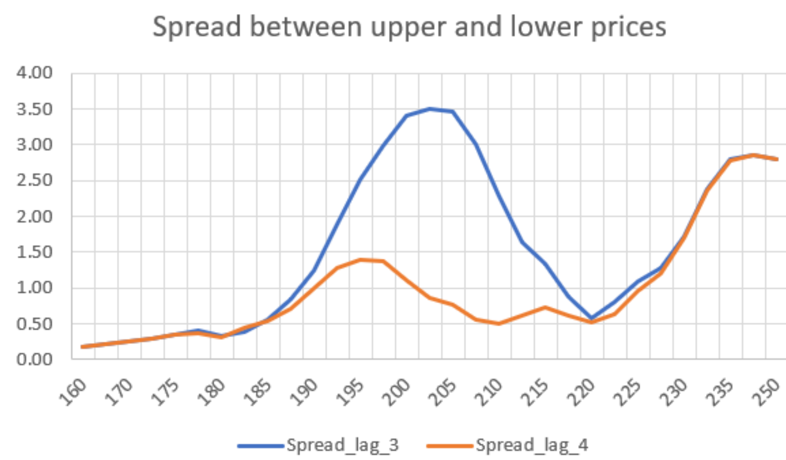


Figure 25: Smoothed curve of the difference between upper and lower prices using Lagrangian UVM

### 4.7.3 Lagrangian UVM with input derivative which have different maturities

We want to price the Apple calls using the Lagragiam UVM method with four calls as inputs (with strike $K = S_0$), with different maturities presented in the table below.

| Inputs calls | strike K | Maturity |
|---|---|---|
| CALL1 | $S_0$ | T=1/365 |
| CALL2 | $S_0$ | T=2/365 |
| CALL3 | $S_0$ | T=4/365 |
| CALL4 | $S_0$ | T=8/365 |

Table 8: Parameters of the hedging portfolio's calls

In this section, we want to show the link between $\lambda$, which represents the proportions to be considered in each input (to compute the upper price), according to their maturities. Indeed, we take the average of $\lambda 1$, $\lambda 2$, $\lambda 3$ and $\lambda 4$ on all Apple calls. These $\lambda$ are shown in the table below. We note that for the maturity of $T = 4/365$, which is the maturity of our portfolio, the $\lambda$ value is the most important: 0.42. It can also be noted that the furthest maturity from $T = 4/365$ is that with the least weight almost 0.

| Average of proport. on Apple calls | Value |
|---|---|
| $\lambda_1$ | 0.032 |
| $\lambda_2$ | 0.077 |
| $\lambda_3$ | 0.417 |
| $\lambda_4$ | 0.00 |

Table 9: Proportions of the hedging portfolio's calls associated to the lagrangian UVM upper price

Now we will be interested in representing the lambdas of each Apple call and not the average over all the calls as done previously. We notice in the figure below 26 that the ratio associated with call3 is the most dominant. To hedge the ATM (at the money) call, we can see that the ratio associated with call3 with the maturity T (the same as the call to hedge) is equal to 1 and the other ratios are worth 0, which is logical because we are hedging the call by itself.

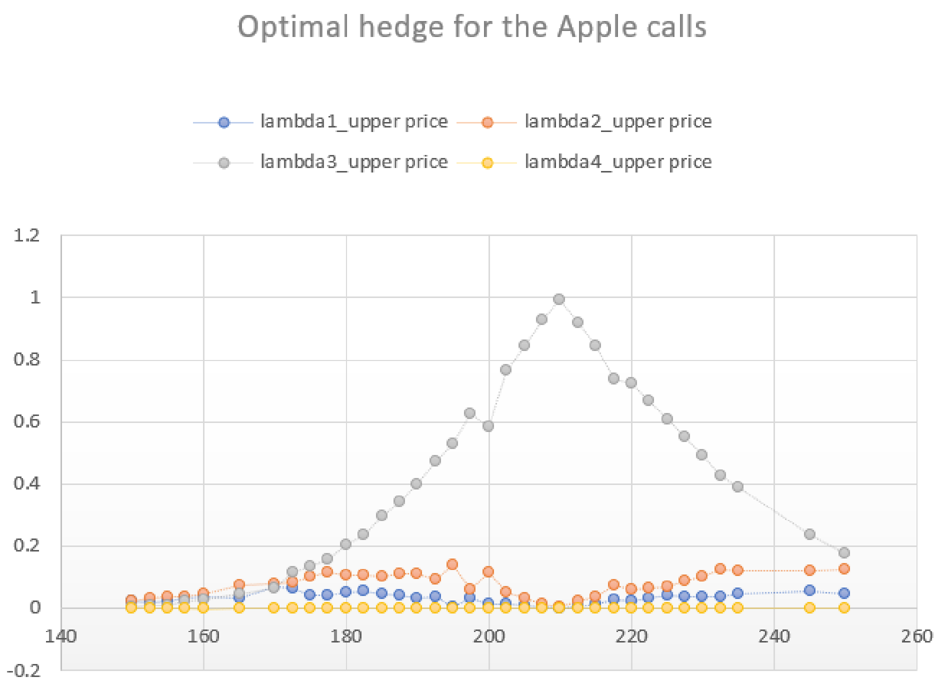


Figure 26: Market and lagragian UVM prices of the calls, using four calls as hedging options

# 5 UVM and finite-difference method

## 5.1 Finite-difference method to solve the BSB equation

In the previous section, we examined the UVM option pricing method using a trinomial model for the underlying asset S. We will now study UVM pricing for barrier options, by approximating the BSB equation using the finite-difference method: by discretizing the time t and the values of the asset S. There are studies and articles in this direction like the papers of [20] and [26]. In this section, we will more focus on the article of [26].

Recall the expression of the BSB equation for a derivative with payoff $F_N(S)$

$$\begin{cases} \frac{\partial W(S,t)}{\partial t} + r\left(S\frac{\partial W(S,t)}{\partial S} - W(S,t)\right) + \frac{1}{2}\sigma^2\left[\frac{\partial^2 W(S,t)}{\partial S^2}\right] S^2 \frac{\partial^2 W(S,t)}{\partial S^2} = 0 \\ W(S,t_N) = F_N(S) \end{cases}$$

with:

$$\sigma\left[\frac{\partial^2 W^+}{\partial S^2}\right] = \begin{cases} \sigma_{\max} & \text{if } \frac{\partial^2 W^+}{\partial S^2} \geqslant 0 \\ \sigma_{\min} & \text{if } \frac{\partial^2 W^+}{\partial S^2} < 0 \end{cases}$$

$$\sigma\left[\frac{\partial^2 W^-}{\partial S^2}\right] = \begin{cases} \sigma_{\max} & \text{if } \frac{\partial^2 W^-}{\partial S^2} \leqslant 0 \\ \sigma_{\min} & \text{if } \frac{\partial^2 W^-}{\partial S^2} > 0 \end{cases}$$

To approximate BSB equation, we can discretize: $\frac{\partial W(S,t)}{\partial t}$, $\frac{\partial W(S,t)}{\partial S}$ ,and $\frac{\partial^2 W(S,t)}{\partial S^2}$. Using Taylor Expansion, we get that:

$$\frac{\partial W(S,t)}{\partial t} = \frac{W(S,t+\Delta t) - W(S,t)}{\Delta t} + O(\Delta t)$$

for $\Delta t << 1$, we use that $\frac{\partial W(S,t)}{\partial t} \approx \frac{W(S,t+\Delta t)-W(S,t)}{\Delta t}$, the error in this case is $\Delta t$.

$$\frac{\partial W(S,t)}{\partial S} = \frac{W(S+\Delta S,t) - W(S-\Delta S,t)}{2\Delta S} + O(\Delta S^2)$$

For $\frac{\partial W(S,t)}{\partial S}$, we used the centered approximation, we can show that in this case the error is of the order of $\Delta S^2$, so this last method is more precise than that used for $\frac{\partial W(S,t)}{\partial t}$

$$\frac{\partial^2 W(S,t)}{\partial S^2} = \frac{W(S+\Delta S,t) + W(S-\Delta S,t) - 2W(S,t)}{\Delta S^2} + O(\Delta S^2)$$

$$\frac{\partial^2 W(S,t)}{\partial S^2} \approx \frac{W(S+\Delta S,t) + W(S-\Delta S,t) - 2W(S,t)}{\Delta S^2}$$

This formulation of the second order derivative is centered, and the error of approximation is of the order of $\Delta S^2$. We obtain this formulation by making the difference of the Taylor expansion of $W(S+\Delta S,t)$ and $W(S-\Delta S,t)$.

Time $t \in [0,T]$ and asset $S \in [S_{min}, S_{max}]$ are discretized into N and M sub-intervals respectively. More precisely, $t_i = i \times \Delta t$, with $\Delta t = \frac{T}{N}$, and $S_j = S_{min} + j \times \Delta S$ with $\Delta S = \frac{S_{max}-S_{min}}{M}$, for $i \in \{0,1,..,N\}$ and $j \in \{0,1,..,M\}$.

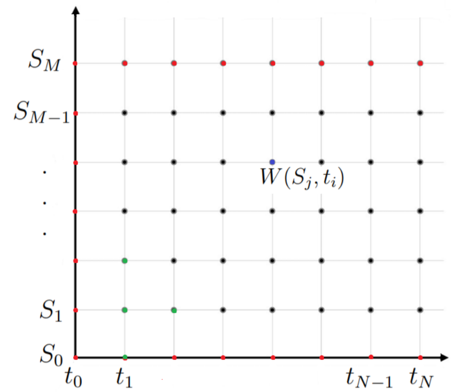


Figure 27: Grid representation: in red the boundary conditions of the price $W(S,t)$ and in green the points of the grid used to calculate the point at the next instant.

Using this discretization, we get that:

$$\frac{\partial W(S,t)}{\partial t} = \frac{W(j,i+1) - W(j,i)}{\Delta t}$$
$$\frac{\partial W(S,t)}{\partial S} = \frac{W(j+1,i) - W(j-1,i)}{2\Delta S}$$
$$\frac{\partial^2 W(S,t)}{\partial S^2} = \frac{W(j+1,i) + W(j-1,i) - 2W(j,i)}{\Delta S^2}$$

where: $W(j,i)$ denotes $W(S_j, t_i)$.

The BSB equation for upper price $W^+$ becomes:

$$\begin{matrix} -\frac{W(j,i+1)-W(j,i)}{\Delta t} + r\left(S_j\left(\frac{W(j+1,i)-W(j-1,i)}{2\Delta S}\right) - W(j,i)\right) + \\ \frac{1}{2}\sigma^2\left[\frac{\partial^2 W(S,t)}{\partial S^2}\right] S_j^2 \frac{W(j+1,i)+W(j-1,i)-2W(j,i)}{\Delta S^2} \end{matrix} = 0$$

With

$$\sigma\left[\frac{\partial^2 W^+}{\partial S^2}\right] = \begin{cases} \sigma_{\max} & \text{if } W(j+1,i) + W(j-1,i) \geqslant 2W(j,i) \\ \sigma_{\min} & \text{if } W(j+1,i) + W(j-1,i) < 2W(j,i) \end{cases}$$

Therefore,

$$\begin{aligned} W(j,i+1) &= W(j,i)\left(1 - r\Delta t - \sigma^2\left[\frac{\partial^2 W(S,t)}{\partial S^2}\right]\frac{\Delta t S_j^2}{\Delta S^2}\right) \\ &+ W(j+1,i)\Delta t\left(-r\frac{S_j}{2\Delta S} + \frac{1}{2}\sigma^2\left[\frac{\partial^2 W(S,t)}{\partial S^2}\right]\frac{S_j^2}{\Delta S^2}\right) \\ &+ W(j-1,i)\Delta t\left(\frac{1}{2}\sigma^2\left[\frac{\partial^2 W(S,t)}{\partial S^2}\right]\frac{S_j^2}{\Delta S^2} + \frac{rS_j}{2\Delta S}\right). \end{aligned}$$

From this last expression, we notice that we can calculate $W(j,i+1)$ at time $t_{i+1}$, from $W(j-1,i)$, $W(j,i)$ and $W(j+1,i)$ at time $t_i$. Boundary conditions and initial condition must be added to compute the values of $W(j,i)$ for $i \in \{1, ..., N\}$ and $j \in \{1, ..., M\}$.

## 5.2 UVM price of a up-and-out Call

To apply the finite difference method to an up-and-out call, we will introduce the boundary conditions:

$$W(S_{min}, 0) = 0$$
$$W(S_{max}, 0) = 0$$

and the initial condition:

$$W(S_j, 0) = (S_j - K)^+$$

The various constants and parameters used to solve BSB equation using the finite-difference on an up-and-out call are presented below:

| Up-and-out call | Value |
|---|---|
| $S_{min}$ | 150 |
| $S_{max}$ | 240 |
| T | 30/365 |
| K | 210 |
| $\sigma_{min}$ | 0.1 |
| $\sigma_{max}$ | 0.2 |
| r | 0.07 |
| N | 30 |
| M | 30 |

Table 10: Parameters of the up-and-out call's range UVM-prices

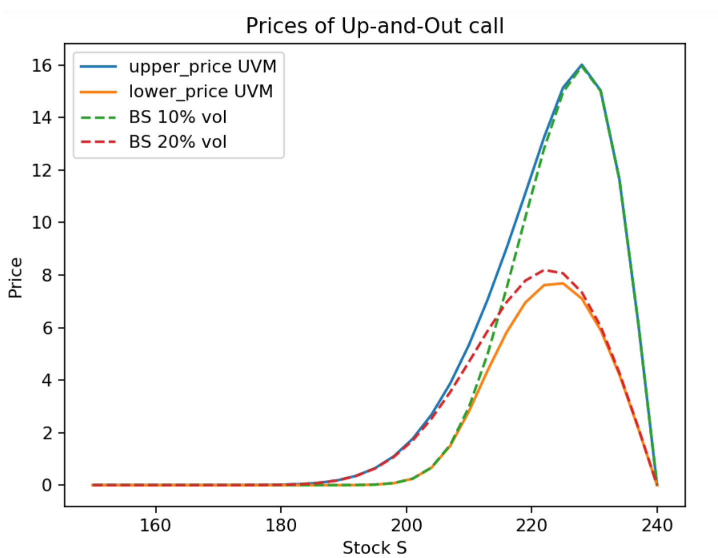


Figure 28: Prices of Up-and-Out call: UVM prices and BS prices for a volatility of 10% and 20%

The figure 28 above represents the values of the up-and-out call, with the characteristics presented in the table, using UVM method. We also represent the Black-Scholes values for a constant volatility of 10% and 20%. We note that the prices of the two methods are different, this is predictable because the gamma of the up-and-out call is not of constant sign (see the figure 29). Therefore, we can't get UVM prices using constant volatility.

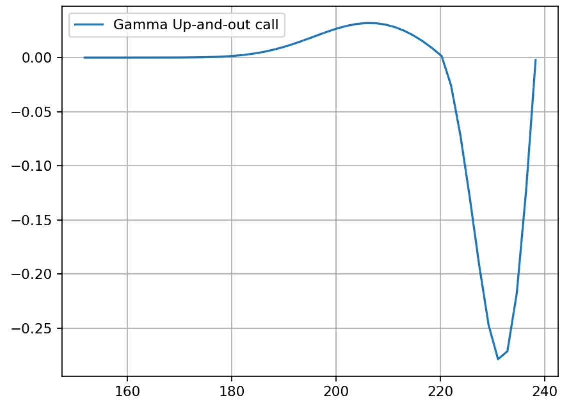


Figure 29: Gamma of the up-and-out call

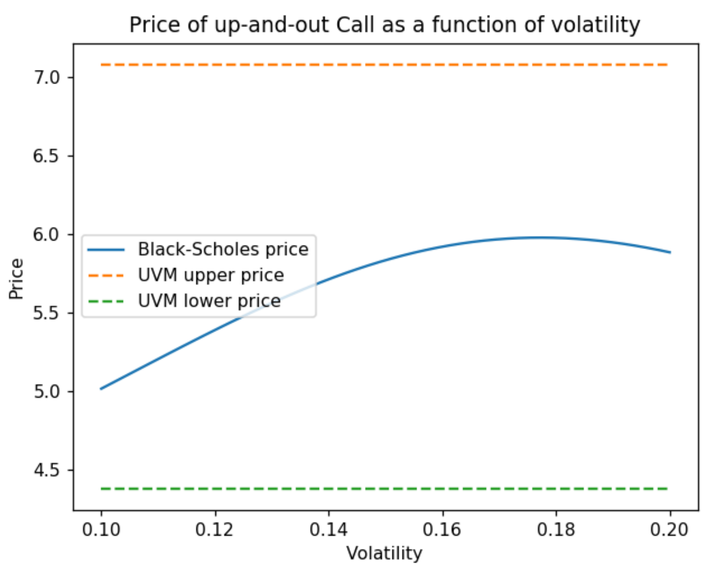


Figure 30: Prices of up-and-out call using BS and UVM methods as a function of the volatility

Now we are interested in implied volatility, from the market price we can find the value of implied volatility by inverting the Black-Scholes formula. We kept the call up-and-out with the same characteristics, and we plot the BS price as a function of the volatility. We also represent the UVM call prices which are of course constant for a volatility band [0.1,0.2] and for $S_0 = 213$. We see that the Black-Scholes price varies between 5 and 6, and on this interval: some prices can have two implied volatilities. For the other prices outside this range, no volatility is associated with it and an arbitrage is possible at these prices, as specified in [32]. Using UVM methods, we have a wider range to determine market prices: between around 4.3 and 7, as we can see in figure 30.

The python script corresponding to the UVM method using the finite-difference is given by the google colab link: `https://colab.research.google.com/drive/1_RB6aJb-hBA_LdpzXnjIkCxSFG2E-JZd?usp=sharing`
(see the link of the data used at the end of the thesis)

## 5.3 Lagragian UVM and finite-difference

### 5.3.1 Finite difference solution for the Lagrangian UVM prices

In this section, we will use the Lagrangian UVM method to evaluate vanilla options by using the finite-differences method. The Lagrangian UVM method, as explained in the first part, is a hedging method using options, and the expression below represents the price of the portfolio composed of a short position on an option with cash flow $F_j$ at time $t_j$.

$$W\left(S_t, t, \lambda_1, \ldots, \lambda_M\right) = \sup_{P \in \mathbb{P}} \mathbf{E}^P \left\{ \sum_{j=1}^{N} e^{-r(t_j - t)} F_j\left(S_{t_j}\right) - \sum_{i=1}^{M} \lambda_i e^{-r(t_i - t)} G_i\left(S_{t_i}\right) \right\} + \sum_{i=1}^{M} \lambda_i C_i \tag{5.1}$$

$$= \sup_{P \in \mathbb{P}} \mathbf{E}^P \left\{ \sum_{i=1}^{N'} e^{-r(t_i - t)} \left( \tilde{F}_i\left(S_{t_i}\right) - \lambda_i \tilde{G}_i\left(S_{t_i}\right) \right) \right\} + \sum_{i=1}^{M} \lambda_i C_i \tag{5.2}$$

We have kept the same notations as for the presentation of the model in the section 3.4.

We notice that the first part of this expression is similar to UVM when considering the global cash flows: $\tilde{F}_i - \lambda_i \tilde{G}_i$

We therefore return to the previous method (UVM), on condition of adding to the value of the portfolio: the weighted sum of the prices of hedging options by the proportions:

$$\sum_{i=1}^{M} \lambda_i C_i$$

.

The last step that remains to be done is to minimize the value of the worst case scenario of the portfolio according to $\lambda_i$. We consider an interval $\lambda_i \in [\Lambda_i^-, \Lambda_i^+]$ which will be fixed by the availability of options in the market. In the case where we are in a long position in derivatives, we need to maximize over $\lambda_i$.

### 5.3.2 Hedging of Up-and-out call with one call

We approximate BSB equation as above but this time with different initial condition. We took the example of the up-and-out call with the following characteristic. We hedge this portfolio by a call of strike $K_{hedging}$

| $\lambda-$UVM for Up-and-out call | Value |
|---|---|
| $S_{min}$ | 0 |
| $S_{max}$ | 240 |
| T | 50/365 |
| K | 109 |
| $K_{hedging}$ | 113 |
| $\sigma_{min}$ | 0.1 |
| $\sigma_{max}$ | 0.2 |
| r | 0.07 |
| N | 100 |
| M | 30 |

Table 11: Parameters of the up-and-out call's range $\lambda$-UVM-prices

The boundary conditions for this example are:

$$W(S_{min}, 0) = 0$$

$$W(S_{max}, 0) = 0$$

The initial condition:

$$W(S_j, 0) = (S_j - K)^+ - \lambda(S_j - K_{hedging})$$

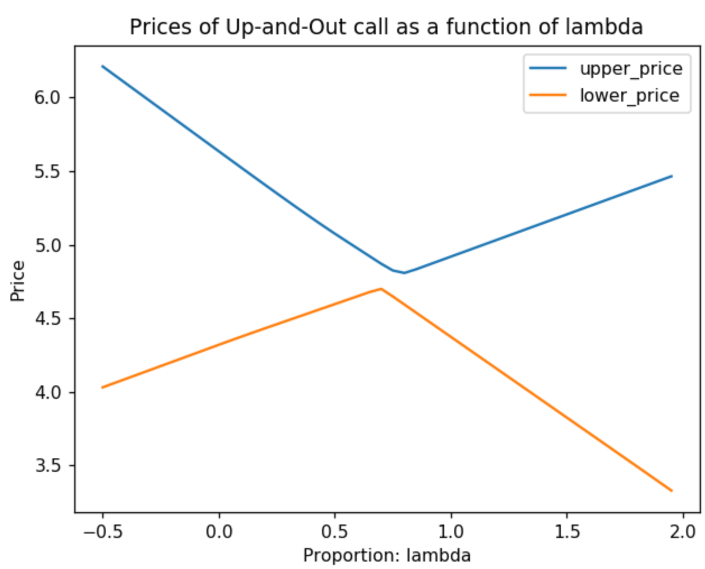


Figure 31: Prices of up-and-out call as a function of lambda

We have drawn the price envelope using lagrangian UVM as a function of the proportion $\lambda$. We notice first that the lambda which checks the last step of the Lagrangian UVM algorithm is approximately $\lambda = 0.7$: for the higher price we look for the lambda which minimizes the price and for the lower price it is the opposite.

We also note that the difference between the upper and lower price curves is larger for $\lambda = 0$ than for $\lambda = 0.7$. When $\lambda = 0$, we find the exact price with the UVM method. Consequently, the Lagrangian UVM method reduces the deviation of the price envelope compared to the UVM method.

### 5.3.3 UVM and Lagrangian methods with two hedging calls

In this previous case, we were only interested in a call as a hedging instrument. Now, we will deal with the case of two calls to hedge an up-and-out call, and we will plot the upper and lower value of the portfolio for the Lagrangian UVM and the classic UVM methods according to the value of the stock S, and compare them.

| $\lambda-$UVM for Up-and-out call | Value |
|---|---|
| $S_{min}$ | 0 |
| $S_{max}$ | 240 |
| T | 50/365 |
| K | 112 |
| $K^1_{hedging}$ | 210 |
| $K^2_{hedging}$ | 220 |
| $\sigma_{min}$ | 0.1 |
| $\sigma_{max}$ | 0.2 |
| r | 0.07 |
| N | 100 |
| M | 30 |

Table 12: Parameters of the up-and-out call's range $\lambda$-UVM-prices

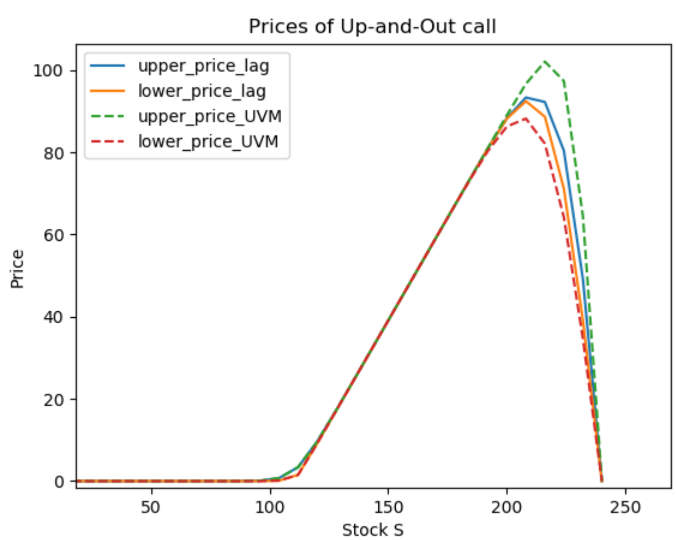


Figure 32: Prices of up-and-out call using Lagrangian UVM and classic UVM methods

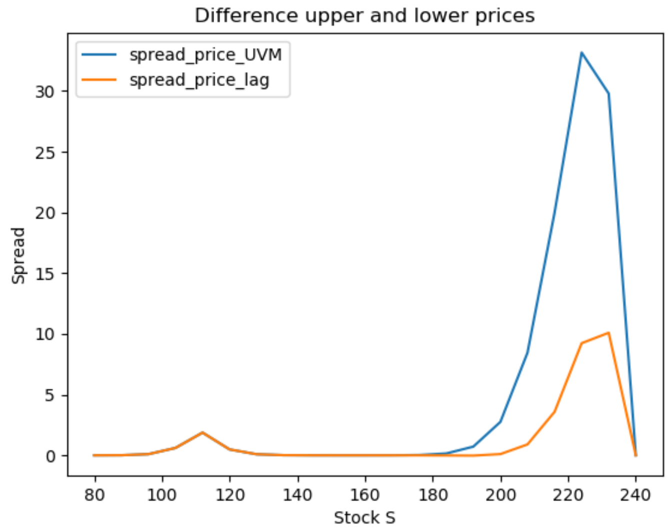


Figure 33: Difference between upper and lower prices using Lagrangian UVM and classic UVM methods

As stated with one hedging instrument, the Lagragian UVM method reduces the difference between the upper and lower price compared to UVM. Indeed, we have represented a barrier call, with the characteristics of the table above, which we have priced with two methods and using two calls as inputs in the case of $\lambda$-UVM. The latter method gives greater precision over the possible range of portfolio's values.

# Conclusion

In this study, we have presented the option pricing method using a volatility range. We determined the possible price range using two methods: the trinomial model and the finite-difference. With this first method, we priced the Apple vanilla options and we highlighted a number of results. It has been shown that the market price is indeed included in the price range given by the UVM method, almost surely. We have compared the classic BS model with the UVM model: we have shown that for a convex or concave price payoff UVM and BS give the same result, while for a mixed convexity option, it was pointed out that the two previous methods provide different results. One of the key results of the UVM method is risk diversification: we have underlined the property of portfolio price sub-additivity which indicates that the overall portfolio is less risky than the sum of the individual portfolios. We also introduced a variant of the UVM method called Lagrangian UVM, we implemented it with different hedging portfolios and we were able to show the importance and the impact of hedging portfolio on the price range. The last point discussed is the implementation of an alternative method to solve BSB equation: finite-difference method for the barrier options.

We have seen two approaches for option pricing using UVM: with trinomial tree and finite difference, there is also another more efficient method using Monte Carlo method especially if we are working with multiple assets.

# Python Script links

The python script corresponding to the UVM using trinomial method and finite-difference is given respectively by the google colab links:
Trinomial method: https://colab.research.google.com/drive/1LCcsn4dDTEzoh4_YK7EdLEhH4q7xF-e6?usp=sharing.
Finite-difference method: https://colab.research.google.com/drive/1_RB6aJb-hBA_LdpzXnjIkCxSFG2E-JZd?usp=sharing.

To access to the data used, we will find the links below:
For the Apple calls:
https://drive.google.com/file/d/1V56y34LAo6FKSXCaTBnBWjnKUsP_gVei/view?usp=sharing.
For the Apple puts:
https://drive.google.com/file/d/1Tlkt1Buj8gNU2BeY6r62Hphg_mhocVSk/view?usp=sharing.
And for the Apple straddles:
https://drive.google.com/file/d/1AlyCT0Jv8HUMIcUot99BL99zt4bZCx6d/view?usp=sharing.

# Acronyms

**BS** Black-Scholes equation. 7

**BSB** Black Scholes Barenblatt equation. 17

**cdf** cumulative distribution function. 13

**PDE** Partial Differential Equation. 11

**UVM** uncertain volatility model. 3

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