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Static and Dynamic Execution Strategies in the Presence of Liquidity Signals

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Thesis submitted as part of the requirements for the award of the MSc in Mathematics and Finance, Imperial College London, 2017-2018

Declaration

The work contained in this thesis is my own work unless otherwise stated.

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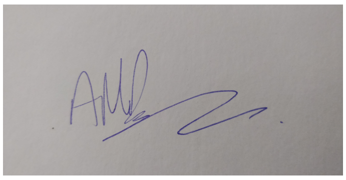


10/9/2018

Acknowledgements

I'd like to thank my project supervisor, Dr Eyal Neuman, for his continual guidance and support throughout this project. His tuition in new areas of mathematics has enabled me to learn things beyond the scope of the taught component of the MSc program and has undoubtedly shaped this research. He has provided numerous stimuli that have inspired the direction of this thesis and has provided many a valuable insight in our meetings, which have proven to be useful, rewarding and interesting in equal measure.

I would moreover like to thank the staff in the Mathematics and Finance department for both their teaching and industrial expertise, without which this project would not have been possible.

Abstract

Since its inception 20 years ago, optimal execution has come a long way with regards modelling and reducing the costs of trading. A recent development is the inclusion of liquidity imbalance signals to construct dynamically updating trading strategies. With this new frontier of strategies in mind, a reasonable question is to assess the extent to which they offer an advantage over the more classical, static strategies.

To do so requires the construction of the optimal static and dynamic strategies corresponding with those that exist already in the literature. As such we propose a static alternative to the dynamic, signal driven strategy in the instantaneous market impact framework as given by Neuman and Lehalle. Moreover we derive a heuristic, dynamic strategy corresponding to the static solution in the transient impact setting again given by Neuman and Lehalle.

Following this, we compare the static and dynamic strategies in both market impact frameworks, suggest improvements to the heuristic approach and outline a data driven methodology for optimal execution.

Contents

1 Introduction

Simple supply and demand dictates that the desire to *purchase* an item *increases* the price whereas the desire to *sell* an item *decreases* the price. Trading in a market causes the price to move against the investor and the amount the price moves is known as the *market impact*. The field of optimal execution seeks to manage precisely this: the adverse effects of supply and demand on asset price.

With financial institutions facing tighter margins and increased trade frequency, market impact is an essential consideration for profitability. This is most important in the case of large trades which have a larger adverse effect on the price. Broadly speaking the most common tactic to address the market impact for large, *parent* orders is to split them into smaller *child* orders and execute them over a longer period of time. The effect of this is two fold in that it does not broadcast the intent of an investor to the market and moreover allows for market impact to be shared across the trades, and hopefully reduced.

The way the parent order is broken down into child orders, and moreover the schedule that details the times that child orders are executed, constitutes the *trading strategy*. The trading strategy will be built around an underlying market impact model that dictates how the market moves in response to said trades amongst other things. Optimal execution concerns finding the optimal strategy given such a market model.

There are many things to consider when constructing such a market model including the underlying asset dynamics, the market impact and the class of *admissible strategies*.

We briefly outline the approaches taken in the literature concerning these facets before more rigorously introducing them in the context of our chosen model in section 2.

1.1 Asset Dynamics

Obviously one must model the underlying dynamics of an asset, however given the way that prices move in response to trading we must consider two different prices: the *unaffected* and *affected* price. The unaffected price, or colloquially the *mid-price*, is the price of the asset as observed by an investor who is *not* trading in the market.

The mid-price of an asset changes continuously in response to buy and sell orders placed by *other* market participants, however it is most common to model the aggregate effect of these involvements rather than the underlying buy and sell orders.

In both the Almgren-Chriss [1] and Bertsimas-Lo [2] frameworks the mid-price is modelled via an *arithmetic* Brownian motion (ABM). Alternatively in Gatheral and Schied's framework [3] they opt for a *geometric* Brownian motion (GBM), hoping to achieve more realistic results. The former argue that over short time horizons the simplicity afforded by ABM does not hinder the result in comparison to the more involved GBM. Brigo and Di Graziano [4] considered a mid-price that follows a displaced diffusion, however concluded this made little difference in practice.

With various asset dynamics producing similar results, more recently the trend has been to move away from aggregate modelling of other market participant's actions, to a more sophisticated approach and consider the underlying order flow. Notably in Cartea and Jaimungal [5] the mid-price is modelled via two order flow processes for buy and sell orders and attempts to capture an important facet of any market: Liquidity.

1.2 Liquidity

Prices in today's markets are established through a limit order book in which buy and sell orders are placed by market participants, thereby quantifying the supply and demand of the asset in question. Liquid assets are considered to have *deep* order books, with a large volume of bids and offers whereas illiquid assets are said to have a *thin* order book with fewer bids and offers.

A large acquisition or liquidation of a position in a deep market may have little effect in pushing the price away from the agent, however that same trade in a thin market could have drastically different effects. Needless to say, the effect of liquidity is of fundamental importance when considering the market impact of a trade. Moreover the *imbalance* in the order book, that is a surplus in buy or sell orders, could be a good indication as to the direction of future price movements.

In response to this, new frameworks attempt to incorporate this notion, including the aforementioned Cartea and Jaimungal [5] and the more recent Neuman and Lehalle [6]. In the latter a liquidity signal is directly incorporated into the drift term of the mid-price dynamics. We will introduce this idea more formally later and explore some of the implications of this modelling choice throughout the project.

1.3 Market Impact

The other component of asset price to consider is the affected or *execution* price. This price incorporates both the mid-price and the market impact and can be considered the price of an asset for an investor who *is* trading in the market.

Different types of market impact attempt to model different fundamental ideas about the order book. Temporary impact affects only the execution price and reflects the idea of *walking an elastic order book*, that is executing a trade that executes at progressively worse price points, only for the order book to be repopulated once the trade has finished. Permanent impact affects only the mid-price, and captures the notion that an individual can exert a pressure on price themselves by walking an *inelastic* order book, that is executing trades at progressively worse prices, with *no* re-population of orders, pushing the mid-price higher/lower.

There are various ways to model these two ideas, the most common of which is linear impact, with market impact parameters dictating the strength of their effect. Another, more realistic school of thought on the issue is that of Gatheral [11]: Transient impact. Transient market impact affects both the execution price and the mid-price of the asset, however in the latter case that effect decays over time. We will consider both temporary impact and transient impact in the remainder of the project and these are introduced more concretely in section 2.

1.4 Admissible Strategy

As with any optimisation problem there is a *feasible* set containing all permissible solutions, thereby specifying the constraints of the problem. Classic consideration for a strategy include

- Adaptability: the strategy relies only on information available at the current time
- Whether time is considered at discrete points as in [2] or continuously as in [1], [3]
- Continuity of inventory level, are instantaneous block trades allowed?
- Is meeting the acquisition/liquidation target a 'hard' constraint or is the investor willing to sacrifice completing the entire trade for the sake of a better execution price throughout?

- Is the strategy *static* or *dynamic*?

The last of these is something we will cover more thoroughly throughout this project and as such devote some attention to the previous literature surrounding it.

1.5 Static and Dynamic Strategies

One key property of a trading strategy is whether it is static or dynamic in nature. A static strategy is determined at time $t = 0$ and left to run without alteration until the end of the trading period. Contrastingly the dynamic strategy is constantly updated to reflect underlying asset price or more generally broader market conditions. Intuitively speaking, market observables should inform decisions of trading speed, for example faster trading in deep markets, and slower trading in thin ones.

Static strategies obviously require less involvement of the agent, however one could reasonably assume that the dynamic strategy allows for more efficient trading. To exemplify this consider the liquidation of a large position in a stock that has just tanked, the static strategy would continue to sell despite the low price since it has no new information about the stock in question. The dynamic strategy may observe this low price or some other market observable and sell slower with the view to sell more later when the price recovers.

At this point we take a brief look at how static and dynamic strategies arise in the existing literature and identify 2 main factors that influence this:

- An *a priori* assumption of a static strategy
- Choices in various components of the model, including asset dynamics, inventory penalties and time constraints

In the classic Almgren-Chriss model [1] an *a priori* assumption of a static strategy is used. Restricting the class of admissible strategies to exclusively static ones, allows a vast simplification in the stochastic control approach that is usually taken in optimal execution and instead we can appeal to the relatively simpler calculus of variations or dynamic programming methodology. In this case the optimal inventory to hold at time t, denoted by X_t, is given by

$$X_t = \frac{\sinh((T-t)\eta)}{\sinh(\eta T)} X_0, \quad 0 \leq t \leq T.$$

In some cases, even without the assumption of a static strategy, the respective optimal strategies can be found to be static, the simplest example of which being the Time Weighted Average Price (TWAP). In this model we assume no permanent market impact, linear temporary impact, strict time constraint and no inventory penalties. This time appealing to stochastic optimisation methodology, the optimal inventory is given by

$$X_t = \frac{T-t}{T} X_0, \quad 0 \leq t \leq T.$$

As can be seen, this strategy relies only on the fraction of the trading window that has elapsed and the initial position size. Even with the addition of inventory penalties and more involved market impact, the resulting strategy remains static (see [8, Chapter 6, page 139-151] for examples). This would seem to go against our intuition, since it is only reasonable that the way an agent trades is influenced by market observables. The reason for such disparity is most likely the result of an overly simplified model.

In the work of Gatheral and Schied [3], no assumption of a static strategy is assumed and the resultant optimal strategy is dynamic. In the work of Brigo and Piat [9], they considered the corresponding static strategy, that is the optimal strategy under the *a priori* assumption that said strategy is static. As discussed above, one would expect that the added computational effort of updating dynamic strategies would give benefits over their deterministic counterparts, however, their work showed little difference in the associated costs, for reasonable model parameters.

Again this minimal difference is likely due to an overly simplified model and as such we consider a more involved market impact framework from Neumann and Lehalle [6] who modify the asset price process to include a liquidity signal $(I_t)_{t \geq 0}$. In this situation the resulting optimal strategy (given in terms of trading speed r_t) is indeed dynamic (note the clear dependence on the current level of the liquidity signal I_t)

$$r_t = -\frac{1}{2\eta} \left(2\bar{v}(t) X_t + I_t \int_t^T \exp\left(-\gamma(s-t) + \frac{1}{\eta} \int_t^s \bar{v}(u) du \right) ds \right).$$

Following the ideas of Brigo and Piat a reasonable question is whether this provides any benefit over the corresponding static strategy.

1.6 Structure

The structure of the remainder of the project is as follows. In section 2 we more formally introduce the ideas covered above as they apply to our model. The model in question is based on that of Neuman and Lehalle [6], and we will explore the effect of the underlying modelling choices with regards both the type of impact and the admissible strategy. In section 3 we consider the case of temporary market impact and derive the optimal *static* strategy and compare it to the dynamic strategy given in [6]. In section 4 we consider the transient impact setting and outline a methodology for constructing a dynamic strategy. We compare this to the static strategy in [6] and moreover offer an alternative methodology for the construction of the dynamic strategy using the more classical approach of stochastic optimisation. In section 5 we suggest some further research including two suggestions for evaluating the alternative dynamic strategy and moreover, briefly outline a methodology for a more data driven approach to optimal execution.

2 Model

The main objective of optimal execution is to find the optimal trading speed, and subsequently the optimal *inventory trajectory*, that minimises (maximises) some notion of cost (revenues) to the trader. Typically this cost will encompass 3 things: the asset dynamics, the market impact and the risk of holding a position for a given period of time. The *trading window* is the period of time in which the trade occurs and is defined in terms of a terminal time T, as $[0, T]$. We will be working in the continuous time framework that is usually adopted (with the notable exception of Bertsimas-Lo [2]).

2.1 Asset Dynamics

A central idea to optimal execution is that of market impact, which is captured through the difference between the mid-price $(P_t)_{t\geq 0}$, and execution price $(S_t)_{t\geq 0}$. In many classic works, mid-price dynamics are modelled by a driftless arithmetic Brownian motion

$$dP_t = \sigma_p dW_t. \tag{1}$$

Following the work in [6] we introduce a liquidity signal I_t which drives the drift of the mid-price dynamics. Here instead of (1), the unaffected mid price P_t is given by

$$dP_t = I_t dt + \sigma_p dW_t, \tag{2}$$

where W is a standard Brownian motion. The signal follows Ornstein-Uhlenbeck dynamics,

$$dI_t = -\gamma I_t dt + \sigma dB_t \quad I_0 = \iota, \tag{3}$$

where again B is a standard Brownian motion, independent of the Brownian motion in the price process. To fully describe the execution price we must first introduce two important state variables: inventory and trading rate.

2.2 Inventory and Trading Rate

Let X_t denote the inventory of the agent at time t, with a position of size x to liquidate or acquire over the time horizon $[0, T]$. Let r_t denote the rate of execution at time t, that is the speed of liquidation or acquisition. As one would expect, inventory and execution speed are related via

$$dX_t = \pm r_t dt. \tag{4}$$

Remark 2.1. By convention we adopt the idea that a positive execution rate corresponds to either selling with a liquidation target or buying with an acquisition target. Contrastingly, a negative execution rate is synonymous with purchasing an asset when one has an overall liquidation target or *vice versa*. It may seem counter-intuitive to raise this point; surely no optimal strategy would purchase the stock it is attempting to liquidate. This however proves not be the case as can be seen in sections 3 and 4.

In light of this, in order to make sure the inventory is correctly expressed in terms of the execution speed, we note the plus/minus in (4) which corresponds to the cases of the acquisition and liquidation problem respectively. Similarly we have a case distinction for the initial inventory level of both problems, that is,

$$X_0 = \begin{cases} x & \text{liquidation problem} \\ 0 & \text{acquisition problem} \end{cases}.$$

For the remainder of the project we will exclusively focus on the liquidation case for the sake of discussion however the methods outlined work for acquisition with minimal revision. $\diamond$

2.3 Market Impact

Here we introduce the two types of market impact we will consider in sections 3 and 4 respectively: temporary impact and transient impact.

2.3.1 Temporary Impact

Temporary impact can be seen as walking an order book that is immediately repopulated, and as such affects only the execution price, but not the mid-price, in the following way,

$$S_t = P_t - f(r_t).$$

The functions f is the *temporary* market impact function and is a function of the execution rate r_t as introduced in [1] and [10]. In this project we adopt the convention that temporary impact is *linear*, that is $f(r_t) = \eta r_t$. As such the execution price is given by

$$S_t = P_t - \eta r_t, \quad \eta \geq 0, \tag{5}$$

where η is the *(temporary) market impact parameter.*

Note how f maps from $\mathbb{R}^+ \mapsto \mathbb{R}^+$ and $\mathbb{R}^- \mapsto \mathbb{R}^-$ to appropriately reflect the adverse effects of supply and demand as mentioned in the introduction. In the case $r_t \in \mathbb{R}^+$ (that is selling with a liquidation target) the execution price is pushed *below* the mid-price, as is expected. In the case $r_t \in \mathbb{R}^-$ (that is we are acquiring despite an overall liquidation target) the execution price is pushed *above* the mid-price.

Remark 2.2 (Permanent Impact)**.** A natural counterpart to temporary impact is permanent impact as introduced in [10]. Permanent impact captures the long term affect of continually walking an order book applying an upward or downward pressure, and as such effects only the mid-price as follows

$$dP_t = g(r_t)dt + \sigma_p dW_t,$$

where W is a standard Brownian Motion, and g is the *permanent* impact function. We will not consider permanent impact in this project. ◇

2.3.2 Transient Impact

It should be noted there is a broader class of market impact, *Transient impact* that encompasses the ideas of both temporary and permanent impact. As described in [11], in a transient market the execution price is given by

$$S_t = P_t + \int_{\{s<t\}} G(t-s)dX_s \tag{6}$$

instead of (5), where G the *decay kernel* and P_t is the mid-price, governed by (2).

Note in the transient setting, permanent and temporary impact are not separate quantities, instead they are concepts captured more deftly in the decay kernel. The instantaneous impact captured by the shift from mid-price, the permanent impact captured by the integral over previously elapsed time to incorporate the effects of previous trades. Empirical studies, specifically those of Potter and Bouchaud [12], show that market impact is *quasi-permanent*, that is, trading has a lasting effect on price level. This is precisely the notion that is captured by the transient model. Moreover transient models can in fact reduce to exclusively temporary or permanent impact models when the decay kernel G is singular or 1 respectively.

Despite transient impact begin more realistic, temporary impact models are often considered for their simplicity and use as a benchmark.

2.4 Risk Metric

Aside from price, another thing to be considered when finding an optimal strategy is the risk the agent is exposed to over the course of the trading window $[0, T]$. Although it may be beneficial to trade slowly to reduce price impact, this requires the agent to hold the asset for longer or hold off on a prompt acquisition. This is especially problematic given the agent's desire to act quickly to either offload a bad position or acquire alpha.

To address this issue of incorporating risk into the selection of a trading strategy, we introduce an *inventory penalty.* There are a variety of risk criteria used in the literature including that proposed by Gatheral and Schied [3], the quadratic variation inspired term of Forsyth et al [13] and the original variance term of Almgren [1]. We will take the latter and more classical approach and use the following variance inspired risk criterion

$$\phi \int_0^t X_s^2 ds, \quad \phi \geq 0, \tag{7}$$

where ϕ is is the agent's *risk aversion.*

2.5 Performance Criteria

Having introduced the inventory, execution speed and price processes, we can construct the agent's *cash process* $(C_t)_{t\leq 0}$ which satisfies

$$dC_t = S_t r_t dt. \tag{8}$$

Remark 2.3. We could equivalently write

$$C_t = C_0 + \int_0^t S_s r_s ds. \tag{9}$$

Since we are dealing with the liquidation problem, cash is a positive quantity, to reflect the positive revenues of the trade. Moreover we often assume $C_0 = 0$ without loss of generality. ◇

We can now construct the *performance criteria* to be maximised. Since we are in the liquidation case, we want to maximise the agent's cash and minimise the risk. As such, we consider the negative of the inventory penalty, and consider the expected *risk adjusted revenues*, that is

$$\mathbb{E}\left[C_T - \phi \int_0^T X_t^2 dt\right] \tag{10}$$

The *value function* is the risk adjusted revenues for the optimal strategy, given by

$$\sup_{\Xi} \mathbb{E}\left[C_T - \phi \int_0^T X_t^2 dt\right] \tag{11}$$

where Ξ is the set of admissible strategies, explained in the following section 2.6.

2.6 Admissible strategy

The notion of an admissible strategy is the modelling choice we will explore the most and this section outlines the cases we will consider. All definitions of admissible strategy in this project share two common traits as given by Neuman and Lehalle [6] amongst others.

1. X_t adapted and left continuous
2. X_t has finite and $\mathbb{P}$-a.s. bounded total variation

However, depending on the rest of the modelling choices, the notion of admissible strategy must also change to accommodate the methodologies employed in finding the optimal strategies. As such we consider the two cases of temporary and transient impact and within each of these the dynamic and static strategies.

2.6.1 Admissibility in Temporary impact case

As briefly mentioned in the introduction, one consideration of an admissible strategy is whether complete liquidation is a 'hard' or 'soft' constraint. In the case of a hard constraint we impose another condition on the set of admissible strategies, namely

$$X_T = 0.$$

Alternatively, in the soft constraint setting, we can relax this condition and instead amend the performance criteria to include a *terminal penalty term*. The penalty term of choice is $X_T(P_T - \rho X_T)$ which penalises holding shares past time T. $X_T P_T$ corresponds to a market order of the remaining X_T shares at mid-price at time T. $-\rho X_T^2$ reflects a penalty and is controlled by parameter $\rho > 0$. With this in mind the alternative performance criteria is given by

$$\mathbb{E}\left[C_T - \phi \int_0^T X_t^2 dt + \underbrace{X_T(P_T - \rho X_T)}_{\text{terminal penalty}}\right] \tag{12}$$

In this setting the strategy need not ensure the liquidation of the full position by time T, instead this is simply discouraged by the penalty term according to the parameter ρ.

In the derivation of the optimal strategy in [6] the latter convention of a soft liquidation constraint is adopted and the resulting strategy is dynamic as can be seen in equation (18).

In order to derive the corresponding static strategy (see section 3) we amend the definition of admissible strategy to be moreover *deterministic* which reduces the problem to one which can be solved via calculus of variations.

As is common in calculus of variation one is required to solve a second order ordinary differential equation, and as such requires initial and terminal conditions. In order to do this we remove the penalty term from the cost functional and instead impose a hard fuel constraint on the strategy, that is

$$X_T = 0 \quad \text{and} \quad X_0 = x. \tag{13}$$

Remark 2.4. As we intend to compare the performance of the static and dynamic strategies, it is reasonable to wonder if this comparison is valid, given the difference in terminal inventory constraints. In fact this *is* a valid alternative, as we can consider the asymptotic properties of the dynamic strategy as $\rho \to \infty$ as in remark 3.5 of [6, Remark 3.5, page 15]. As such, we have effectively imposed a hard terminal inventory constraint in the dynamic setting and consequently the two strategies are indeed comparable. ◇

Another key component of the admissible strategy is that block trades are *not* allowed, that is the inventory process $(X_t)_{t\geq 0}$ is assumed continuous in time. This allows us to make use of the standard integration by parts formula which is used in the derivation of the static strategy.

2.6.2 Admissibility in Transient Impact case

In the case of transient market impact, for both the static and dynamic strategies, we take the following definition of an admissible strategy.

1. X_t adapted and left continuous
2. X_t has finite and $\mathbb{P}$-a.s. bounded total variation
3. $X_T = 0$ and $X_0 = x$

In the static case we also make an *a priori* assumption that $(X_t)_{t\geq 0}$ is deterministic. This is of vital importance as it provides us with an important result (Theorem 4.3) that is utilised in the derivation of the optimal strategy. For the dynamic counterpart we partition the trading window, on which each interval the *'sub-strategy'* is deterministic. The dynamic nature of the strategy arises from the fact that each sub-strategy is based on random quantities unknown at time 0, and is updated on the fly. This concept is outlined more concretely in section 4.2

Moreover, unlike the temporary case, we *do* allow for discontinuities in the inventory trajectory, allowing for block trades to occur. As such we will have to amend our notion of the cash process to allow for this, which can be seen later in section 4.

Notation 2.5. Throughout the remainder of the project we will abuse notation slightly and use Ξ to denote the set of admissible strategies allowing context to dictate which of the above definitions is being employed. ◇

3 Temporary Market Impact

3.1 Static Strategy

Under the assumption of a deterministic strategy as laid out in section 2.6.1 we can find the optimal inventory trajectory in the case of temporary market impact.

Theorem 3.1. *The deterministic trajectory* $X_t^* \in \Xi$ *that maximises* (11) *is given by*

$$X_t^* = (x - C)\frac{\cosh(kt)\sinh(kT) - \cosh(kT)\sinh(kt)}{\sinh(kT)} + C\mathrm{e}^{-\gamma t}\left(1 - \mathrm{e}^{-\gamma(T-t)}\frac{\sinh(kt)}{\sinh(kT)}\right)$$

$$0 \leq t \leq T$$

Proof. Under linear temporary impact we can write (9) as

$$C_t = \int_0^t S_s r_s ds = \int_0^t P_s r_s ds - \eta \int_0^t r_t^2 ds.$$

Integration by parts and substitution of asset dynamics (2) gives

$$\int_0^t P_s r_s ds = P_0 x - P_t X_t + \int_0^t X_s dP_s = P_0 x - P_t X_t + \int_0^t X_s I_s ds + \sigma_p \int_0^t X_s dW_s,$$

to give the following expression for the investors cash at the final time T:

$$C_T = P_0 x - P_T X_T + \int_0^T X_t I_t dt + \sigma_p \int_0^T X_t dW_t - \eta \int_0^T \dot{X}_t^2 dt.$$

Note for the sake of a parsimonious representation we will use $\dot{X}$ instead of r in this case. Under the assumption of X_t^* deterministic and moreover $X_T = 0$, the risk adjusted revenues simplify significantly.

$$\begin{aligned}
\mathbb{E}\left[C_T - \phi \int_0^T X_t^2 dt\right] &= \mathbb{E}\left[P_0 x - P_T X_T + \int_0^T X_t I_t dt + \sigma_p \int_0^T X_t dW_t \right. \\
&\qquad \left. -\eta \int_0^T \dot{X}_t^2 dt - \phi \int_0^T X_t^2 dt\right] \\
&= P_0 x + \int_0^T X_s \mathbb{E}\left[I_t\right] dt + \mathbb{E}\left[\sigma_p \int_0^T X_t dW_t\right] \\
&\quad - \eta \int_0^T \dot{X}_t^2 dt - \phi \int_0^T X_t^2 dt \\
&= \underbrace{P_0 x}_{\substack{\text{Cost without} \\ \text{impact}}} + \underbrace{\int_0^T X_t \mathbb{E}\left[I_t\right] dt}_{\text{Signal impact}} - \underbrace{\eta \int_0^T \dot{X}_t^2 dt}_{\substack{\text{Temporary} \\ \text{impact}}} - \underbrace{\phi \int_0^T X_t^2 dt}_{\text{Risk aversion}}
\end{aligned} \tag{14}$$

Since I_t has Ornstein Uhlenbeck dynamics one can evaluate $\mathbb{E}\left[I_t\right]$, moreover removing the constant $P_0 x$ term reduces our objective functional to

$$\iota \int_0^T X_t \mathrm{e}^{-\gamma t} dt - \eta \int_0^T \dot{X}_t^2 dt - \phi \int_0^T X_t^2 dt.$$

Appealing to a Calculus of Variation methodology, let $h(t)$ be a perturbation of the path X_t, such that $h(0) = h(T) = 0$ and define both

$$\begin{aligned}
X_t^\epsilon &:= X_t + \epsilon h(t) \\
H(\epsilon) &:= -\int_0^T \left(\phi (X_t^\epsilon)^2 - \iota X_t^\epsilon \mathrm{e}^{-\gamma t} + \eta (\dot{X}^\epsilon)_t^2\right) dt.
\end{aligned}$$

Differentiating H with respect to ϵ, and evaluating at $\epsilon = 0$ gives

$$\frac{dH}{d\epsilon} = -\int_0^T \left(2\phi (X_t + \epsilon h(t)) h(t) - \iota h(t) \mathrm{e}^{-\gamma t} + 2\eta (\dot{X}_t + \epsilon \dot{h}(t)) \dot{h}(t)\right) dt,$$

$$\left.\frac{dH}{d\epsilon}\right|_{\epsilon=0} = -\int_0^T \left(2\phi X_t h(t) - \iota h(t) \mathrm{e}^{-\gamma t} + 2\eta \dot{X}_t \dot{h}(t)\right) dt. \tag{15}$$

Noting, by straightforward application of integration by parts and $h(T) = h(0) = 0$

$$\int_0^T \dot{X}_t \dot{h}(t) dt = \left.\dot{X}_t h(t)\right|_0^T - \int_0^T \ddot{X}_t h(t) dt = -\int_0^T \ddot{X} h(t) dt,$$

we can then write (15) as follows

$$\left.\frac{dH}{d\epsilon}\right|_{\epsilon=0} = -\int_0^T \left(2\phi X_t - \iota e^{-\gamma t} - 2\eta \ddot{X}_t\right) h(t)dt.$$

The optimal path is given by $\left.\frac{dH}{d\epsilon}\right|_{\epsilon=0} = 0$ and since h is arbitrary it must be the case that X_t^* satisfies the following inhomogeneous second order ODE

$$\ddot{X}_t - k^2 X_t = -\frac{\iota}{2\eta} e^{-\gamma t} \quad k := \sqrt{\frac{\phi}{\eta}} \tag{16}$$

Solving the associated homogeneous ODE, gives a complimentary solution

$$X_t^c = A\cosh(kt) + B\sinh(kt).$$

Moreover making an ansatz at the particular solution of $X_t^p = Ce^{-\gamma t}$ we find $C = \frac{\iota}{2\eta(k^2-\gamma^2)}$ and as such

$$X_t^* = A\cosh(kt) + B\sinh(kt) + Ce^{-\gamma t}$$

Using the conditions of an admissible strategy as outlined in section 2.6.1, namely $X_T = 0$ and $X_0 = x$, we can solve for A and B to give

$$\begin{aligned} A &= x - C \\ B &= \frac{\cosh(kT)(C-x) - Ce^{-\gamma T}}{\sinh(kT)}. \end{aligned}$$

Hence the optimal strategy X_t is given by

$$X_t^* = x\frac{\sinh(k(T-t))}{\sinh(kT)} + Ce^{-\gamma t}\left(1 - \frac{1}{\sinh(kT)}\left(e^{-\gamma(T-t)}\sinh(kt) + e^{\gamma t}\sinh(k(T-t))\right)\right)$$

□

Using the parameter values given in table 1, we plot some example inventory trajectories for varying values of the initial signal as can be seen in figure 1. As one would expect the negative signal, indicative of a negative pressure in the order book and potential predictor of a falling stock, prompts the investor to liquidate quicker to avoid future losses. Similarly for a positive initial signal, the investor is prompted to sell slower with the hope of being able to liquidate more at a higher price point in the future.

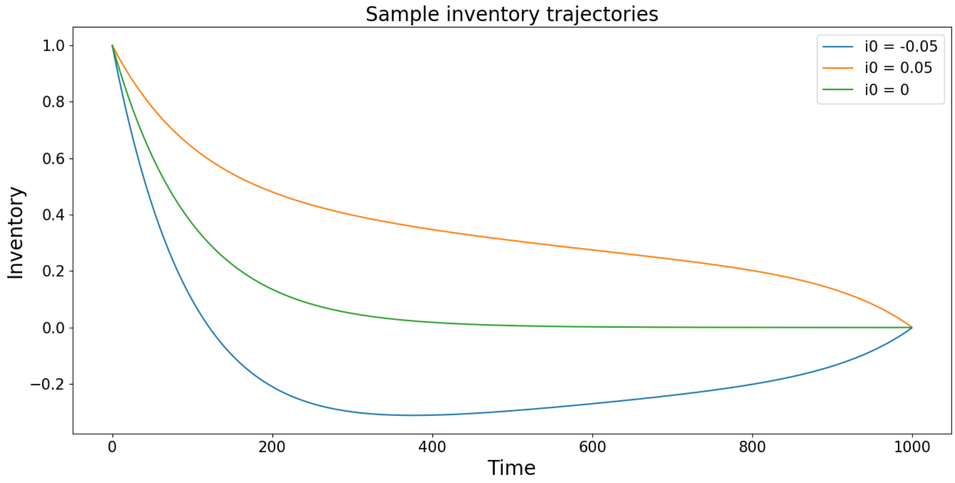


Figure 1: Trajectories of the deterministic strategy in the temporary setting for varying initial signal values, namely $\iota = 0$ (green), $\iota = 0.05$ (orange) and $\iota = -0.05$ (blue).

γ	0.1	σ_p	0.3
θ	0	x	1
σ	0.01	η	0.05
ι	0	ϕ	0.05
P_0	100	T	10

Table 1: Default parameter values as used throughout the remainder of this work

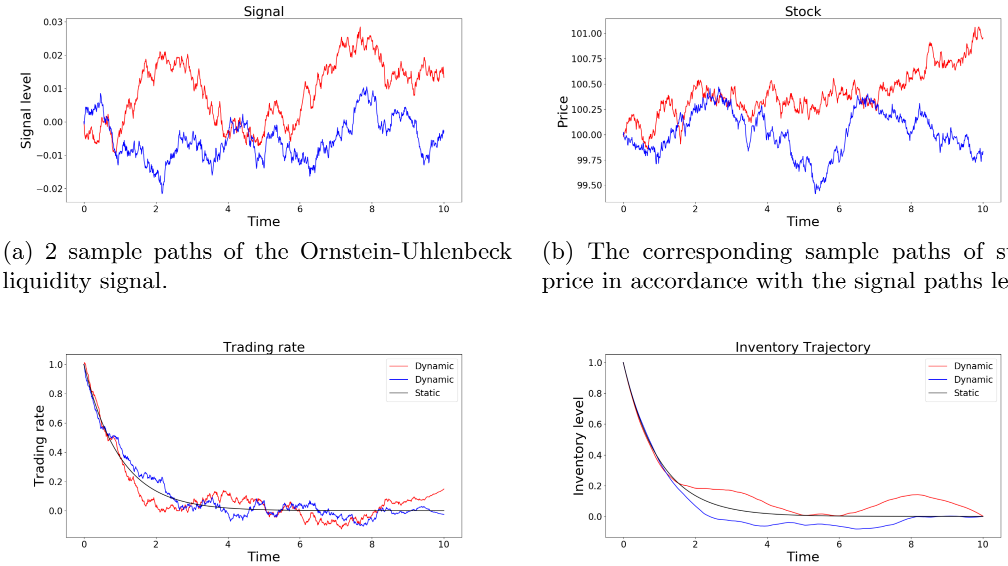
Remark 3.2. For $\iota = 0$ the optimal solution corresponds with Almgren-Chriss solution, namely

$$X_t^* = x\frac{\sinh(k(T-t))}{\sinh(kT)}, \quad 0 \le t \le T,$$

with k defined as in (16). Since this strategy is static, any new information with regards the liquidity will not affect the trading strategy. Moreover since the initial view of the liquidity signal is 0, the agent assumes no upward or downward market pressure for the remainder of the trading session, and as such the mid-price is assumed to be driven only by the Brownian Motion, hence reducing to the classic Almgren-Chriss model. ⋄

Using an intermediary result of the above proof (namely (14)) we can write the following and hence evaluate the corresponding expected risk adjusted revenues of a the optimal strategy.

$$\sup_{\Xi} \mathbb{E}\left[C_T - \phi \int_0^T X_t^2 dt\right] = P_0 x + \iota \int_0^T X_t^* e^{-\gamma t} dt - \eta \int_0^T (\dot{X}_t^*)^2 dt - \phi \int_0^T (X_t^*)^2 dt \quad (17)$$



(a) 2 sample paths of the Ornstein-Uhlenbeck liquidity signal.

(b) The corresponding sample paths of stock price in accordance with the signal paths left.

(c) Trading rates corresponding to the 2 sample paths above (red and blue) and corresponding static trading rate (black)

(d) Inventory trajectories corresponding to the 2 sample paths above (red and blue) and corresponding static trajectory (black)

Figure 2: 2 sample simulations of the signal, corresponding stock path and subsequent optimal trading rates and inventories (red and blue) and corresponding static strategy (black).

3.2 Dynamic Strategy

Dropping the strict fuel constraint, adding in a penalty function and appealing to the asymptotic nature of the solution as outlined in section 2.6.1 (and justified in remark 2.4), the corresponding adapted strategy as given in [6] is as follows,

$$r_t = -\frac{1}{2\eta}\left(2\bar{v}_2(t)X_t + I_t\int_t^T \exp\left(-\gamma(s-t) + \frac{1}{\eta}\int_t^s \bar{v}(u)du\right)ds\right). \tag{18}$$

where $\bar{v}_2$ is given by

$$\bar{v}_2(t) = \sqrt{\eta\phi}\frac{1+\mathrm{e}^{2k(T-t)}}{1-\mathrm{e}^{2k(T-t)}},$$

and k is as in (16).

One can see in figure 2 the dynamic trading strategies (2c) and corresponding inventory trajectories (2d) for 2 sample paths of the liquidity signal (2a) and corresponding asset (2b). Moreover the static trajectory with the same parameters is included for reference. There are notably similarities with the static strategy regarding the response to the value of the signal. Namely, a positive signal encourages a slower liquidation, with the aim of

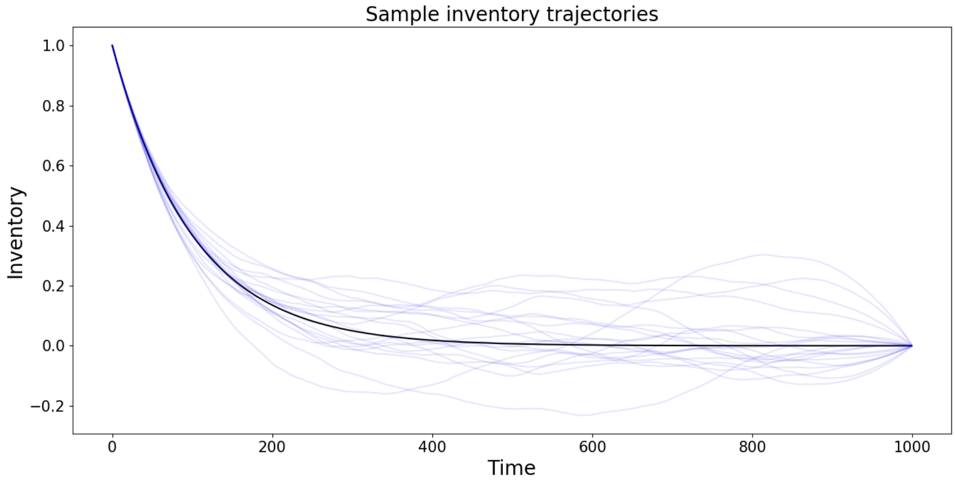


Figure 3: Sample of 20 optimal inventory trajectories as specified by the dynamic strategy (blue) and the corresponding static inventory trajectory (black).

selling at a higher price and a negative signal encourages faster selling, to avoid liquidation at a lower price. The key difference being the strategy's ability to *continuously* observe the signal and thus adjust the trading rate continuously as opposed to just once.

Something else of note that can be observed most clearly in figure 2d is the acquisition of the asset in the face of a liquidation target. Although this behaviour can be observed in the classic, static Almgren-Chriss model, it is usually the result of excessive risk aversion that causes rapid trading. By virtue of the smoothness of the solution, it sells too much and 'overshoots', acquiring a net short position before having to consequently buy back. An example of this can be seen in Brigo and Piat's work [9, Figures 23-24, pages 44-45], albeit within a different modelling framework.

In the dynamic approach described by (18) the acquisition is crucially not a corrective measure to address an overshoot, but a direct response to the signal level. This is the key difference between the static and dynamic strategies, and in what follows we attempt to assess how beneficial this phenomenon is.

By modifying proposition 3.1 of [6, Proposition 3.1, page 14] similarly to above, that is considering the limit as ρ goes to infinity, we have an expression for the associated expected risk adjusted revenues of the strategy.

$$
\begin{aligned}
V(t,\iota,c,x,p) &= c + xp + v_0(t,\iota) + xv_1(t,\iota) + x^2 v_2(t,\iota) \\
\bar{v}_1(t,\iota) &:= \frac{\iota}{\mathrm{e}^{k(T-t)} - \mathrm{e}^{-k(T-t)}} \left(\mathrm{e}^{k(T-t)} \frac{1-\mathrm{e}^{-(\gamma+k)(T-t)}}{\gamma+k} + \mathrm{e}^{-k(T-t)} \frac{1-\mathrm{e}^{-(\gamma-k)(T-t)}}{\gamma-k} \right) \\
\bar{v}_0(t,\iota) &:= \frac{1}{4\eta} \int_t^T \mathbb{E}_t \left[\bar{v}_1(s, I_s) \right] ds.
\end{aligned} \tag{19}
$$

where $\bar{v}_2$ is defined as in (18). Here ι, c, x, p are the values of the signal, cash, inventory and mid-price processes respectively at time t. In our analysis we will focus on the expected revenues at time $t = 0$ and since we are dealing with the liquidation problem we take c to be 0. Adjusting (19) in accordance with this, allows us to compare the associated time 0 expected revenues of both the static and dynamic strategies.

3.3 Comparison of Static and Dynamic Strategy

The dynamic strategy and corresponding inventory trajectory can be seen as a perturbation of the static (and in this case Almgren-Chriss per remark 3.2) solution as can be seen more clearly in figure 3. We return to this idea later in section 5 in a short proposal for a data driven approach to optimal execution.

In order to assess whether the dynamic strategy performs better than the static one we can compare the value functions for a range of parameter values.

In figure 4 we see the expected risk adjusted revenues for the static and dynamic strategies, as specified by (17) and (19) respectively, as a function of the terminal time T. Over shorter time horizons the static and dynamic strategies have similar risk adjusted revenues since the requirement to liquidate is a high priority and thus the limiting factor of the strategy. However in the case of longer time horizons the risk adjusted revenues of the dynamic strategy continue to rise where the static one stays constant. This difference is precisely down to the access the dynamic strategy has to incoming market information via the liquidity signal which allows for increased revenues. With time no longer a limiting factor, the dynamic strategy can afford to make directional plays in the market. Conversely with no new information in the static case the optimal course of action is to avoid making directional plays in the market.

In figure 5a, we see the risk adjusted revenues are quadratic in the position size, which

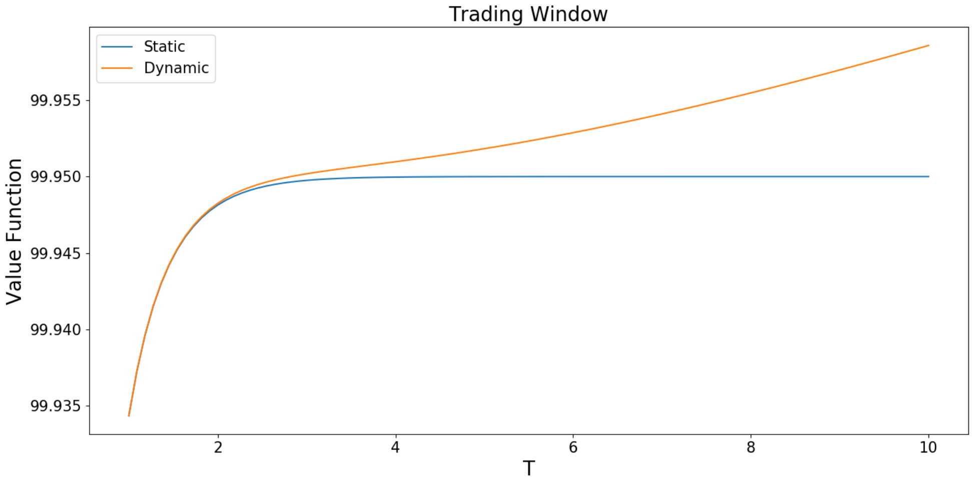
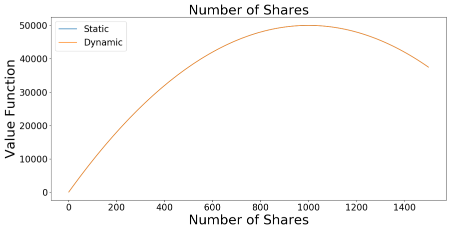


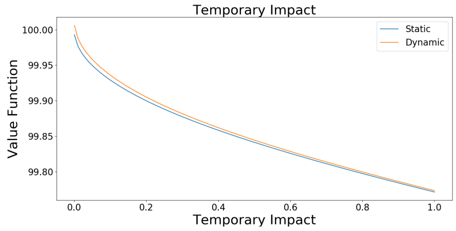
Figure 4: Time 0, expected, risk adjusted revenues as a function of the size of trading window T for the static (blue) and dynamic (orange) strategies

is unsurprising in the dynamic case given (19). In fact, we see this is also the case for the corresponding static strategy. For extremely large positions the risk term dominates the value function and as such the adjusted revenues are large and negative to reflect the extreme risk of such large positions.

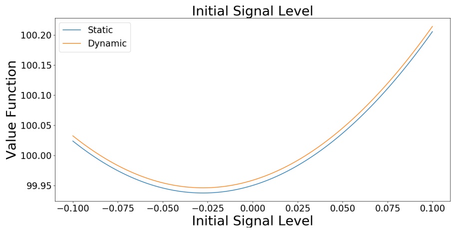
Figures 5b - 5d display the effect of a varying the temporary impact parameter η, initial signal level ι and risk aversion parameter ϕ. We observe only a marginal difference in the risk adjusted revenues in the static and dynamic case, however the dynamic strategies do yield slightly better results as is to be expected when optimising over a larger search space. In figure 5e, for $\iota = 0$ the dynamic strategy has no dependence on the parameter of mean reversion, γ, since these two parameters appear exclusively together. In this case, low mean reversion of the signal is more beneficial to the trader, since there is more time to execute in accordance with prevailing market sentiment. With faster mean reversion the drift of the stock price is more often closer to 0 and as such the dynamic strategy gains less information, making it more comparable to the static strategy. For $\iota \neq 0$ we observe a similar situation to above, with the dynamic strategy only showing marginal improvements over the static counterpart.



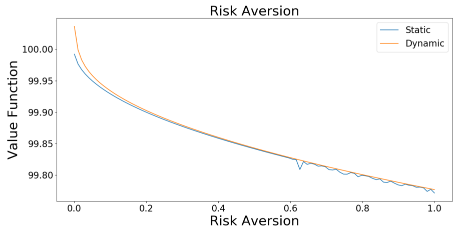
(a) Risk adjusted revenues as a function of size of position size



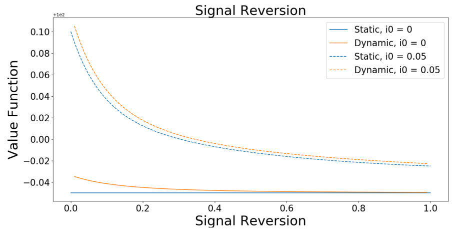
(b) Risk adjusted revenues as a function of market impact parameter



(c) Risk adjusted revenues as a function of intial signal value



(d) Risk adjusted revenues as a function of risk aversion



(e) Risk adjusted revenues as a function of signal reversion speed for 2 different values of ι. The solid lines correspond to $\iota = 0$ and the dashed lines corresponding to $\iota = 0.05$.

Figure 5: A comparison of the time 0, expected risk adjusted revenues of the static (blue) and dynamic (orange) strategies over varying parameter values in accordance with (17) and (19).

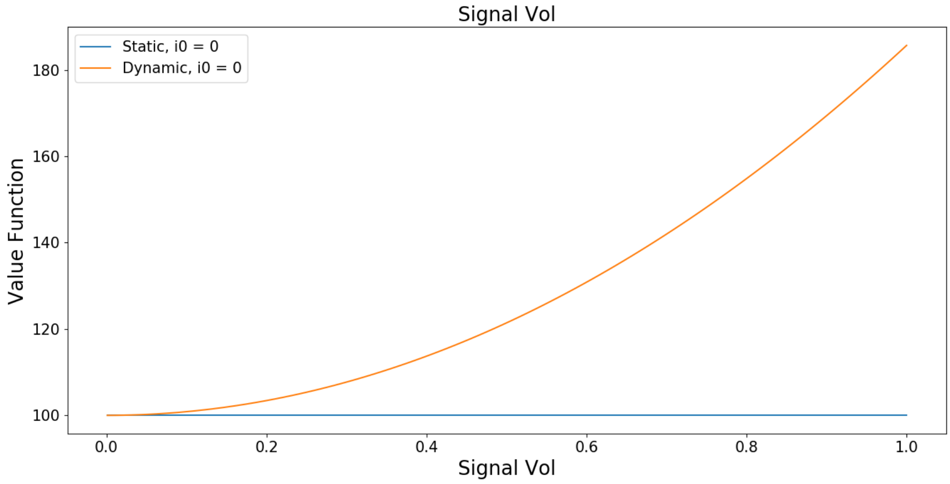
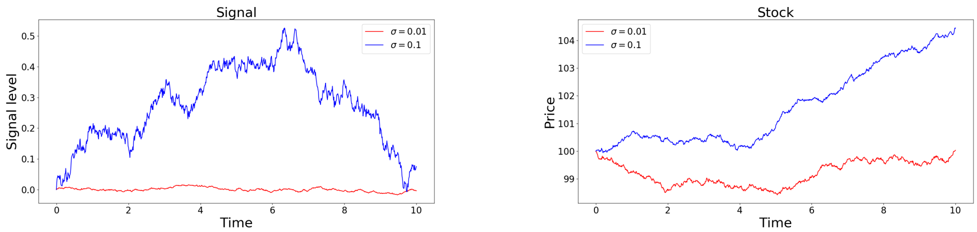


Figure 6: Time 0, expected, risk adjusted revenues as a function of the size of signal volatility T for the static (blue) and dynamic (orange) strategies



(a) Simulations of liquidity signal

(b) Corresponding simulations of asset

Figure 7: Simulation of a signal, and the corresponding stock, for different signal volatilities

In figure 6 the dynamic value function appears most different to its static counterpart since the static strategy is independent of the volatility of the signal. In situations where the liquidity signal is very volatile, the new market information becomes more valuable in deciding how best to trade. This is down to the relationship between the signal volatility and the stock price which can be observed in figure 7. Larger signal volatility for fixed mean reversion is indicative of a signal of greater magnitude, and as such the stock price is more heavily driven by the liquidity signal. In these situations where a more obvious relationship between signal and stock price exists, observing said signal will of course be very valuable to a trader in terms of their potential to increase risk adjusted revenues. Contrastingly, with low volatility, the signal has little effect on the stock price and as such, the access to the signal is seen to be of little benefit to the dynamic strategy.

4 Transient market impact

Recall equation (6), which details the execution price within the transient market impact setting:

$$S_t = P_t + \int_0^t G(t-s)dX_s.$$

In the above work in the temporary market impact setting we have assumed the strategies to be continuous in time, we now however allow for discontinuities in the trading trajectory or *block trades*. With this in mind we have a new expression for the revenues associated with a trading strategy X given by

$$\int_0^T S_t dX_t + \frac{G(0)}{2} \sum (\Delta X_t)^2.$$

Using the same mid-price dynamics as in the temporary market impact case (section 3) we can subsequently write the cost of the strategy in terms of the signal and decay kernel.

$$\begin{aligned}
\int_0^T S_t dX_t + \frac{G(0)}{2} \sum (\Delta X_t)^2 &= \int_0^T P_t dX_t + \int_0^T \int_0^t G(t-s)dX_s dX_t + \frac{G(0)}{2} \sum (\Delta X_t)^2 \\
&= \int_0^T \left(P_0 + \int_0^t I_s ds + \sigma_p \int_0^t dW_t \right) dX_t \\
&\quad + \int_0^T \int_0^t G(t-s)dX_s dX_t + \frac{G(0)}{2} \sum (\Delta X_t)^2 \\
&= \int_0^T \int_0^t I_s ds dX_t + \sigma_p \int_0^T \int_0^t dW_t dX_t \\
&\quad + \int_0^T \int_0^t G(t-s)dX_s dX_t + \frac{G(0)}{2} \sum (\Delta X_t)^2 - P_0 X_0
\end{aligned}$$

Where the final equality uses the hard fuel constraint, $X_T = 0$ as specified in section 2.6. Using the above and Lemma 2.3 of [14, Lemma 2.3, page 6], we can write the expected cost of the strategy as

$$\begin{aligned}\mathbb{E}_0\left[\int_0^T S_t dX_t + \frac{G(0)}{2}\sum(\Delta X_t)^2\right] &= \mathbb{E}_0\left[\int_0^T\int_0^t I_s ds dX_t + \sigma_p\int_0^T\int_0^t dW_t dX_t - P_0X_0\right.\\ &\quad\left.+\int_0^T\int_0^t G(t-s)dX_s dX_t + \frac{G(0)}{2}\sum(\Delta X_t)^2\right]\\ &= \mathbb{E}_0\left[\int_0^T\int_0^t I_s ds dX_t - P_0X_0\right.\\ &\quad\left.+\frac{1}{2}\int_0^T\int_0^T G(|t-s|)dX_s dX_t\right]\end{aligned}$$

Notation 4.1. Here $\mathbb{E}_t[\cdot] = \mathbb{E}\left[\cdot|\mathcal{F}_t^W\right]$ where $\mathcal{F}^W$ is the natural filtration of the Ornstein-Uhlenbeck process $(I_t)_{t\leq 0}$. ◇

Remark 4.2. At this point, one could add a classic risk aversion term $\phi\int_0^T X_t^2 dt$, however for simplicity we will exclude this. It should be noted the following results do however hold with the risk aversion term and we refer the reader to [6] for such results. ◇

Recalling the *a priori* assumption of a deterministic strategy as outlined in section 2.6.2 and removing the constant $-P_0X_0$ term, for any $X \in \Xi$ we can define the cost functional as

$$U([0,T]) := \int_0^T\int_0^t \mathbb{E}_0[I_s]\, ds dX_t + \frac{1}{2}\int_0^T\int_0^T G(|t-s|)dX_s dX_t \tag{20}$$

Before we proceed, it is necessary to quote some important results found in [6]. Theorem 2.4 of Neuman, Lehalle [6, Theorem 2.4, page 8] (a generalisation of Theorem 2.11 of Gatheral et al [14, Theorem 2.11, page 8]) is of vital importance in solving this optimisation problem and is stated as follows. Note that without the assumption of a deterministic strategy the following result fails.

Theorem 4.3. *$X^* \in \Xi$ minimises (20) over Ξ if and only if there exists $\lambda \in \mathbb{R}$ such that*

$$\int_0^t \mathbb{E}_0[I_s]\, ds + \int_0^T G(|t-s|)dX_s = \lambda \qquad \forall t \in [0,T]$$

4.1 Deterministic strategy

For the following we use an exponential decay kernel given by

$$G(t) = \kappa\rho\mathrm{e}^{-\rho t}.$$

Moreover, since $(I_t)_{t\leq 0}$ is an Ornstein Uhlenbeck process, we can evaluate the expectation

$$\mathbb{E}_0\left[I_s\right] = \iota \mathrm{e}^{-\gamma s} \quad \text{where} \quad I_0 = \iota.$$

Using the above results, we have a unique deterministic optimal strategy (given by corollary 2.7 of [6, Corollary 2.7, page 9]).

Theorem 4.4. *The minimiser of cost functional* (20) *is given by.*

$$X_t^* = x + \mathbb{1}_{\{t>0\}}A + Ct + \frac{B}{\gamma}\left(1 - \mathrm{e}^{-\gamma t}\right)\mathbb{1}_{\{t>T\}}$$

Where A, B, C, D are given by

$$\begin{aligned}
A &:= \frac{1}{2+T\rho}\left(\frac{1}{2\kappa\rho^2\gamma}\left((\rho+\gamma)\left(1+T\rho-\frac{\rho-\gamma}{\gamma}(1-\mathrm{e}^{-\gamma t})\right)-(\rho-\gamma)\mathrm{e}^{-\gamma T}\right)-x\right)\\
B &:= \iota\frac{\rho^2-\gamma^2}{2\kappa\rho^2\gamma}\\
C &:= \rho A - \iota\frac{\rho+\gamma}{2\kappa\rho\gamma}\\
D &:= A - \frac{\iota}{2\kappa\rho^2\gamma}\left(\rho+\gamma-(\rho-\gamma)\mathrm{e}^{-\gamma T}\right)
\end{aligned}$$

Moreover the corresponding λ introduced and guaranteed by Theorem 4.3 is given by

$$\lambda = 2\kappa C + \frac{\iota}{\gamma}$$

With this in mind we can evaluate the cost functional for the optimal strategy X_t^* as follows

$$\begin{aligned}
U^*([0,T]) &:= \int_0^T\int_0^t \mathbb{E}_0\left[I_s\right] ds dX_t^* + \frac{1}{2}\int_0^T\int_0^T G(|t-s|)dX_s^*dX_t^*\\
&= \frac{1}{2}\int_0^T\int_0^t \mathbb{E}_0\left[I_s\right] ds dX_t^* + \frac{1}{2}\int_0^T\left[\int_0^t \mathbb{E}_0\left[I_s\right] ds + \int_0^T G(|t-s|)dX_s^*\right]dX_t^*\\
&= \frac{1}{2}\int_0^T \frac{\iota}{\gamma}\left(1-\mathrm{e}^{-\gamma t}\right)dX_t^* + \frac{1}{2}\int_0^T \lambda dX_t^*\\
&= \frac{1}{2}\left(\frac{\iota}{\gamma}+\lambda\right)\int_0^T dX_t^* - \frac{\iota}{2\gamma}\int_0^T \mathrm{e}^{-\gamma t}dX_t^*\\
&= -\frac{1}{2}\left(\frac{\iota}{\gamma}+\lambda\right)X_0 - \frac{\iota}{2\gamma}\{A + \frac{B}{2\gamma}\left(1-\mathrm{e}^{-2\gamma T}\right) + \frac{C}{\gamma}\left(1-\mathrm{e}^{-\gamma T}\right) + D\mathrm{e}^{-\gamma T}\}
\end{aligned} \tag{21}$$

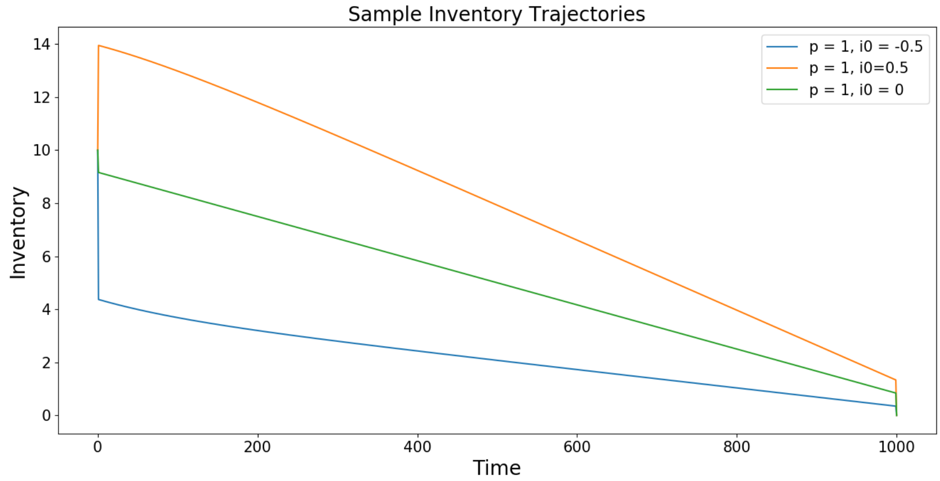


Figure 8: Trajectories of the deterministic strategy in the transient setting

4.2 Dynamic Strategy I

In line with Remark 2.9 of [6, Remark 2.9, page 10] we propose a dynamic version of the trading strategy given above in the natural way, that is we allow the agent to update the trading strategy at some intermediate times according the new information available. To formalise this chose the number of intermediate points m at which the strategy can be updated. Partition the trading window $[0, T]$ into m intervals of size $\frac{T}{m}$ as follows.

$$[0, T] = \{0 = t_0 < t_1 < \cdots < t_{m-1} < t_m = T\} \quad \text{where} \quad t_i := \frac{iT}{m}$$

Moreover consider the cost functional (20), now defined on the interval $[t, T]$, with expectation conditional on the information at time t, $\mathcal{F}_t^W$.

$$U([t, T]) = \int_t^T \int_t^\tau \mathbb{E}_t [I_s] \, ds dX_\tau + \frac{1}{2} \int_t^T \int_t^T G(|\tau - s|) dX_s dX_\tau - P_0 X_0 \tag{22}$$

We define $\tilde{X}_s^i$ to be the strategy that minimises the cost functional $U([t_i, T])$ and $\tilde{X}_s$ as follows

$$\tilde{X}_s = \begin{cases} \tilde{X}_s^0 & \text{for } s \in [t_0, t_1] \\ \tilde{X}_s^1 & \text{for } s \in [t_1, t_2] \\ & \vdots \\ \tilde{X}_s^{m-1} & \text{for } s \in [t_{m-1}, T] \end{cases}$$

Theorem 4.5. *The optimal strategy for a given $t \in [0, T]$ is given by*

$$\tilde{X}_s^t = A\mathbb{1}_{\{t\in[t,s]\}} + \frac{B}{\gamma}(\mathrm{e}^{-\gamma t} - \mathrm{e}^{-\gamma s}) + C(s-t) + D\mathbb{1}_{\{T\in[t,s]\}} + x$$

where A, B, C, D are given in the proof.

Proof. Appealing to Theorem 4.3, $\tilde{X}^t \in \Xi$ minimises $U([t,T])$ over Ξ if and only if there exists $\lambda^t \in \mathbb{R}$ such that

$$\int_t^\tau \mathbb{E}_t\left[I_s\right] ds + \int_t^T G(|\tau - s|) d\tilde{X}_s^t = \lambda^t \qquad \forall \tau \in [t, T]$$

For an exponential decay kernel and Ornstein-Uhlenbeck signal we can rewrite this as

$$\frac{\iota}{\gamma}\left(\mathrm{e}^{-\gamma t} - \mathrm{e}^{-\gamma\tau}\right) + \kappa\rho\int_t^T \mathrm{e}^{-\rho|\tau-s|} d\tilde{X}_s^t = \lambda^t, \quad \text{where} \quad \iota = I_t. \tag{23}$$

As in Obizhaeva and Wang [7] we make an ansatz at the solution of the form

$$d\tilde{X}_s^t = A\delta_t + (B\mathrm{e}^{-\gamma s} + C)dt + D\delta_T \tag{24}$$

where δ_a is the Dirac delta at a. Substituting the ansatz into equation (23), and evaluating the resulting integral gives

$$\begin{aligned} &\frac{\iota}{\gamma}\left(\mathrm{e}^{-\gamma t} - \mathrm{e}^{-\gamma\tau}\right) + A\kappa\rho\mathrm{e}^{-\rho(\tau-t)} + D\kappa\rho\mathrm{e}^{-\rho(T-\tau)} + C\kappa\left(2 - \mathrm{e}^{-\rho(\tau-t)} - \mathrm{e}^{-\rho(T-\tau)}\right) \\ &\quad + B\kappa\rho\left[\frac{1}{\rho-\gamma}\left(\mathrm{e}^{-\gamma\tau} - \mathrm{e}^{-\gamma t}\mathrm{e}^{-\rho(\tau-t)}\right) + \frac{1}{\rho+\gamma}\left(\mathrm{e}^{-\gamma\tau} - \mathrm{e}^{-\gamma T}\mathrm{e}^{-\rho(T-\tau)}\right)\right] = \lambda^t \end{aligned} \tag{25}$$

Since (23) holds for all $\tau \in [t, T]$ it must be the case that

$$\lambda^t = \frac{\iota}{\gamma}\mathrm{e}^{-\gamma t} + 2C\kappa$$

Recall the hard constraint on terminal liquidation as detailed in section 2.6.2 which can be alternatively written as

$$\int_t^T d\tilde{X}_\tau^t = -x, \tag{26}$$

where x is the remaining inventory at time t.

Collecting terms in (25) and substituting our ansatz (24) into the fuel constraint (26) we construct the following system of equations

$$\begin{aligned}
-x &= A + \frac{B}{\gamma}\left(\mathrm{e}^{-\gamma t} - \mathrm{e}^{-\gamma T}\right) + C(T-t) + D \\
0 &= B\kappa\rho\left(\frac{1}{\rho-\gamma} + \frac{1}{\rho+\gamma}\right) - \frac{\iota}{\gamma} \\
0 &= A\kappa\rho - B\frac{\kappa\rho}{\rho-\gamma}\mathrm{e}^{-\gamma t} - C\kappa \\
0 &= D\kappa\rho - B\frac{\kappa\rho}{\rho+\gamma}\mathrm{e}^{-\gamma T} - C\kappa
\end{aligned} \tag{27}$$

Solving the above system gives

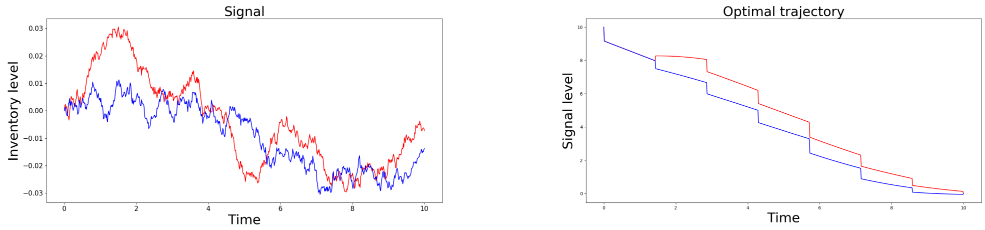
$$\begin{aligned}
B &= \frac{\iota}{2\rho^2\kappa\gamma}\left(\rho^2 - \gamma^2\right) \\
A &= \frac{-1}{2+\rho(T-t)}\{B\left(\frac{1}{\gamma}\left(\mathrm{e}^{-\gamma t} - \mathrm{e}^{-\gamma T}\right) - \frac{\rho}{\rho-\gamma}\mathrm{e}^{-\gamma t}(T-t) + \frac{\mathrm{e}^{-\gamma T}}{\rho+\gamma} - \frac{\mathrm{e}^{-\gamma t}}{\rho-\gamma}\right) + x\} \\
C &= A\rho - \frac{B\rho}{\rho-\gamma}\mathrm{e}^{-\gamma t} \\
D &= B(\frac{\mathrm{e}^{-\gamma T}}{\gamma+\rho} - \frac{\mathrm{e}^{-\gamma t}}{\rho-\gamma}) + A
\end{aligned}$$

Finally integrating our ansatz from t to s gives

$$\tilde{X}_s^t = A\mathbb{1}_{\{t\in[t,s]\}} + \frac{B}{\gamma}(\mathrm{e}^{-\gamma t} - \mathrm{e}^{-\gamma s}) + C(s-t) + D\mathbb{1}_{\{T\in[t,s]\}} + x$$

□

Figure 9 shows the simulated signal and corresponding optimal inventory trajectory. The signal informs the strategy by indicating the size and direction of the block trade as well as the subsequent rate of trading prior to the next update. In cases where the signal is low, the investor will make large block liquidations. Contrarily for higher signal values, block liquidations are smaller or even block acquisitions in some cases, despite the liquidation target. Again we note that as with the temporary impact case, these



(a) Simulations of signal (b) Corresponding optimal inevntory trajectory

Figure 9: Sample inventory trajectories of the dynamic strategy in the transient impact case

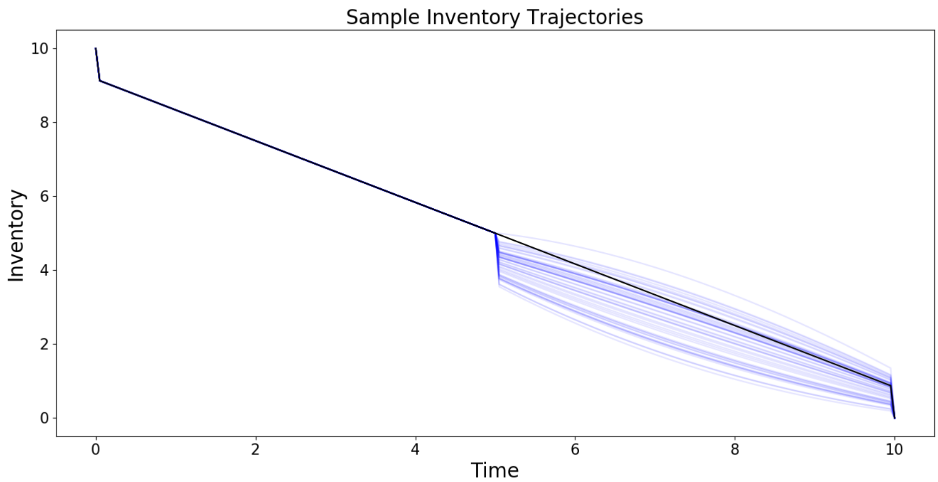


Figure 10: 50 sample inventory trajectories in the case of the dynamic strategy, for $m = 2$ (blue) and the corresponding static inventory trajectory (black).

acquisitions do not constitute a corrective measure to remedy excessive trading, and are simply an attempt to capture alpha and offset the cost of market impact.

4.3 Comparison of Static and Dynamic Strategy

Note that by construction, the static and dynamic strategies coincide on $[t_0, t_1]$, since they are optimising over the same objective function. This can be clearly seen in figure 10.

With regards finding the associated expected time 0 revenues of the trade, we are unable to appeal to the same methodology as in equation (21) since we cannot similarly apply theorem 4.3 by virtue of the limits of integration. With this in mind the analytical result becomes *very* cumbersome for the case where $m = 2$ and is worse still for larger m. As such we appeal to a Monte Carlo scheme to numerically evaluate the expected revenues

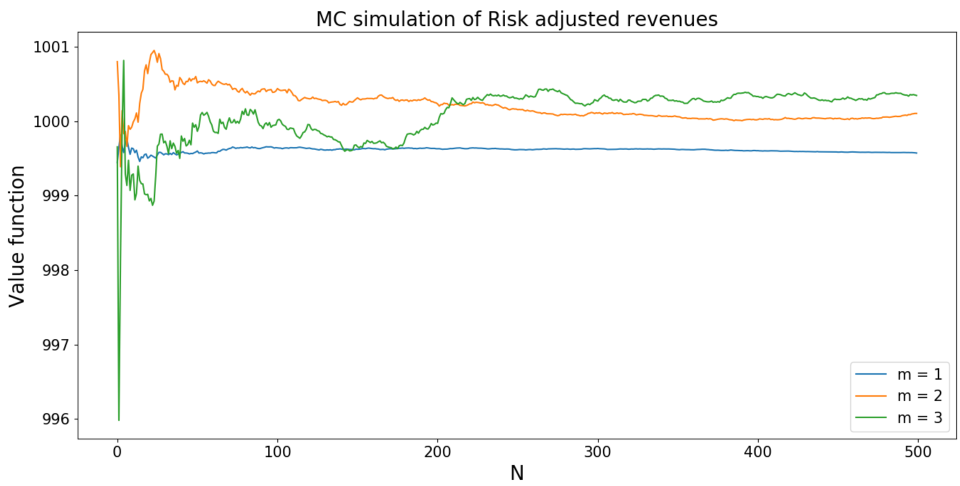


Figure 11: Monte Carlo simulation of risk adjusted revenues for $m = 1, 2, 3$

of the dynamic strategy.

$$\begin{aligned}\mathbb{E}_0\left[\int_0^T S_t d\tilde{X}_t + \frac{G(0)}{2}\sum(\Delta\tilde{X}_t)^2\right] &= \mathbb{E}_0\left[\int_0^T\int_0^t I_s ds d\tilde{X}_t \right.\\ &\left. +\frac{1}{2}\int_0^T\int_0^T G(|t-s|)d\tilde{X}_s d\tilde{X}_t\right] - P_0X_0\\ &\approx \frac{1}{N}\sum_{i=1}^N A_i - P_0X_0\end{aligned}$$

where A_i is given below

$$A_i := \int_0^T\int_0^t I_s ds d\tilde{X}_t + \frac{1}{2}\int_0^T\int_0^T G(|t-s|)d\tilde{X}_s d\tilde{X}_t$$

Moreover we numerically evaluate A_i in the standard way, constructing a partition on $[0, T]$ into n pieces of size $h := T/n$

$$A_i \approx \sum_{j=1}^n\left[\sum_{k=1}^j I_{t_k}h + \frac{1}{2}\sum_{k=1}^n G(|t_j - t_k|)\left(\tilde{X}_{t_k} - \tilde{X}_{t_{k-1}}\right)\right]\left(\tilde{X}_{t_j} - \tilde{X}_{t_{j-1}}\right)$$

Figure 11 shows the convergence of the Monte Carlo simulations for the expected revenues in the case where $m = 1, 2, 3$. As can be seen, the revenues are highest for the strategy with the most updates and with the most access to new market information.

Figure 12 shows the simulated expected revenues as a function of the number of up-

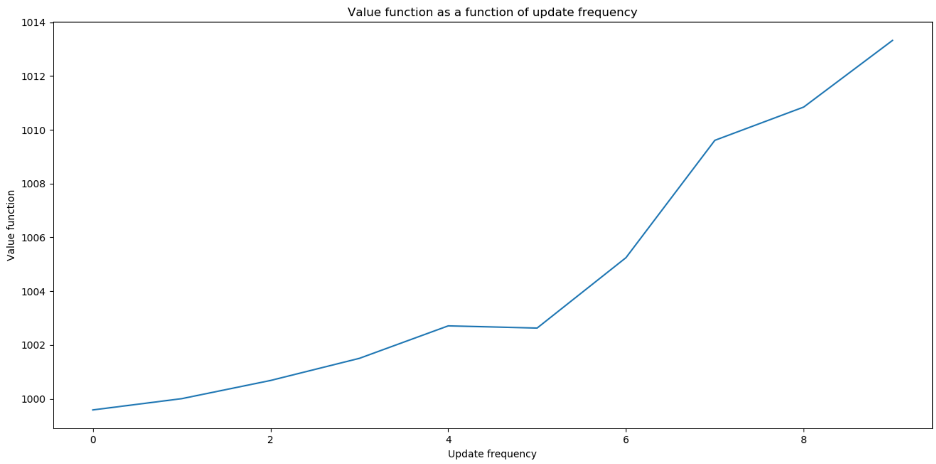


Figure 12: Revenues as a function of number of strategy updates.

dates m. This continues to display the aforementioned trend of increasing revenues with more access to market information and is in line with the expected behaviour of dynamic strategies in comparison with static ones ($m = 1$).

Remark 4.6 (Criticism)**.** One thing worth baring in mind with this model is the interaction of the transient nature of the model and the way the strategy is updated. In using the optimal strategy according to equation (22) the decaying market impact from trading that occurs before time t is *not* taken into consideration when formulating a new optimal strategy. As such, the updated strategy will underestimate the associated market impact and hence overestimate the amount of liquidation. The effect of this is seen in figure 10, where the sample dynamic trajectories are seen to be lower than the static strategy, demonstrating the above bias. ◇

4.4 Alternative derivation of a Dynamic Strategy

To rectify the issue outlined in remark 4.6 one might consider a stochastic optimisation approach as is commonplace in the optimal execution setting ([1], [5], [6] amongst others). Specifically, we address the issue of preserving past transient impact by incorporating a new state variable μ into the model described in [6].

$$\mu_t = \int_0^t G(t-s) r_s ds.$$

With this notation, recall the execution price in the transient setting,

$$S_t = P_t - \int_0^t G(t-s) r_s ds = P_t - \mu_t.$$

As above, we consider an exponential decay kernel,

$$G(t) = \kappa \alpha \mathrm{e}^{-\alpha t}.$$

Note, here the parameter ρ used for terminal penalty so we will use α as the decay parameter. Applying Leibniz Integral Rule we have the following.

$$\begin{aligned} \frac{d\mu}{dt} &= G(t-t) r_t + \int_0^t \frac{\partial}{\partial t} \left(G(t-s) \right) r_s ds \\ &= G(0) r_t + \int_0^t G'(t-s) r_s ds \\ &= r_t - \alpha \int_0^t \kappa \alpha \mathrm{e}^{-\alpha t} \\ &= r_t - \alpha \mu_t \end{aligned} \tag{28}$$

or equivalently

$$d\mu_t = \left(r_t - \alpha \mu_t \right) dt.$$

Remark 4.7. Note in the following instead of using the integral of the price process with respect to inventory we use the investors cash, C as a state variable. This works to simplify the resulting calculation and moreover motivates the choice of ansatz given by (35). ⋄

Let us define the performance criteria, as introduced in section 2.5, which captures the risk adjusted revenues of the trade in addition to a terminal penalty as per remark 2.4.

$$V^r(t, \iota, c, x, p, \mu) \coloneqq \mathbb{E}_{t,\iota,c,x,p,\mu} \left[C_T^r - \phi \int_t^T (X_s^r)^2 ds + X_T^r (P_T^r - \rho X_T^r) \right]. \tag{29}$$

Notation 4.8. $\mathbb{E}_{t,\iota,c,x,p,\mu}$ is the expectation conditioned on $I_t = \iota$, $C_t = c$, $X_t = x$, $P_t = p$, $\mu_t = \mu$. For brevity we will equivalently use $\mathbb{E}_{t,\vec{x}_t}[\cdot]$, where $\vec{x}_t$ is defined as the vector of parameters $\vec{x}_t \coloneqq (I_t, C_t, X_t, P_t, \mu_t)$. Moreover the notation $\vec{x}_\tau^r \coloneqq (I_\tau^r, C_\tau^r, X_\tau^r, P_\tau^r, \mu_\tau^r)$ is used to denote the state variables having evolved from time t to τ in accordance with control r. ⋄

Moreover we can introduce the *value function*

$$V(t, \iota, c, x, p, \mu) \coloneqq \sup_r V^r(t, \iota, c, x, p, \mu). \tag{30}$$

Remark 4.9. Similarly to above, when discussing the performance criteria or value function we will often write $V^r(t, \vec{x})$ and $V(t, \vec{x})$. ◇

From here we will appeal to the usual methodology of stochastic optimisation as detailed in [15]. Specifically, formulating the *dynamic programming principle* (DPP) and the *Hamilton-Jacobi-Bellman* (HJB) equation.

Theorem 4.10 (Dynamic Programming Principle)**.** *The corresponding DPP for the above stochastic control problem is given by*

$$V(t, \vec{x}) = \sup_r \mathbb{E}\left[V(\tau, \vec{x}^r_\tau) - \phi \int_t^\tau (X^r_s)^2 ds\right].$$

Proof. Consider the evolution of the state variables $\vec{x}$ from time t to time τ according to an arbitrary admissible strategy r. By application of the tower property we have the following:

$$\begin{aligned}
V^r(t, \vec{x}) &= \mathbb{E}_{t,\vec{x}}\left[C^r_T - \phi \int_t^T (X^r_s)^2 ds + X^r_T(P^r_T - \rho X^r_T)\right] \\
&= \mathbb{E}_{t,\vec{x}}\left[\mathbb{E}_{\tau,\vec{x}^r_\tau}\left[C^r_T - \phi \int_t^T (X^r_s)^2 ds + X^r_T(P^r_T - \rho X^r_T)\right]\right] \\
&= \mathbb{E}_{t,\vec{x}}\left[\mathbb{E}_{\tau,\vec{x}^r_\tau}\left[C^r_T - \phi \int_\tau^T (X^r_s)^2 ds + X^r_T(P^r_T - \rho X^r_T)\right] - \phi \int_t^\tau (X^r_s)^2 ds\right] \\
&= \mathbb{E}_{t,\vec{x}}\left[V^r(\tau, \vec{x}^r_\tau) - \phi \int_t^\tau (X^r_s)^2 ds\right].
\end{aligned}$$

Since the above holds for an arbitrary control r it follows that

$$\begin{aligned}
V^r(t, \vec{x}) &= \mathbb{E}_{t,\vec{x}}\left[V^r(\tau, \vec{x}^r_\tau) - \phi \int_t^\tau (X^r_s)^2 ds\right] \\
&\leq \mathbb{E}_{t,\vec{x}}\left[V(\tau, \vec{x}^r_\tau) - \phi \int_t^\tau (X^r_s)^2 ds\right].
\end{aligned}$$

Taking supremum over both sides gives

$$V(t, \vec{x}) \leq \sup_r \mathbb{E}_{t,\vec{x}}\left[V(\tau, \vec{x}^r_\tau) - \phi \int_t^\tau (X^r_s)^2 ds\right].$$

We now show the reverse inequality. Let $\epsilon > 0$ be given and, under the assumption of a continuous control space, there exists r^ϵ such that

$$V(t,\vec{x}) \geq V^{r^\epsilon}(t,\vec{x}) \geq V(t,\vec{x}) - \epsilon$$

We now construct the following control, that follows r^ϵ after τ and the arbitrary control r before τ,

$$\hat{r}^\epsilon_t = \begin{cases} r_t & \text{for } t \leq \tau \\ r^\epsilon_t & \text{for } t > \tau. \end{cases}$$

By an analogous tower property argument to above we have,

$$V(t,\vec{x}) \geq V^{\hat{r}^\epsilon}(t,\vec{x}) = \mathbb{E}_{t,\vec{x}}\left[V^{\hat{r}^\epsilon}(\tau,\vec{x}^{\hat{r}^\epsilon}_\tau) - \phi\int_t^\tau (X^{\hat{r}^\epsilon}_s)^2 ds\right].$$

Note that, by construction of $\hat{r}^\epsilon$ and r^ϵ, the following hold:

1. $\vec{x}^{\hat{r}^\epsilon}_\tau = \vec{x}^r_\tau$, since $\hat{r}^\epsilon$ and r are equivalent before τ
2. $V^{\hat{r}^\epsilon} = V^{r^\epsilon}$, since $\hat{r}^\epsilon$ and r^ϵ are equivalent after τ
3. $V^{r^\epsilon} \geq V - \epsilon$

Combining the above we can deduce the following,

$$\begin{aligned} V(t,\vec{x}) &\geq \mathbb{E}_{t,\vec{x}}\left[V^{\hat{r}^\epsilon}(\tau,\vec{x}^{\hat{r}^\epsilon}_\tau) - \phi\int_t^\tau (X^{\hat{r}^\epsilon}_s)^2 ds\right] \\ &= \mathbb{E}_{t,\vec{x}}\left[V^{r^\epsilon}(\tau,\vec{x}^r_\tau) - \phi\int_t^\tau (X^r_s)^2 ds\right] \\ &\geq \mathbb{E}_{t,\vec{x}}\left[V(\tau,\vec{x}^r_\tau) - \phi\int_t^\tau (X^r_s)^2 ds\right] - \epsilon. \end{aligned} \tag{31}$$

Noting the above holds for arbitrary r it must indeed hold for the supremum. Moreover taking $\epsilon \to 0$ gives,

$$V(t,\vec{x}) \geq \sup_r \mathbb{E}_{t,\vec{x}}\left[V(\tau,\vec{x}^r_\tau) - \phi\int_t^\tau (X^r_s)^2 ds\right].$$

□

Theorem 4.11 (Hamilton-Jacobi-Bellman equation)**.** *The corresponding HJB equation* *for the above stochastic control problem is given by*

$$0 = \left(\partial_t + \mathcal{L}_t^I + \mathcal{L}_t^P - \alpha\mu\partial_\mu\right) V - \phi x^2 \ + \sup_r \Big(\big((p-\mu) r\partial_c - r\partial_x + r\partial_\mu\big) V \Big)$$

$$V(T, \vec{x}_T) = c + x(p - \rho x)$$

where $\mathcal{L}_t^I$ *and* $\mathcal{L}_t^P$ *are the generators of* I *and* P *respectively, given in the proof.*

Proof. For $\tau = t + h$ for small h, Theorem 4.10 reads,

$$V(t, \vec{x}_t) = \sup_r \mathbb{E}\left[V(t+h, \vec{x}^r_{t+h}) - \phi \int_t^{t+h} (X_s^r)^2 ds \right].$$

Assuming sufficient regularity of the value function we apply Itô's formula to obtain

$$\begin{aligned} V(t+h, \vec{x}^r_{t+h}) = V(t, \vec{x}_t) + \int_t^{t+h} \left(\partial_t + \mathcal{L}_s^I + S_s r_s \partial_c - r_s \partial_x + \mathcal{L}_s^P + (r_s - \alpha\mu_s)\partial_\mu\right) V ds \\ + \int_t^{t+h} \sigma \partial_\iota V dB_s + \int_t^{t+h} \sigma_p \partial_p V dB_s \end{aligned} \quad (32)$$

where $\mathcal{L}^I$ and $\mathcal{L}^P$ are the generators of I and P respectively, given by

$$\begin{aligned} \mathcal{L}_s^I &= -\gamma I_s \partial_\iota + \frac{1}{2}\sigma^2 \partial_\iota^2 \\ \mathcal{L}_s^P &= I_s \partial_p + \frac{1}{2}\sigma_p \partial_p^2 \end{aligned}$$

Using the second inequality of Theorem 4.10 and (32) gives

$$\begin{aligned} V(t, \vec{x}) &= \sup_r \mathbb{E}\left[V(t+h, \vec{x}^r_{t+h}) - \phi \int_t^{t+h} (X_s^r)^2 ds \right] \\ &\geq \mathbb{E}\left[V(t+h, \vec{x}^r_{t+h}) - \phi \int_t^{t+h} (X_s^r)^2 ds \right] \\ &= \mathbb{E}\Bigg[V(t, \vec{x}_t) + \int_t^{t+h} \left(\partial_t + \mathcal{L}_s^I + S_s r_s \partial_c - r_s \partial_x + \mathcal{L}_s^P + (r_s - \alpha\mu_s)\partial_\mu\right) V ds \\ &\qquad + \int_t^{t+h} \sigma \partial_\iota V dB_s + \int_t^{t+h} \sigma_p \partial_p V dB_s - \phi \int_t^{t+h} (X_s^r)^2 ds \Bigg] \\ &= \mathbb{E}\Bigg[V(t, \vec{x}_t) + \int_t^{t+h} \left(\partial_t + \mathcal{L}_s^I + S_s r_s \partial_c - r_s \partial_x + \mathcal{L}_s^P + (r_s - \alpha\mu_s)\partial_\mu\right) V ds \\ &\qquad - \phi \int_t^{t+h} (X_s^r)^2 ds \Bigg] \end{aligned}$$

Hence we have

$$0 \geq \mathbb{E}\left[\int_t^{t+h} \left(\partial_t + \mathcal{L}_s^I + S_s r_s \partial_c - r_s \partial_x + \mathcal{L}_s^P + (r_s - \alpha\mu_s)\partial_\mu\right) V ds \right.$$
$$\left. -\phi \int_t^{t+h} (X_s^r)^2 ds\right]$$

Dividing by h, taking the limit as $h \to 0$ and appealing to the mean value theorem we have

$$0 \geq \left(\partial_t + \mathcal{L}_t^I + (p-\mu) r \partial_c - r\partial_x + \mathcal{L}_t^P + (r - \alpha\mu)\partial_\mu\right) V - \phi x^2.$$

Since the above holds for arbitrary r we have

$$0 \geq \sup_r \left(\left(\partial_t + \mathcal{L}_t^I + (p-\mu) r \partial_c - r\partial_x + \mathcal{L}_t^P + (r - \alpha\mu)\partial_\mu\right) V - \phi x^2\right)$$
$$\geq \left(\partial_t + \mathcal{L}_t^I + \mathcal{L}_t^P - \alpha\mu\partial_\mu\right) V - \phi x^2 \ + \sup_r \left(\left((p-\mu) r \partial_c - r\partial_x + r\partial_\mu\right) V\right)$$

Since this result holds for all r, it must also hold for the optimal control r^*, and as such the above argument holds with *equality* and we arrive at the above result. Moreover the terminal condition follows immediately from the definition of the value function (29). □

4.4.1 Adding temporary impact

In this case the supremum is linear in r meaning the optimisation problem is unbounded. To rectify this we will consider the addition of a temporary market impact term to the execution price. Although this may initially seem odd, to have both a temporary impact term *and* a transient impact term, we will arrive at a quadratic supremum that can henceforth be solved. Moreover one could then consider the limiting solution as $\eta \to 0$. As such let us define the execution price S,

$$S_t = P_t - \mu_t - \eta r_t.$$

Moreover we can write the investor's cash as

$$C_T - C_t = \int_t^T (P_s - \mu_s - \eta r_s)\, r_s ds. \tag{33}$$

Remark 4.12. We note that in this new optimisation problem, the performance criteria and value function are the same as before, the only difference being in our definition of the state variable C_t as given in (33). As such the dynamic programming principle is as in Theorem 4.10 and we can reuse most of the above argument for formulating the new HJB equation.

◇

Theorem 4.13 (Hamilton-Jacobi-Bellman equation)**.** *The corresponding HJB equation for the new stochastic control problem, where execution price includes a temporary market impact, is given by*

$$0 = \left(\partial_t + \mathcal{L}_t^I + \mathcal{L}_t^P - \alpha\mu\partial_\mu\right) V - \phi x^2 \;+ \sup_r \Big(\big((p - \mu - \eta r) r\partial_c - r\partial_x + r\partial_\mu\big) V \Big)$$
$$V(T, \vec{x}_T) = c + x(p - \rho x),$$

where $\mathcal{L}_t^I$ and $\mathcal{L}_t^P$ are as above.

Proof. Arguing as in Theorem 4.11, we have

$$0 \geq \mathbb{E}\left[\int_t^{t+h} \left(\partial_t + \mathcal{L}_s^I + S_s r_s \partial_c - r_s\partial_x + \mathcal{L}_s^P + (r_s - \alpha\mu_s)\partial_\mu\right) V ds \right.$$
$$\left. -\phi \int_t^{t+h} X_s^2 ds\right].$$

Again we note that the argument is identical to that of Theorem 4.11, up to a redefinition of state variables. Dividing by h, taking the limit as $h \to 0$ and again appealing to the mean value theorem we have

$$0 \geq \left(\partial_t + \mathcal{L}_t^I + (p - \mu - \eta r) r\partial_c - r\partial_x + \mathcal{L}_t^P + (r - \alpha\mu)\partial_\mu\right) V - \phi x^2.$$

Since the above holds for arbitrary r we have

$$\begin{aligned} 0 &\geq \sup_r \left(\left(\partial_t + \mathcal{L}_t^I + (p - \mu - \eta r) r\partial_c - r\partial_x + \mathcal{L}_t^P + (r - \alpha\mu)\partial_\mu\right) V - \phi x^2\right) \\ &\geq \left(\partial_t + \mathcal{L}_t^I + \mathcal{L}_t^P - \alpha\mu\partial_\mu\right) V - \phi x^2 \;+ \sup_r \Big(\big((p - \mu - \eta r) r\partial_c - r\partial_x + r\partial_\mu\big) V \Big) \end{aligned}$$

We use an identical argument to above to show the above holds with equality and arrive at the desired result. □

Crucially now the supremum is *quadratic* in r meaning it can be easily solved to give the optimal trading speed

$$r^* = \frac{((p-\mu)\partial_c - \partial_x + \partial_\mu)\, V}{2\eta \partial_c V}.$$

Moreover this allows us to simplify the HJB equation of 4.11 to give

$$0 = \left(\partial_t + \mathcal{L}_t^I + \mathcal{L}_t^P - \alpha\mu\partial_\mu\right) V - \phi x^2 \ + \frac{\left((p-\mu)\partial_c V - \partial_x V + \partial_\mu V\right)^2}{4\eta\partial_c V}$$
$$V(T, \vec{x}_T) = c + x(p - \rho x) \quad (34)$$

Examining the terminal condition $V(T, \vec{x}_T) = c + x(p - \rho x)$ we suggest the following ansatz for the value function

$$V(t, \vec{x}) = c + xp + v(t, \iota, x, \mu), \quad (35)$$

where v is to be determined in accordance with the terminal condition of V. Specifically $v(T, \iota, x, \mu) = \rho x^2$. Substituting this ansatz into (34) we have

$$0 = \left(\partial_t + \mathcal{L}_t^I - \alpha\mu\partial_\mu\right) v + \iota x - \phi x^2 \ + \frac{\left(\partial_\mu v - \mu - \partial_x v\right)^2}{4\eta} \quad (36)$$

5 Further Research

5.1 Alternative derivation of a Dynamic Strategy

In section 4.4 we proposed an alternative derivation for the dynamic strategy in the transient market impact setting. This poses two new avenues for future research to consider. The first of which is to find analytic or numerical solutions to PDE (36) and then consider the limiting case as $\eta \to 0$.

Alternatively one could work with the earlier PDE (recall below) of theorem 4.11 and construct a numerical solution.

$$0 = \left(\partial_t + \mathcal{L}_t^I + \mathcal{L}_t^P - \alpha\mu\partial_\mu\right) V - \phi x^2 \; + \sup_r \Big(\big((p-\mu) r \partial_c - r\partial_x + r\partial_\mu\big) V \Big)$$
$$V(T, \vec{x}_T) = c + x(p - \rho x)$$

As mentioned in section 4.4.1 the linear supremum does pose a problem regarding the boundedness of the problem, however this may be rectified using an approach outlined in [16]. In this approach we assume the control r is bounded in absolute value by some M. Then we can find the optimal control r^* given by

$$r^* = \begin{cases} M & \text{for } (p-\mu)\partial_c - \partial_x + \partial_\mu > 0 \\ -M & \text{for } (p-\mu)\partial_c - \partial_x + \partial_\mu < 0 \end{cases}$$

As such one could devise a numerical scheme involving 2 PDEs, one for each of the above cases, to find a numerical solution and moreover consider the results as $M \to \infty$.

5.2 A Supervised Learning approach to adapted strategies

With execution moving away from the classical methods to a more signal driven approach, we would expect institutions to be trading with these ideas in mind. As such, instead of proposing ever more complicated models that react to certain signals, an alternative may be to observe how market participants deal with large liquidations and acquisitions and attempt to *learn* from them. As with any machine learning problem, one must first address the source of the data, which in this case is the NASDAQ OMX exchange. The NASDAQ OMX is a particularly useful data source as, up until 2014, it published both the trades executed and moreover *who* executed them. With this, one could establish individual market participants' full inventory trajectories, which we will denote $(\hat{X}_t)_{t\geq 0}$.

As observed in figure 3, the trajectories of the dynamic strategy can be seen as perturbations of the corresponding static strategy. With this in mind if we assume the realised inventory trajectories, $(\hat{X}_t)_{t\geq 0}$, are perturbations around the optimal inventory of some classical static model, $(X^*_t)_{t\geq 0}$, we could extract a process of *aggression coefficients* $(\beta_t)_{t\geq 0}$ via

$$\beta_t = \frac{\hat{X}_t}{X^*_t} \quad 0 \leq t \leq T.$$

The problem can then be framed as a time indexed series of supervised learning problems, with β_t as the target variable. In the same light as Cartea and Jaimungal and Neuman and Lehalle in their consideration of liquidity and imbalance signals as a key driver of execution strategy, possible features could be the depth, spread and price in addition to other directional signals derived from level I order book data. Once trained, to subsequently evaluate the resulting strategy one could take a simulation based approach similar to that of section 4.3, for a comparison with the corresponding static model.

Conclusion

To begin this project, we gave a brief survey on commonly adopted modelling choices in the existing optimal execution literature, with specific focus on how static and dynamic strategies arise. Following the work of Brigo and Piat, we wished to compare corresponding static and dynamic strategies with less stylised asset dynamics, namely those introduced by Neuman and Lehalle in the form of a driving Ornstein-Uhlenbeck liquidity signal. We considered two different market impact models, namely temporary impact and transient impact.

Under the former framework we derived a static execution strategy via a calculus of variations approach and observed the dynamic strategy could be thought of as perturbations around its static counterpart. We moreover considered the asymptotic properties of the dynamic strategy with respect to the terminal penalty parameter, to establish a basis for comparison. We observe how the optimal dynamic strategy may chose to purchase *despite* a liquidation target but note this isn't a corrective measure as seen in classic static models, instead this is in response to the liquidity signal in an effort to increase revenues. In most cases there is little to no difference between the two strategies, with the dynamic strategy providing only marginally increased revenues, however this was not the case when considering two specific model components: signal volatility and size of trading window. For small signal volatilities, the dynamic strategy offered little benefit, however for larger values, access to new information greatly increased the respective performance criteria. For shorter time horizons the strategies performed similarly, with little difference in the revenues. Over longer time horizons the dynamic strategy utilised new market information to make directional plays in the market where the static strategy could not, thus increasing risk adjusted revenues.

In the transient setting we introduced some key results in the current literature before using these to develop the heuristic approach in constructing a dynamic strategy as proposed by Neuman and Lehalle. We explicitly derived this strategy in terms of an update frequency and, as in the temporary case, note that the optimal strategy can look to purchase an asset even with an overall liquidation target. We found the expected risk adjusted revenues of the dynamic startegy via Monte Carlo simulation, which coincided with our intuition that increasing the update frequency would increase the revenues. Following this we discussed a criticism of this heuristic approach and subsequently proposed an alternative that addresses these concerns, highlighting how future research may proceed with this in mind.

Finally we respond to the desire for a more data driven approach to optimal execution and consider supervised learning as a different resolution to the optimal execution problem.

References

[1] Robert Almgren and Neil Chriss. Optimal execution of portfolio transactions. *Journal of Risk*, 3:5–40, 2001.

[2] Dimitris Bertsimas and Andrew W Lo. Optimal control of execution costs. *Journal of Financial Markets*, 1(1):1–50, 1998.

[3] Jim Gatheral and Alexander Schied. Optimal trade execution under geometric brownian motion in the almgren and chriss framework. *International Journal of Theoretical and Applied Finance*, 14(03):353–368, 2011.

[4] Damiano Brigo and Giuseppe Di Graziano. Optimal trade execution under displaced diffusions dynamics across different risk criteria. *Journal of Financial Engineering*, 1(02):1450018, 2014.

[5] Álvaro Cartea and Sebastian Jaimungal. Incorporating order-flow into optimal execution. *Mathematics and Financial Economics*, 10(3):339–364, 2016.

[6] Charles-Albert Lehalle and Eyal Neuman. Incorporating signals into optimal trading. *arXiv preprint arXiv:1704.00847*, 2017.

[7] Anna A Obizhaeva and Jiang Wang. Optimal trading strategy and supply/demand dynamics. *Journal of Financial Markets*, 16(1):1–32, 2013.

[8] Álvaro Cartea, Sebastian Jaimungal, and José Penalva. *Algorithmic and high-frequency trading*. Cambridge University Press, 2015.

[9] Damiano Brigo and Clement Piat. Static vs adapted optimal execution strategies in two benchmark trading models. 2016.

[10] Gur Huberman and Werner Stanzl. Optimal liquidity trading. *Review of Finance*, 9(2):165–200, 2005.

[11] Jim Gatheral. No-dynamic-arbitrage and market impact. *Quantitative finance*, 10(7):749–759, 2010.

[12] Marc Potters and Jean-Philippe Bouchaud. More statistical properties of order books and price impact. *Physica A: Statistical Mechanics and its Applications*, 324(1-2):133–140, 2003.

[13] Peter A Forsyth, J Shannon Kennedy, ST Tse, and Heath Windcliff. Optimal trade execution: a mean quadratic variation approach. *Journal of Economic Dynamics and Control*, 36(12):1971–1991, 2012.

[14] Jim Gatheral, Alexander Schied, and Alla Slynko. Transient linear price impact and fredholm integral equations. *Mathematical Finance: An International Journal of Mathematics, Statistics and Financial Economics*, 22(3):445–474, 2012.

[15] Huyên Pham. *Continuous-time stochastic control and optimization with financial applications*, volume 61. Springer Science & Business Media, 2009.

[16] Monique Jeanblanc-Picqué and Albert N Shiryaev. Optimization of the flow of dividends. *Russian Mathematical Surveys*, 50(2):257, 1995.