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by Manola Meconcelli

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Imperial College London

Department of Mathematics

Credit skew term structure model with Cheyette style diffusion.

Author: Manola Meconcelli (CID: 02138534)

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Declaration

The work contained in this thesis is my own work unless otherwise stated.

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Abstract

In this work we present a model for the stochastic term structure of the credit spread: we choose to model instantaneous credit spread by using the Cheyette representation of the Heath-Jarrow-Morton (HJM) framework, that leads to a finite-dimension Markovian system of equations. The goal is to derive affine formulae for the default factor and risky bonds in order to first calibrate the Credit Default Swaps (CDSs) on the default curve and then use the calibrated Constant Maturity Default Swaps Rate to price related Swap contracts. First, we assume zero correlation between the instantaneous credit spread and the short rate to derive those formulae and then we introduce a convexity adjustment, inspired by the Hull-White extended Vasicek model. Without such correction, we wouldn't be able to calibrate well CDSs linked to systemic companies or long-maturity CDSs. This is shown in the first part of the numerical results, where we analyse the correlation impact on CDSs pricing by repricing some of the CDSs on the default curve. Further numerical examples on the model performance, using the standard Monte Carlo method as benchmark, complete the paper.

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Introduction

Credit Default Swaps are extremely popular protection contracts and the most frequently traded credit derivatives. In addition, they can be seen as a speculation instruments on the probability of default and are therefore one of the most sensitive indicators of corporate financial health and contribute significantly to the completeness of the market. A mathematical interpretation on why this is the case will be lately analysed in 3.1.10. Their popularity goes back to 1991-94 when JP Morgan introduced them and their market has grown much faster than other derivatives market since its establishment. It is worth mentioning that the CDSs market size has risen more than 50 times from 2001 to 2008, and has remained steady after the crisis at around 2 trillions. The reason of this can be found not only in their risk transfer utilization but mostly in the speculative one, that has been blamed for lifting up CDSs spreads of entities in precarious situation and for being responsible for the deterioration of some countries sovereign debt, such as Greece. However, it has to be said that Greek bond spreads have been firmly followed by CDS spreads in 2010, showing no direct indication of one piloting the other in any remarkable way. Again, the study that has been underlined in [1] shows that in Asia, CDS trading has had positive effects on bond market development as markets liquidity sources.

CDSs market may both act as medium for systemic risk, when protection sellers, such as major banks, funds or insurers, are defectively capitalised or, inversely, contribute to mitigating systemic risk when a central clearing facility with adequate reserves is involved (as it has been imposed by regulators after the financial crisis).

A Credit Default Swap (CDS) is a swap contract and agreement in which the protection buyer of the CDS makes a series of payments (often referred to as the CDS "spread", see 3.1.6) to the protection seller and, in exchange, receives a payoff if a credit instrument (typically a bond or loan) experiences a credit event, including, for instance, a failure to pay, restructuring and bankruptcy. Commercial banks are usually the buyers of CDS protection, whilst insurance companies and highly rated dealers used to be protection sellers before the financial crisis. CDSs usually have maturity from one to ten years, with most of the liquidity condensed in the middle horizon.

CDSs are liquid products and therefore are used as calibration instruments for pricing models for more complex products, such as Constant Maturity Default Swaps (CMDS) Rates and products derived by the latter, like Swaps. The liquidity of the CDSs market allows the construction of an implied term structure of default probability (see 3.1.10).

This paper aims to present a model for the stochastic term structure of the credit spreads. Inspired by the results in [2], [3] and [4], in this paper we choose to model istantaneous credit spread by using the Cheyette representation of the HJM model, that leads to a Markovian system of equations that describe the credit spread process together with the rate one. The Markovianity is clearly a desiderable property for pricing credit spread sensitive products because it leads to analytical formulae for the risky bonds and default factors. More precisely, we choose the istantaneous credit spread as the state variable and model its nature by a diffusion process of Cheyette type. A lognormal type of model for the intensity has been studied and used intensively in both the literature and industry (see [5]). The reason of this extensive use can be found in the fact that the log normality of the model ensures the positivity of the survival probabilities. However, this model doesn't give analytical expression neither for risky bonds, conditional accrual or conditional premium. We instead want to find an analytic formula for these three variable in order to have a stable CMDS rate. The analytical formula, and even better an affine formula, is instead insured by using the Cheyette model, thanks to the Markovianity property. From a numerical point of view, this will lead to much faster and stable pricing models. However, a significant drawback for the usage of Cheyette model in credit is that this could lead to negative survival probabilities. This is not an issue for the rate case, since in the past years we have frequently seen negative rates on the market, due to monetary policies aimed at controlling deflation. On the other hand, this is clearly a significant issue for the credit case since we are talking about probabilities, which cannot be negative. It is therefore of fundamental importance to ensure the positivity of the survival probabilities with an appropriate discretisation scheme, that will be investigated in further works. Notice that although a first approach in the further development of the model would be to assume independence between the interest rate and the credit intensity, dependence will become desirable and will be introduced making use of the Hull-White extended Vasicek model for credit.

In Chapter 1, we shall provide a preliminary introduction to the Cheyette model in the rates case, preceded by a broad introduction of the more general Heath-Jarrow-Morton framework. Here, the concepts and heuristics of forward rate, instantaneous forward rate and bond price will be presented and commented. In Chapter 2 we will present the credit spread modelling in a similar way to what has been done in the previous chapter. The credit intensity λ is indeed the analogue to the short rate r. This means that it is used as a driver to model those rates that are observed in the market. We will proceed with the probabilistic framework, together with the concept of survival measure and a new related sigma algebra that takes into account for the default. We will then move on with the Heath-Jarrow-Morton dynamics description for the credit spread and Markovian dynamics. In particular, the most significant formulae related to CDS calibration and CMDS pricing will be derived, such as the risky bond and default factor reconstitution formulae. In Chapter 3 we will present the CDS contract and CMDS rate, their cash flows and their calibration/pricing formulae, using the results obtained in Chapter 2. Moreover, in Chapter 4 we will get into the details of the calibration algorithm that has been implemented and finally propose a convexity adjustment term, using the Hull-White extended Vasicek model. In the end, the numerical results will be presented and commented in Chapter 5, preceded by a correlation analysis first related to the CDSs on the default curve and then to the CMDS rate that has been computed to reprice a Swap contract. This analysis will confirm that the independence assumption between rates and credit is valid for short term maturity CDSs and for non-systemic entities.

Chapter 1

Cheyette Model

In this chapter, we present the theoretical framework that underlies the experiments. We first discuss the setup of Heath, Jarrow, and Morton (1992) framework and then turn to a specific subclass of it: the Cheyette model. Before getting into the details of the credit case, it is fundamental to recall some of the main results, highly studied in the literature of the past 30 years (see [6], [5] and [7]), related to the interest rate case.

1.1 The market dynamic

In the rest of the paper, we will denote the risk free rate with $r(t)$ and we will assume the usual Black-Scholes dynamics for the risk-free bank account B:

$$\begin{cases} dB(t) = r_t B(t)dt \\ B(0) = B_0. \end{cases}$$

Moreover, we will denote with $\mathbb{E}$ the expectation with respect to the risk neutral measure $\mathbb{Q}$, that has the bank account B as associated numeraire.

Definition 1.1.1 (Discount factor)**.** The discount factor is a random variable at time t and becomes known at time T. It is given by the following formula:

$$D(t,T) := \exp\left(-\int_t^T r_s ds\right) = \frac{B(t)}{B(T)}.$$

We consider the probability space $(\Omega, \mathcal{F}_t, \mathbb{Q})$ where $(\mathcal{F}_t)_t$ denotes the usual filtration that represents the information flow of all the involved default-free market quantities.

Definition 1.1.2 (Zero Coupon Bond)**.** A Zero Coupon Bond with maturity T is a contract that guarantees the payment of one unit of currency at time T:

$$B(t,T) := \mathbb{E}\left[D(t,T)|\, \mathcal{F}_t\right].$$

1.2 The Heath-Jarrow-Morton framework

In this section, we focus on the Heath-Jarrow-Morton (HJM) framework, that enables to overcome the issues occurring in short rate models, such as the ineffective volatility structure that derives from them (see [7]). The HJM model gives indeed freedom in volatility and correlation modelling across the curve, while keeping the possibility of obtaining analytical formulas for bond prices and derivatives. The latter is fundamental, from a practical point of view. Moreover, every short rate model can be equivalently formulated in terms of the forward rate: we will discuss the specific example of Hull-White extended Vasicek in Section 4.1.

There are several ways to present the whole yield curve: the way chosen by the HJM model is in terms of the forward rates. The HJM setup is a continuous term structure model that grants the description of the arbitrage-free dynamics of the full term structure of forward rates $f(t,T)$. It was developed in 1992 by Heath, Jarrow and Morton as an alternative tool for modeling the dynamics of the entire term structure of interest rates. The general setup is based on two assumptions: firstly, it is not possible to gain risk less profit (no-arbitrage condition), secondly, it assumes completeness of the financial markets (every derivative contract is replicable).

The HJM approach has two main positive aspects: it gives a lot of freedom to model different parts of the curve and allows to easily check whether the model is arbitrage free. Moreover, it describes the evolution of the full term structure, unlike the short rate models which only provide a description of the dynamics of the short rate $r(t)$. On the other hand, the forward rate (similarly, the risky version of it) is not a real market content but a fictional rate, in contrast to market models that are driven by real market rates, which are associated to market payoffs, such as Forward Rate Agreements. On top of it, forward rates are derived quantities, as we can see from (1.2.1), and this explains the complexity of their tractability.

The HJM framework comes with two key challenges: first, the difficulty of applying the framework to the market practice. Secondly, the complex computational process brought in by the high dimensional stochastic process of the underlying security. The first obstacle was overcome by the introduction of LIBOR market model. The second difficulty has been improved by the inception of the Cheyette model, which imposes a specific time-dependent structure on the volatility function (see (1.3.2)). This shift converts the dimensional structure, from a high, potentially infinite one, to a finite dimensional framework, and ensures the desirable property of markovianity of the state variables.

Definition 1.2.1 (Istantaneous Forward Rate)**.** The forward rate between time t and T has the following formula:

$$f(t,T) := -\frac{\partial \ln B(t,T)}{\partial T}. \tag{1.2.1}$$

The forward rate between t and T is the rate you can lock in at time t for investing at time T. The reason why this is the case, can be found in [5]. The idea is to build a portfolio at time t that allows us to engage in a forward investment at time $T > t$. In particular, we take a short position of size 1 in $B(t,T)$ and a long position of size $\dfrac{B(t,T)}{B(t,T+\delta)}$ in $B(t,T+\delta)$ where δ is a small number. Since the first investment generates an income of $B(t,T)$ whilst the second costs $B(t,T)$, the two investment offset each others and the netting cost of the portfolio is 0. At time T we will need to pay 1. At time $T+\delta$ we receive $\dfrac{B(t,T)}{B(t,T+\delta)}$. Overall, we have invested 1 at T and more than 1 at $T+\delta$. The continuously compounded rate of interest is therefore:

$$\frac{1}{\delta}\frac{B(t,T)}{B(t,T+\delta)} = -\frac{\log B(t,T+\delta) - logB(t,T)}{\delta}. \tag{1.2.2}$$

In order to avoid arbitrage, the only forward rate for investing at time T, locked at time t is the limit for $\delta \to 0$ of the expression above, called *instantaneous forward rate*.

Remark 1.2.2. By definition, the corresponding bond price is given by:

$$B(t,T) = \exp\left(-\int_t^T f(t,s)ds\right). \tag{1.2.3}$$

Remark 1.2.3. It is clear that we can determine bond prices from forward rates from (1.2.3) and viceversa from (1.2.1). However, whilst bond prices are easy to determine from market data, since the integration in (1.2.3) is not sensitive to small changes in forward rate (it's stable), we can't say the same thing for the differentiation in (1.2.3).

Remark 1.2.4. It is worth noticing that the instantaneous forward rate reduces to the short rate, in the limit of $T \to t$:

$$\lim_{T\to t} f(t,T) = r_t.$$

This is the rate we can lock in at time t for borrowing at time t. Moreover, the dynamics of the instantaneous forward rate is given by the following:

$$dr(t) = df(t,t) + \lim_{u\to t}\frac{d}{du}f(t,u)dt.$$

Theorem 1.2.5 (HJM dynamics for rates)**.** *Under HJM, the dynamics of the forward rate is:*

$$df(t,T) = \mu_f(t,T)dt + \sigma_f(t,T)dW_t^f \tag{1.2.4}$$

where W^f is a Brownian motion under the risk-neutral measure $\mathbb{Q}$. If no arbitrage has to hold, the drift of the forward rate has to be restricted to the following form:

$$\mu_f(t,T) = \sigma_f(t,T)\int_t^T \sigma_f(t,s)ds. \tag{1.2.5}$$

Proof. See [5]. □

We notice that the no-arbitrage property of the dynamics of interest rates is expressed as a link between the local drift and the local diffusion coefficient. In other terms, given the volatility, there is no freedom in choosing the drift. As a consequence, the term structure of volatility determines the forward rate at all times.

Corollary 1.2.6. *The integrated dynamics of the instantaneous forward rate under the risk-neutral measure is:*

$$f(t,T) = f(0,T) + \int_0^t \sigma_f(u,T)\int_u^T \sigma_f(u,s)dsdu + \int_0^t \sigma_f(s,T)dW_s^f. \tag{1.2.6}$$

Proof. The proof is straightforward: we integrate (1.2.4) and substitute the no-arbitrage condition (1.2.5). □

The instantaneous short rate at time t doesn't need to be modeled with a diffusion process but can be derived from the instantaneous forward rate (1.2.6), as the following corollary shows.

Corollary 1.2.7. *The short rate dynamics, given the forward rate dynamics, is given by the following formula:*

$$r(t) = f(t,t) = f(0,T) + \int_0^t \sigma_f(u,t)\int_u^t \sigma_f(u,s)dsdu + \int_0^t \sigma_f(s,t)dW_s^f. \tag{1.2.7}$$

Proof. The proof is obtained directly by taking the limit of $T \to t$ in (1.2.6). □

Remark 1.2.8. If the volatility σ_f is a deterministic function, then both the forward and short rate are normally distributed. The mean computation is straightforward, whilst we use the Ito's isometry to compute the variance.

Since the time t appears inside the integral function and as extreme of integration in the expression (1.2.7), the latter is not a Markov process and the entire path may be necessary to capture the dynamics of the term structure. However, if a particular volatility structure is chosen, the instantaneous forward rate fulfills the Markov property. For instance, we could consider a volatility with separability condition, such as the one proposed by Cheyette in [2], that we will further investigate in the following Section.

Using expression (1.2.6), combined with (1.2.3) we obtain the following formula, that gives an expression for the bond price using the forward rate dynamics.

Corollary 1.2.9. *The bond price can be expressed using the forward rate:*

$$B(t,T) = \frac{B(0,T)}{B(0,t)}\exp\left\{\int_t^T\int_0^t \sigma_f(u,s)\left[-\int_u^T \sigma_f(u,p)dp\right]duds - \int_t^T\left[\int_0^t \sigma_f(u,s)dW(u)\right]ds\right\}. \tag{1.2.8}$$

Proof. The derivation of the formula is straightforward: we use formula (1.2.3), integrating (1.2.6). □

Similarly to what happens with the short rate in (1.2.7), the bond price is not Markovian, in general. The prices at time t depends indeed from the entire path taken from 0 to t. From a pricing perspective, this expression is of a little use. It would be ideal to obtain an expression for the bond price (and similarly, for the short rate) that depends on a few state variables. To accomplish this goal, as previously announced, we need to add some restrictions on the volatility form. This leads us to the Cheyette model.

1.3 The Cheyette Interest Rate Model

The main problem with models developed according to the general HJM framework is that a finite dimension Markovian structure together with a no-arbitrage condition are not satisfied simultaneously, in general. Cheyette argues in [2] that if a particular structure is imposed on the forward rate volatility term, then an HJM model can be expressed entirely by a finite dimension Markovian SDEs system, making it computationally suitable. From this point of view, the class of Cheyette rate models is nothing more than a subset of the general class of HJM models that allows the achievement of an exogenous model of the yield curve. A Markovian Gaussian model is obtained by imposing the separability condition on the deterministic volatility structure of forward rates. We can obtain a more general class, named quasi-Gaussian models, where the separability condition is still imposed, whilst the deterministic requirement is relaxed. The Cheyette Model is a quasi-Gaussian, quadratic volatility model of interest rates introduced to overcome the limitations previously presented of the Heath-Jarrow-Morton framework. By imposing the following special time dependent structure on the forward rate stochastic volatility function, the Cheyette approach leads to dynamics which are Markovian and finite dimensional, in contrast to the general HJM model.

Definition 1.3.1 (Cheyette Volatility)**.**

$$\begin{aligned} \sigma_f(t,T) &:= \tfrac{\alpha_f(T)}{\alpha_f(t)}\sigma_f(t,t) \\ &= \tfrac{\alpha_f(T)}{\alpha_f(t)}\sigma_r(t) \end{aligned} \tag{1.3.1}$$

where $\alpha_f(t) = \mathrm{e}^{-\int_0^t \kappa_f(u)du}$.

Because of its exponential structure, the model in (1.3.1) is called *exponential Cheyette model*.

Remark 1.3.2. It is worth noticing that the factor $\sigma_f(t,t)$ cannot control the volatility with respect to the remaining lifetime $(T-t)$. The latter is addressed by $\frac{\alpha_f(T)}{\alpha_f(t)}$. Moreover, the choice of k_f to be constant implies no flexibility of the volatility structure with respect to the remaining lifetime and this may be an issue in fluctuating markets. For this reason, k_f is usually chosen to be piecewise constant. In this way, we clearly improve the flexibility issue but we also increase the number of degree of freedom of the volatility parametrization.

Remark 1.3.3. If k_f is constant, the parametrization becomes:

$$\sigma_f(t,T) = \sigma_r(t)\mathrm{e}^{-(T-t)k_f}. \tag{1.3.2}$$

In his famous article from 1994 [2], Cheyette states that the evolution of the term structure under the risk neutral measure and the assumptions stated above can be reduced to the following bi-dimensional system of Markovian SDEs.

Theorem 1.3.4 (Markovian SDEs for the short rate)**.** *The evolution of the term structure of the short rate, assuming* (1.3.2) *and under the risk neutral measure, is ruled by two state variables* (X_f, Φ_f)*:*

$$\begin{cases} r(t) = f(0,t) + X_f(t) \\ \mathrm{d}X_f(t) = (-\kappa_f X_f(t) + \Phi_f(t))\,\mathrm{d}t + \sigma_r(t)\mathrm{d}W^f(t), \quad X_f(0) = 0 \\ \mathrm{d}\Phi_f(t) = (\sigma_r(t)^2 - 2\kappa_f\Phi_f(t))\mathrm{d}t, \quad \Phi_f(0) = 0. \end{cases} \tag{1.3.3}$$

Proof. Let's recall (1.2.7):

$$r(t) = f(t,t) = f(0,T) + X_f(t)$$

where

$$X_f(t) := \int_0^t \sigma_f(u,t)\int_u^t \sigma_f(u,s)dsdu + \int_0^t \sigma_f(s,t)dW_s^f.$$

We now differentiate $X_f(t)$ using the Leibnitz integral rule for the deterministic part, and the analogous rule for the stochastic integral. This leads us to the following formula:

$$\begin{aligned}\mathrm{d}X_f(t) &= \left(-\kappa_f \left(\int_0^t \sigma_f(u,t) \int_u^t \sigma_f(u,s) ds du + \int_0^t \sigma_f(s,t) dW_s^f\right) + \int_0^t \sigma_f^2(u,t)\mathrm{d}u\right)\mathrm{d}t \\ &+\sigma_f(t,t)\mathrm{d}W^f(t). \\ &= \left(-\kappa_f X_f(t) + \Phi_f(t)\right)\mathrm{d}t + \sigma_r(t)\mathrm{d}W^f(t).\end{aligned}$$

The Leibnitz integral rule can be used again to find the differential equation expression for $\Phi_f(t)$. □

Remark 1.3.5. $\Phi_f(t)$ is the accumulated variance of the forward rate up to t:

$$\Phi_f(t) = \int_0^t \sigma_f^2(u,t)\mathrm{d}u. \tag{1.3.4}$$

Remark 1.3.6. k_f is the rate's mean reversion.

A great advantage that is reached using Cheyette model is a tractable, analytical form for the zero coupon bond between t and T, that only depends on the state variables' value at time t.

Theorem 1.3.7 (Bond reconstitution formula)**.** *The zero coupon bond at time t with maturity T is given as follows:*

$$B(t,T) = \frac{B(0,T)}{B(0,t)} \exp\left(-0.5\beta_f^2(t,T)\Phi_f(t) - \beta_f(t,T)X_f(t)\right) \tag{1.3.5}$$

where

$$\beta_f(t,T) = \int_t^T \mathrm{e}^{-\kappa_f(u-t)}\mathrm{d}u.$$

Proof. Details of the proof can be found in [2]. □

Chapter 2

The credit spread

The credit intensity $\lambda(t)$ is the analogue to the short rate $r(t)$. This means that it is used as a driver to model those rates that are observed in the market.

There are two main kind of models for the credit intensity: intensity (reduced form) models and firm value (structural) models. The first category treats the credit intensity as it was a short rate, in the sense that it can be assumed to be constant, time varying, or stochastic. In the last case, its dynamics is described using an SDE. The problem with reduced form models is that they give no economic idea of why default happens. Therefore, if we hadn't any other instruments to calibrate the intensity model, they would be totally useless. This can happen, for instance, if a new company enters the market and you want to sell a CDS. In this case, you wouldn't be able to imply any default probabilities. Whereas, with firm value models, you can use balance sheet data to come up with some default probabilities.

For the purpose of this paper, we will follow an intensity model, in particular we will use a Cheyette stochastic model for the credit intensity.

2.1 Probabilistic framework

Before getting into the details of the credit intensity and the related quantities' definitions, we shall first introduce the probabilistic framework that lies behind.
The broadest space were we want to define the random time of default τ is $(\Omega, \mathcal{G}, \mathbb{Q})$ where $\mathbb{Q}$ is the usual risk neutral measure. This space is assumed to be big enough to support a random variable U uniformly distributed on $[0,1]$ and independent of $(\mathcal{F}_t)_t$, where $(\mathcal{F}_t)_t$ denotes the usual filtration that represents the information flow of all the involved default-free market quantities. We also consider a new filtration $(\mathcal{G}_t)_t$ that includes the information flow related to the default time τ together with all the default-linked quantities:

$$\mathcal{G}_t := \mathcal{F}_t \vee \sigma(\{\tau < s\}, s \leq t). \tag{2.1.1}$$

Clearly:

$$\mathcal{F}_t \subset \mathcal{G}_t \subset \mathcal{G}.$$

We now introduce a process λ which is non-negative, $\mathcal{F}_t$, measurable at time t and has integrable sample paths in $(\Omega, \mathcal{G}, \mathbb{Q})$. This process λ is the stochastic intensity of the default time τ, which is defined as follows.

Definition 2.1.1 (Stochastic Intensity)**.**

$$\tau := \inf\left\{t \geq 0 : \exp\left(-\int_0^t \lambda_s ds\right) \leq U\right\}. \tag{2.1.2}$$

Notice that the filtration G_t is generated by the short rate r and intensity λ, but not by the default indicators $\mathbf{1}_{\{\tau > T\}}$.

Remark 2.1.2. λ is therefore the (generalised) inverse of the cumulative intensity of an exponential random variable with mean 1 and independent of λ.

Another important quantity is the hazard rate.

Definition 2.1.3 (Hazard Rate)**.** The hazard rats is the following stochastic process:

$$\Lambda(t) := \int_0^t \lambda_s ds.$$

Remark 2.1.4. The function $t \to \Lambda(t)$ is increasing.

Remark 2.1.5. In other terms, the hazard rate analyzes the likelihood that an entity such as a company or government will survive to a certain point in time based on its survival to an earlier time 0. This can be seen, mathematically speaking, from (2.1.3).

Remark 2.1.6. The following equalities derive directly from the definition of τ:

$$\begin{aligned} &\mathbb{Q}\left(\tau > t \mid \mathcal{F}_t\right) = \mathbb{Q}\left(\int_0^t \lambda_s ds < U \mid \mathcal{F}_t\right) = \exp\left(-\int_0^t \lambda_s ds\right) \\ &\mathbb{Q}\left(\tau > t\right) = \mathbb{E}\left[\exp\left(-\int_0^t \lambda_s ds\right)\right]. \end{aligned} \tag{2.1.3}$$

We can already see from (2.1.3) that the survival probability of an intensity model has the same structure of a bond, where the risk-free rate has been replaced with the intensity λ. This simple remark underlines why credit intensities and rates are modelled in a very similar way. Moreover, it is an important result since it tells us that survival probabilities have the same structure as discount factors. The default intensity λ plays the same role as interest rates. This property will allow us to view default intensities as credit spreads.

Remark 2.1.7. When λ is large, the hazard rate grows faster and therefore the probability of τ to be small increases. Notice that default always occurs in sample paths for which the hazard rate is infinite.

Remark 2.1.8 (λ as a local default probability)**.** It is interesting to notice that λ can be interpreted as a local default probability as well as a credit spread:

$$\begin{aligned} \mathbb{Q}(\tau \in [t, t+dt) \mid \tau > t, \lambda) &= \frac{\mathbb{Q}(\tau \in [t,t+dt) \cap \tau > t \mid \lambda)}{\mathbb{Q}(\tau > t \mid \lambda)} \\ &= \frac{\mathbb{Q}(\tau \in [t,t+dt) \mid \lambda)}{\mathbb{Q}(\tau > t \mid \lambda)} \\ &= \frac{d_t \mathbb{Q}(\tau \leq t \mid \lambda)}{\mathbb{Q}(\tau > t \mid \lambda)} \\ &= \frac{d_t\left(1 - e^{-\int_0^t \lambda_s ds}\right)}{e^{-\int_0^t \lambda_s ds}} \\ &= \lambda_t dt. \end{aligned} \tag{2.1.4}$$

Definition 2.1.9 (Risky rate)**.** The risky rate is given by the sum of the risk-free rate and the default factor λ:

$$r_{risky} := r + \lambda.$$

2.2 The survival measure

In order to price default contingent claims, such as CDSs, we need to introduce a new measure $\bar{\mathbb{Q}}$, called *survival measure*. The numeraire associated to it is the defaultable money market account $\bar{B}(t)$.

Definition 2.2.1 (Defaultable Money Market Account)**.** The defaultable money market account has the following formula:

$$\bar{B}(t) := \mathbb{1}_{(\tau > t)} \exp\left(\int_0^t r_{risky}(s) ds\right).$$

$\bar{\mathbb{Q}}$ is defined throughout its Radon-Nikodym density.

Proposition 2.2.2 (Radon-Nikodym density for $\bar{\mathbb{Q}}$)**.** *The Radon-Nikodym density for $\bar{\mathbb{Q}}$ is given by the following formula:*

$$\left.\frac{\mathrm{d}\bar{\mathbb{Q}}}{\mathrm{d}\mathbb{Q}}\right|_{\mathcal{G}_t} = \exp\left(\int_0^t \lambda(u) \mathrm{d}u\right) \mathbb{1}_{(\tau > t)}.$$

Proof.

$$\left.\frac{d\bar{\mathbb{Q}}}{d\mathbb{Q}}\right|_{\mathcal{F}_t} = \frac{\bar{B}(t)}{\bar{B}(0)}\frac{B(0)}{B(t)}\mathbb{1}_{(\tau>t)} = \exp\left(\int_0^t \lambda(u)du\right)\mathbb{1}_{(\tau>t)}.$$

□

Remark 2.2.3. It is worth noticing that the two measures $\bar{\mathbb{Q}}$ and $\mathbb{Q}$ are not equivalent, since the former assigns zero probability to the events after default. However, $\bar{\mathbb{Q}}$ is absolutely continuous to $\mathbb{Q}$ and therefore Girsanov theorem is applicable. In this way, we can deduce that the drift term of the factor $X_f(t)$ in (1.3.3) doesn't change if we want to study the dynamics of it under $\bar{\mathbb{Q}}$.

Finally, we give a useful formula for credit derivatives pricing.

Proposition 2.2.4. *Let X be a contingent claim. Its price at time t is given by the following formula:*

$$\mathbb{E}_Q\left(\exp\left(-\int_t^T r(u)du\right)X\,\mathbb{1}_{(\tau>t)} \mid \mathcal{G}_t\right) = \mathbb{E}_{\bar{Q}}\left(\exp\left(-\int_t^T r_{risky}(u)du\right)X \mid \mathcal{G}_t\right).$$

Proof. A change of measure combined with (2.2.2) proves the statement. □

2.3 Preliminary definitions

In this section, we will give further preliminary definition before getting into the details of the intensity dynamics under the Cheyette model.

Analogously to what we did in the interest rate case, it is fundamental to present the most simple derivative in the credit market: the risky bond. A risky bond is a bond that is issued by a company that could default and therefore might not be able to pay back the notional at maturity. In this case, at time T (maturity), you will get 1, if no default, or a recovery (*Rec*), if default happens. The price of this bond will be denoted by $\bar{B}(t,T)$ to distinguish it from the usual risk-less zero coupon bond $B(t,T)$. It is important to notice that the default risk has transformed the constant payoff of 1 to a binary payoff.

Definition 2.3.1 (Risky Bond)**.**

$$\mathbf{1}_{\{\tau>t\}}\bar{B}(t,T) := \mathbb{E}\left\{D(t,T)\mathbb{1}_{\{\tau>T\}} \mid \mathcal{G}_t\right\} \tag{2.3.1}$$

where D is the usual stochastic discount factor between two dates and $\mathcal{G}_t$ is the market filtration generated by the default free filtration $\mathcal{F}_t$ and the filtration generated by the default time at every possible instant between 0 and t:

$$\mathcal{G}_t = \mathcal{F}_t \vee \sigma(\{\tau < s\}, s \le t) \tag{2.3.2}$$

The filtration $\mathcal{G}_t$ tells us if the company has defaulted before t and when.

Remark 2.3.2. In (2.3.1), $Rec = 0$. If we want to include the recovery, the payoff changes in case *Rec* is paid at default time τ (2.3.3) or is paid at maturity T (2.3.4):

$$\mathbb{1}_{\{\tau>t\}}\bar{B}(t,T) := \mathbb{E}\left\{D(t,T)\mathbb{1}_{\{\tau>T\}} + Rec\cdot D(t,\tau)\mathbb{1}_{\{\tau\le T\}} \mid \mathcal{G}_t\right\} \tag{2.3.3}$$

$$\mathbb{1}_{\{\tau>t\}}\bar{B}(t,T) := \mathbb{E}\left\{D(t,T)\mathbb{1}_{\{\tau>T\}} + Rec\cdot D(t,T)\mathbb{1}_{\{\tau\le T\}} \mid \mathcal{G}_t\right\}. \tag{2.3.4}$$

Let's now price a risky bond with zero recovery.

Proposition 2.3.3 (Risky Bond Price)**.** *The price of a risky bond with zero recovery is the same price of a risk-free bond where the risk-free rate has been replaced by the risky-rate:*

$$\mathbb{1}_{\{\tau>t\}}\bar{B}(t,T) = \mathbb{1}_{\{\tau>t\}}\mathbb{E}\left[\exp\left(-\int_t^T r_{risky}(s)ds\right) \mid \mathcal{F}_t\right]. \tag{2.3.5}$$

Proof. See A.1. □

Another definition that is particularly significant for credit derivatives is the default factor.

Definition 2.3.4 (Default Factor)**.** The default factor is given by the following formula:

$$S(t,T) := \frac{\bar{B}(t,T)}{B(t,T)}. \tag{2.3.6}$$

The following proposition will be useful for practical purposes that will be presented in Chapter 2.

Proposition 2.3.5. *The default risk factor has the following expression*

$$S(t,T) = \mathbb{E}^T \left[\exp\left(- \int_t^T \lambda_s ds \right) \right]. \tag{2.3.7}$$

where T denotes the T-forward measure.

Proof. The proof is straightforward from the change of measure from the risk neutral measure to the T-forward one. $\bar{B}(t,T)$ indeed becomes:

$$\begin{aligned}
\bar{B}(t,T) &= \mathbb{E}\left[\exp\left(-\int_t^T r(s) + \lambda(s) ds\right) \mid \mathcal{F}_t\right] \\
&= \mathbb{E}\left[\exp\left(-\int_t^T r(s) ds\right) \exp\left(-\int_t^T \lambda(s) ds\right) \mid \mathcal{F}_t\right] \\
&= \tfrac{B(t,T)}{B(T,T)} \mathbb{E}^T\left[\exp\left(-\int_t^T \lambda(s) ds\right) \mid \mathcal{F}_t\right] \\
&= B(t,T)\mathbb{E}^T\left[\exp\left(-\int_t^T \lambda(s) ds\right) \mid \mathcal{F}_t\right].
\end{aligned}$$

The proof then follows directly from (2.3.6). □

2.4 Markovian SDEs for Credit spread

We are now ready to present the istantaneous credit spread modeling using Cheyette framework. Analogously to what we did with the rate section, we first write down the diffusion process followed by the forward credit spread for every maturity T:

$$\mathrm{d}g(t,T) = \mu_g(t,T)\mathrm{d}t + \sigma_g(t,T)\mathrm{d}W^g(t). \tag{2.4.1}$$

where $W^g(t)$ is a brownian motion under $\mathbb{Q}$.

Similarly to what we did for rates in Theorem 2.4.1, we state the following theorem, that gives the expression for the drift term of the forward rate in order to have no arbitrage where both credit and rates are considered.

Theorem 2.4.1 (HJM dynamics for credit spread)**.** *Under HJM, the dynamics of the credit spread is:*

$$\mathrm{d}g(t,T) = \mu_g(t,T)\mathrm{d}t + \sigma_g(t,T)\mathrm{d}W^g(t). \tag{2.4.2}$$

If no arbitrage has to hold, the drift of the credit spread has to be restricted to the following form:

$$\begin{aligned}
\mu_g(t,T) = \sigma_g(t,T) &\int_t^T \sigma_f(t,u)\mathrm{d}u \\
&+ \sigma_f(t,T) \int_t^T \sigma_g(t,u)\mathrm{d}u + \sigma_g(t,T) \int_t^T \sigma_g(t,u)\mathrm{d}u.
\end{aligned} \tag{2.4.3}$$

Proof. See [4]. □

Remark 2.4.2. In (2.4.3), in a completely analogous way as in (2.4.1), the drift of the risky forward rate is described by the risk-free and risky forward rates. Again, the term structure of risk-free and risky volatilities determine the risky forward rate at all times.

Throughout the rest of the paper, we will do the following assumption, similarly to what we did in (1.3.1) for rates:

$$\sigma_g(t,T) := \sigma_\lambda(t)\mathrm{e}^{-(T-t)\kappa_g}. \tag{2.4.4}$$

Theorem 2.4.3. *The instantaneous credit spread process, when correlated to the rates process, can then be represented as a four-dimensional Markov process* $(X_\lambda(t), \Phi_\lambda(t), \Phi_{\lambda r}(t), \Upsilon_{\lambda r}(t))$ *when the no-arbitrage conditions* (2.4.3) *and* (1.2.5) *are satisfied:*

$$\begin{cases} \lambda(t) = g(0,t) + X_g(t) \\ \mathrm{d}X_g(t) = (-\kappa_g X_g(t) + \Phi_g(t) \\ + \left(2 - \frac{\kappa_f - \kappa_g}{\kappa_g}\right) \Phi_{gf}(t) - \frac{\kappa_f - \kappa_g}{\mathrm{e}^{-\kappa_g t}} \Upsilon_{gf}(t) \Big) \, \mathrm{d}t + \sigma_\lambda(t) \mathrm{d}W^g(t) \end{cases} \tag{2.4.5}$$

where

$$\begin{array}{l} \Phi_g(t) := \int_0^t \sigma_g(u,t) du \\ \Phi_{gf}(t) := \int_0^t \sigma_g(u,t)\sigma_f(u,t) du \\ \Upsilon_{gf}(t) := \frac{-1}{k_g} \int_0^t \sigma_g(u,t)\sigma_f(u,t) e^{-k_g u} du. \end{array} \tag{2.4.6}$$

Proof. The proof can be found in [3]. □

Remark 2.4.4. k_g is the mean reversion of the credit intensity.

Theorem 2.4.5. *In the hypothesis of Theorem 2.4.3, when credit and rates are independent, the system* (2.4.5) *becomes:*

$$\begin{cases} \lambda(t) = g(0,t) + X_g(t) \\ \mathrm{d}X_g(t) = (-\kappa_g X_g(t) + \Phi_g(t)) \, \mathrm{d}t + \sigma_\lambda(t) \mathrm{d}W^g(t), \quad X_f(0) = 0 \\ \mathrm{d}\Phi_g(t) = (\sigma_\lambda(t)^2 - 2\kappa_g \Phi_g(t)) \mathrm{d}t, \quad \Phi_f(0) = 0. \end{cases} \tag{2.4.7}$$

Proof. The proof is analogous to the one of Theorem 1.3.4. It is nothing more than a specific case of Theorem 2.4.3. □

Another approach is to model the risky rate r_{risky} directly as a 2 factor Cheyette model:

$$r_{risky}(t) = g(0,t) + r(t) + \lambda(t).$$

In this case the systems (1.3.3) and (2.4.5) become:

$$\begin{cases} dr_t = (Y_{11}(t) + Y_{12}(t) - k_f r_t) dt + \sigma_r(t) dW_t \\ d\lambda_t = (Y_{22}(t) + Y_{21}(t) - k_g \lambda_t) dt + \sigma_\lambda(t) dW_t \\ dY_{11}(t) = (\sigma_r^2(t) - 2k_f Y_{11}(t)) dt \\ dY_{22}(t) = (\sigma_\lambda^2(t) - 2k_g Y_{22}(t)) dt \\ dY_{12}(t) = dY_{21}(t) = (\rho\sigma_r(t)\sigma_\lambda(t) - (k_g + k_f) Y_{12}(t)) dt \end{cases} \tag{2.4.8}$$

where ρ is the correlation between credit and rates, k_g is the mean reversion of the default intensity, k_g is the mean reversion of rates, $\sigma_\lambda(t)$ is the local volatility input (LV_λ) for the default intensity times the intensity and $\sigma_r(t)$ is the local volatility input (LV_r) for the short rates:

$$\begin{array}{l} \sigma_\lambda(t) = LV_\lambda(\lambda(t), t) \cdot \lambda(t) \\ \sigma_r(t) = LV_r(r(t), t). \end{array}$$

Remark 2.4.6. The dynamic of the credit spread is determined by the state variables above. The differential equations in (2.4.8) describing the dynamic of the state variables can be solved independently. Due to this improvement, the simulation of the forward rate becomes extremely efficient.

Remark 2.4.7. In the expression (2.4.8), Y_{11}, Y_{22} correspond, respectively to the processes Φ_f, Φ_g in (1.3.3) and (2.4.7). $Y_{12} = Y_{21}$ is instead a cross term that arises from the correlation ρ between credit and rates and it corresponds to Φ_{gf}. In the latter, though $\rho = 1$. Indeed,

$$\begin{aligned} d\Phi_{gf}(t) &= (\sigma_g(t,t)\sigma_f(t,t) + \textstyle\int_0^t \frac{\partial\sigma_g(u,t)}{\partial t}\sigma_f(u,t) + \frac{\partial\sigma_f(u,t)}{\partial t}\sigma_g(u,t))dt \\ &= (\sigma_\lambda(t)\sigma_r(t) - (k_f + k_g)\textstyle\int_0^t \sigma_f(u,t)\sigma_g(u,t)du)dt \\ &= (\sigma_\lambda(t)\sigma_r(t) - (k_f + k_g)\Phi_{gf}(t))dt. \end{aligned}$$

2.5 Default risk factor reconstitution formula

Let's first have a look at the simplified case where rates and credit are independent.

Proposition 2.5.1. *When credit intensity and interest rate are independent, the default factor has the following form:*

$$S(t,T) = \mathbb{E}\left[\exp\left(-\int_t^T \lambda_s ds\right) \mid \mathcal{F}_t\right]. \tag{2.5.1}$$

Proof.

$$\begin{aligned} \bar{B}(t,T) \quad &= \mathbb{E}\left[\exp\left(-\textstyle\int_t^T r(s) + \lambda(s)ds\right) \mid \mathcal{F}_t\right] \\ &= \mathbb{E}\left[\exp\left(-\textstyle\int_t^T r(s)ds\right)\exp\left(-\textstyle\int_t^T \lambda(s)ds\right) \mid \mathcal{F}_t\right] \\ \text{(indep. } \lambda\text{, r)} \quad &= \mathbb{E}\left[\exp\left(-\textstyle\int_t^T r(s)ds\right) \mid \mathcal{F}_t\right]\mathbb{E}\left[\exp\left(-\textstyle\int_t^T \lambda(s)ds\right) \mid \mathcal{F}_t\right] \\ &= B(t,T)\cdot\mathbb{E}\left[\exp\left(-\textstyle\int_t^T \lambda(s)ds\right) \mid \mathcal{F}_t\right]. \end{aligned}$$

The proof then follows directly from (2.3.6). □

Therefore, we have the following theorem:

Theorem 2.5.2. *(Default factor reconstitution formula) If credit intensity and rates are independent, the default factor between t and T has the following form:*

$$S(t,T) = \frac{S(0,T)}{S(0,t)}\exp\left(-0.5\beta_g^2(t,T)\Phi_g(t) - \beta_g(t,T)X_g(t)\right) \tag{2.5.2}$$

where

$$\beta_g(t,T) = \int_t^T e^{-\kappa_g(u-t)}du.$$

Proof. The proof is identical to the proof of 1.3.7 since λ and r have an analogous dynamics. □

Remark 2.5.3. Because of the 4 processes $(X_\lambda(t), \Phi_\lambda(t), \Phi_{\lambda r}(t), \Upsilon_{\lambda r}(t))$ are Markov, then, there must exists a function $g(t,T,y_1,y_2,y_3,y_4)$ such that:

$$S(t,T) = g(t,T,X_g(t),\Phi_g(t),\Phi_{gf}(t),\Upsilon_{gf}(t)).$$

As expected, in the general form for the default factor, we will have some additional cross factors that multiplies the expression (2.5.2).

Moreover, using the notation in (2.4.8) we are able to prove that, not only the default factor can be written as a function of 4 state variables, but more important, it has an affine form, when we consider σ_λ, σ_r to be deterministic. As already said, this is a desirable property from a computational point of view.

Theorem 2.5.4. *The default factor has an affine form, even in the case of correlation between credit intensity and rates.*

Proof. The proof is done for the case in which σ_λ, σ_r are deterministic. The theorem, however, remains true even in the case of stochastic σ_λ, σ_r. Starting from the SDE for λ in (2.4.8), we obtain the following result:

$$\lambda_t = \lambda_0 e^{-k_g t} + e^{-k_g t}\left[\int_0^t e^{k_g s}(Y_{22}(s) + Y_{21}(s))ds + \int_0^t e^{k_g s}(Y_{22}(s) + Y_{21}(s))dW_s\right] \qquad (2.5.3)$$

where

$$\begin{aligned} Y_{22}(t) &= \textstyle\int_0^t \sigma_\lambda(u)du \\ Y_{12}(t) &= Y_{12}(0)e^{-(k_g+k_f)t} + \rho \textstyle\int_0^t e^{-(k_g+k_f)(t-s)}\sigma_\lambda(s)\sigma_r(s)ds. \end{aligned} \qquad (2.5.4)$$

Therefore, $\lambda_t \mid \lambda_0$ is normally distributed. Analogously, $\lambda_T \mid \lambda_t$ is normally distributed. As a consequence, $\int_t^T \lambda_u du \mid \lambda_0$ is normally distributed as well and therefore

$$X_\lambda := -\int_t^T \lambda_u du \mid \lambda_0 \sim \mathcal{N}\left(M_\lambda, V_\lambda^2\right)$$

for appropriate M_λ and V_λ. Finally,

$$S(t,T) = \mathbb{E}\left[e^{X_\lambda}\right] = e^{M_\lambda + \frac{V_\lambda^2}{2}}.$$

□

2.6 Risky bond reconstitution formula

Similarly to what we did in the previous section, we first have a look at the independence case.

Theorem 2.6.1. *(Risky Bond reconstitution formula) If credit intensity and rates are independent, the risky bond between t and T has the following form:*

$$\begin{aligned} \bar{B}(t,T) &= \tfrac{\bar{B}(0,T)}{\bar{B}(0,t)} \exp\left(-0.5\beta_g^2(t,T)\Phi_g(t) - \beta_g(t,T)X_g(t)\right) \\ &\cdot \exp\left(-0.5\beta_f^2(t,T)\Phi_f(t) - \beta_f(t,T)X_f(t)\right) \end{aligned} \qquad (2.6.1)$$

where

$$\begin{aligned} \beta_g(t,T) &= \textstyle\int_t^T \mathrm{e}^{-\kappa_g(u-t)}\mathrm{d}u \\ \beta_f(t,T) &= \textstyle\int_t^T \mathrm{e}^{-\kappa_f(u-t)}\mathrm{d}u. \end{aligned}$$

Proof. The formula derives directly from the definition of default factor, together with (2.5.2) and (1.3.5). □

Remark 2.6.2. Because of the 4 processes $(X_\lambda(t), \Phi_\lambda(t), \Phi_{\lambda r}(t), \Upsilon_{\lambda r}(t))$ are Markov, then, there must exist a function $f(t, T, y_1, y_2, y_3, y_4)$ such that:

$$\bar{B}(t,T) = f(t, T, X_g(t), \Phi_g(t), \Phi_{gf}(t), \Upsilon_{gf}(t)).$$

Notice that we know the expression of $B(t,T)$ already from (1.3.5). Similarly to what happens for the default factor, in the general form for the bond, we will have some additional cross factors that multiplies the expression (2.6.1).

Similarly, the risky bond as well has an affine form in the general case.

Proposition 2.6.3. *The risky bond has an affine form, even in the case of correlation between credit intensity and rates.*

Proof. The proof is done for the case in which σ_λ, σ_r are deterministic. The theorem, however, remains true even in the case of stochastic σ_λ, σ_r. The proof is totally analogous to the proof of Theorem 2.5.4. From that, we know $\lambda_t \mid \lambda_0$ is normally distributed. As a consequence, $\int_0^T \lambda_u du \mid \lambda_0$

is normally distributed as well. Starting now from the SDE for r in (2.4.8), we obtain the following result:

$$r_t = \lambda_0 e^{-k_f t} + e^{-k_f t}\left[\int_0^t e^{k_f s}(Y_{11}(s) + Y_{21}(s))ds + \int_0^t e^{k_f s}(Y_{11}(s) + Y_{21}(s))dW_s\right] \qquad (2.6.2)$$

where

$$\begin{aligned} Y_{11}(t) &= e^{-k_f t}\textstyle\int_0^t \sigma_r(u)e^{k_f t}du \\ Y_{12}(t) &= Y_{12}(0)e^{-(k_g+k_f)t} + \rho\textstyle\int_0^t e^{-(k_g+k_f)(t-s)}\sigma_\lambda(s)\sigma_r(s)ds. \end{aligned} \qquad (2.6.3)$$

Therefore, $r_t \mid r_0$ is normally distributed. Analogously, $r_T \mid r_t$ is normally distributed. As a consequence, $\int_t^T r_u du \mid r_t$ is normally distributed as well. Since the sum of and therefore

$$X_r := -\int_t^T r_u du \mid r_0 \sim \mathcal{N}\left(M_r, V_r^2\right)$$

for appropriate M_r and V_r. Finally,

$$B(t,T) = \mathbb{E}\left[e^{X_r}\right] = e^{M_r + \frac{V_r^2}{2}}.$$

Now, by the definition of default factor (2.3.6), we can easily derive the form of the bond price:

$$\bar{B}(0,T) = S(t,T)\cdot B(t,T) = e^{M_r + \frac{V_r^2}{2} + M_\lambda + \frac{V_\lambda^2}{2}}. \qquad (2.6.4)$$

□

Chapter 3

CDS calibration and CMDS pricing

3.1 Credit Default Swaps

A Credit Default Swap is a credit security that provides insurance against default risk. There are three figures involved this contract: the buyer (A), the seller (B) and the borrower (C). The buyer (A) buys a CDS from the seller to have insurance against the borrower (C), who borrows money from the buyer (A) of the CDS and will pay it back with accrued interest, in a specific amount of time, if no default occurs. This doesn't come for free: the seller (B) receives a premium from the buyer (A) for agreeing on taking any risk, paid at several intermediate dates between the start and end of the protection period. On the other hand, if the default happens before the protection maturity, the protection buyer (A) is expected to receive a portion of the notional amount, given by $L_{GD} := (1 - Rec)$, called *loss given default*, where Rec is the *recovery rate*.

Getting into the details of what just stated above, the borrower (C) is usually a company or government that issues a bond. The buyer (A) buys this bond from (C) together with protection on it from (B). This protection is useful because, in case of default of company (C), the only amount that (A) will obtain, without having bought any protection, is what is called the *recovery rate* times the principal of the bond and accrued interest on it. Buying protection instead, (A) will ensure itself to obtain the same amount that he would have obtained if no default occurred and no protection was bought, minus, of course, the price of the protection. This is given by the fact that, in case of default, (A) will receive the *recovery rate* Rec from (C) and the *loss given default* $L_{GD} := (1 - Rec)$ from (B).

There are several types of CDSs depending on how the protection payment rates are settled. We will see some examples later on in the chapter.

Definition 3.1.1 (Recovery rate)**.** The recovery rate is the percentage amount of principal and accrued interest, expressed as a percentage of face value, that the buyer of the bond would receive back, if the issuer of the bond defaults. The recovery rate can also be interpreted as the value of a company/government/security when it emerges from bankruptcy or default. An ordinary value for the recovery rate is around 50%.

Remark 3.1.2 (The recovery rate pre-post 2008)**.** It is worth mentioning that the recovery rate pre-crisis used to oscillate between 40% and 50% for financials, whilst Lehman Brothers recovery rate was about 8.625% in immediate auction. Clearly, the higher the recovery rate, the lower the default rate.

Definition 3.1.3 (Loss given default)**.** The loss given default is given by 1 - *recovery rate*. It's the percentage amount of principal and accrued interest, expressed as a percentage of face value, that the buyer of the bond would receive back from the CDS seller, if the issuer of the bond defaults and a CDS on the bond has been previously bought.

The modeling of the CDS price is based on the probability of default as well as the recovery rate, given a credit event happens.

We now give a more formal definition of CDS, using the concepts of *protection leg*, *premium leg* and *accrual term*.

Definition 3.1.4 (CDS)**.** A CDS, as previously said, is a protection contract against default, where the protection seller (B) and the protection buyer (A) agree on the exchanges in (3.1.1), (3.1.2) and (3.1.2). The protection clearly depends on the reference credit entity (C), whose time of default is $\tau \in [T_a, T_b]$ where T_a, T_b are, respectively, the start and and end dates of the contract, therefore the protection. The *premium leg* is the flow of the cash that goes from the protection buyer to the protection seller. This is given, from the point of view of the protection seller, by the sum of the discount factors times the premium times the year fraction over all the payments dates, if defaults occurs after the specific coupon date:

$$PremiumLeg(T_a, T_b) := \sum_{i=a+1}^{b} D(t, T_i)\, \alpha_i R \mathbb{1}_{\{\tau > T_i\}}. \tag{3.1.1}$$

The *protection leg* is the protection L_{GD} that the protection buyer receives in case of default at time τ. This is given by the following formula, seen from the point of view of the protection seller (that explains the minus sign):

$$ProtectionLeg(T_a, T_b) := -\mathbb{1}_{\{T_a < \tau \leq T_b\}} D(t, \tau) L_{GD}. \tag{3.1.2}$$

Finally, there could be an *accrual term*, which is given by the fact that, given an interval $[T_i, T_{i+1}]$, if there is no default, premium times the time fraction between the two extremes of the interval is paid to the protection leg. However, if there is a default $\tau_c \in [T_i, T_{i+1}]$, then the protection buyer should pay the premium times the time fraction between the default and T_i rather than the full year fraction. The payment in this case is given by the loss given default, discounted from the time of default, if the default is in the protection window. This leads to the following formula, seen from the point of view of the protection seller:

$$AccrualTerm(T_a, T_b) := D(t, \tau)\left(\tau - T_{\beta(\tau)-1}\right) R \mathbb{1}_{\{T_a < \tau < T_b\}}. \tag{3.1.3}$$

Remark 3.1.5. The choice of the rate R and the payment dates T_i gives birth do different types of CDSs. The type of CDS described in 3.1.4 is called *running CDS* (RCDS).

To sum up, the cash flows for a running CDS, seen from the protection seller point of view, can be summarised by the following formula:

$$\begin{aligned} \Pi_{\mathrm{RCDS}a,b}(t) := D(t, \tau)\left(\tau - T_{\beta(\tau)-1}\right) R \mathbb{1}_{\{T_a < \tau < T_b\}} \\ + \textstyle\sum_{i=a+1}^{b} D(t, T_i)\, \alpha_i R \mathbb{1}_{\{\tau > T_i\}} \\ - \mathbb{1}_{\{T_a < \tau \leq T_b\}} D(t, \tau) L_{GD}. \end{aligned} \tag{3.1.4}$$

The first term corresponds to the *discounted accrued rate at default*, that compensates the protection seller for the protection provided from the last T_i before default until the default τ. The second term is the *CDS rate premium* and corresponds to the premium received by the protection seller for the protection being provided. Clearly, this term vanishes if default occurs before the CDS maturity date T_b. Finally, the third term is the payment of *protection at default* that the buyer receives from the seller if default happens before the final time of protection T_b.

The pricing formula for the CDS is the expectation of the discounted cash flows in (3.1.4). Clearly, the latter depends on the default time and interest-rate dynamics assumptions. The CDS price at time 0 is therefore given by:

$$\mathrm{CDS}_{a,b}(0, R, L_{GD}) = \mathbb{E}\left[\Pi_{\mathrm{RCDS}_{a,b}}(0)\right]. \tag{3.1.5}$$

Similarly, the CDS price at time t is given by:

$$\mathrm{CDS}_{a,b}(t, R, L_{GD}) = \mathbb{E}\left[\Pi_{\mathrm{RCDS}_{a,b}}(t) \middle| \mathcal{G}_t\right]. \tag{3.1.6}$$

where $\mathcal{G}_t$ is given in (2.3.2).

Remark 3.1.6 (Spread)**.** CDS trading in general is based on a *spread*, which represents the cost a protection buyer has to pay to the protection seller (the premium paid for protection) in exchange for protection. More specifically, it's the amount the protection buyer must pay the protection seller annually over the length of the contract, expressed as a percentage of the notional amount. The value of the CDS increases for the protection buyer if the spread increases.

In the literature, it has been widely assumed the independence between rates and credit. Indeed, unless the company of the CDS is a systemic financial one, the zero correlation assumption is a good approximation (see data in Section 5.2 for more details). Before getting into the details of the CDS price based on the Cheyette model for intensity, we first give CDS model-independent formulae. For the default leg, the idea is to spread the default in every possible small interval: rather than monitoring the default between T_a and T_b, we break the interval in small intervals $[t, t+dt]$. Moreover, we assume independence between the discount factor and the default time. This leads to the following expression.

Theorem 3.1.7. *Assuming independence between rates and credit, we have the following pricing formula for a running CDS:*

$$\begin{aligned}&\mathrm{CDS}_{a,b}\left(t, R, L_{GD}; \mathbb{Q}(\tau \leq \cdot)\right) = -L_{GD}\left[\int_{T_a}^{T_b} B(0,t) d_t \mathbb{Q}(\tau \leq t)\right] + \\ &R\left[\int_{T_a}^{T_b} B(0,t)\left(t - T_{\beta(t)-1}\right) d_t \mathbb{Q}(\tau \leq t) + \sum_{i=a+1}^{b} B\left(0, T_i\right) \alpha_i \mathbb{Q}\left(\tau \geq T_i\right)\right].\end{aligned} \tag{3.1.7}$$

Proof. Recall that

$$\delta_\tau(t) = \begin{cases} 1 & \text{if } t = \tau \\ 0 & \text{if } t \neq \tau \end{cases}$$

We first want to prove the part of the formula related to the protection leg.

$$\begin{aligned}
\mathrm{ProtecLeg}_{a,b}(L_{GD}) \quad & = \mathbb{E}\left[\mathbb{1}_{\{T_a < \tau \leq T_b\}} D(0,\tau) L_{GD}\right] \\
& = L_{GD} \mathbb{E}\left[\int_{t=0}^{\infty} \mathbb{1}_{\{T_a < t \leq T_b\}} D(0,t) \delta_\tau(t) dt\right] \\
(Fubini) \quad & = L_{GD} \int_{t=T_a}^{T_b} \mathbb{E}\left[D(0,t) \delta_\tau(t) dt\right] \\
(Independence\ of\ r\ and\ \tau) \quad & = L_{GD} \int_{t=T_a}^{T_b} \mathbb{E}[D(0,t)] \mathbb{E}\left[\delta_\tau(t) dt\right] \\
& = L_{GD} \int_{t=T_a}^{T_b} B(0,t) \mathbb{Q}(\tau \in [t, t+dt)) \\
& = L_{GD} \int_{t=T_a}^{T_b} B(0,t) d_t \mathbb{Q}(\tau \leq t).
\end{aligned}$$

We now prove the expression of the premium leg.

$$\begin{aligned}
\mathrm{PremiumLeg}_{\ a,b}(R) \quad & = \mathbb{E}\left[D(0,\tau)\left(\tau - T_{\beta(\tau)-1}\right) R \mathbb{1}_{\{T_a < \tau < T_b\}}\right] \\
& + \sum_{i=a+1}^{b} \mathbb{E}\left[D\left(0, T_i\right) \alpha_i R \mathbb{1}_{\{\tau \geq T_i\}}\right] \\
& = \mathbb{E}\left[\int_{t=0}^{\infty} D(0,t)\left(t - T_{\beta(t)-1}\right) R \mathbb{1}_{\{T_a < t < T_b\}} \delta_\tau(t) dt\right] \\
& + \sum_{i=a+1}^{b} \mathbb{E}\left[D\left(0, T_i\right)\right] \alpha_i R \mathbb{E}\left[\mathbb{1}_{\{\tau \geq T_i\}}\right] \\
& = \int_{t=T_a}^{T_b} \mathbb{E}\left[D(0,t)\left(t - T_{\beta(t)-1}\right) R \delta_\tau(t) dt\right] \\
& + \sum_{i=a+1}^{b} B\left(0, T_i\right) \alpha_i R \mathbb{Q}\left(\tau \geq T_i\right) \\
& = \int_{t=T_a}^{T_b} \mathbb{E}[D(0,t)]\left(t - T_{\beta(t)-1}\right) R \mathbb{E}\left[\delta_\tau(t) dt\right] \\
& + \sum_{i=a+1}^{b} B\left(0, T_i\right) \alpha_i R \mathbb{Q}\left(\tau \geq T_i\right) \\
& = R \int_{t=T_a}^{T_b} B(0,t)\left(t - T_{\beta(t)-1}\right) \mathbb{Q}(\tau \in [t, t+dt)) + \\
& + R \sum_{i=a+1}^{b} B\left(0, T_i\right) \alpha_i \mathbb{Q}\left(\tau \geq T_i\right) \\
& = R \int_{t=T_a}^{T_b} B(0,t)\left(t - T_{\beta(t)-1}\right) d_t \mathbb{Q}(\tau \leq t) + \\
& + R \sum_{i=a+1}^{b} B\left(0, T_i\right) \alpha_i \mathbb{Q}\left(\tau \geq T_i\right).
\end{aligned}$$

□

Remark 3.1.8. The formula above is model independent, given the zero coupon bond and survival probabilities observed in the market at time 0.

Remark 3.1.9. Notice that the probability in line 5 is nothing more than the probability density of the random variable τ around t. This can be rewritten as the differential of the cumulative distribution function of τ.

Remark 3.1.10. It is important to notice that the default probability of the reference credit appears in the formula. This is indeed the kind of information that is contained in the protection leg of a CDS. This explains why, as outlined in the introductory part, CDSs can be also seen as a speculation instrument on the probability of default. Using a stripping algorithm on the CDSs on the default curve would allows us to get the *market implied default probabilities* (and survival probabilities).

Remark 3.1.11. Another point is worth noticing is that the expectation that we are computing is a risk-neutral expectation. Therefore, the probability $\mathbb{Q}$ is the risk-neutral probability of default and not the real-world probability $\mathbb{P}$.

Let's now have a look at some variations on the standard running CDS we previously discussed.

Definition 3.1.12 (Forward starting CDS)**.** A forward starting CDS is a credit default swap whose protection starts at some future start date, provided that no default event occurs up till its start date.

Definition 3.1.13 (Postponed payments running CDS)**.** A postponed payments running CDS (PRCDS) is a credit default swap where, in case of default, the protection is not paid on default but on the first date $T_i > \tau$. As a consequence, the accrual term (3.1.3) becomes zero.

A constant maturity default swap (CMCDS) is a CDS where the rate is given by a specific floating-rate CDS rate. Before getting into the details of this contract, we first need to define what a floating rate CDS is.

Definition 3.1.14 (Floating-rate CDS)**.** A floating-rate CDS is a credit default swap where the floating rate $R_i(T_{i-1})$, determined at time T_{i-1}, is paid at each T_i by the protection buyer to the protection seller in change of protection between the interval $[T_{i-1}, T_i]$, if no default has occurred. In particular, in each $[T_{i-1}, T_i]$, $R_i(T_{i-1})$ makes the exchange fair, so the net present value of the premium leg minus the protection leg is zero.

$$[FloatingCDSrate]\ R_i(T_{i-1}) := \frac{L_{GD}\left[\int_{T_{i-1}}^{T_i} B(0,t) d_t \mathbb{Q}(\tau \leq t)\right]}{B\left(0, T_{i-1}\right) \alpha_{i_1} \mathbb{Q}\left(\tau \geq T_{i-1}\right) + B\left(0, T_i\right) \alpha_i \mathbb{Q}\left(\tau \geq T_i\right)}.$$

3.2 CDS Calibration

As already said, CDS are liquid instruments, so they don't need a model to be priced. However, a model is needed to price other more complicated products, such as CMDS rates and derivatives depending on the latter. When the default curve is built from default swaps then the centering ensures that the prices of these default swaps is matched.

We now write the floating and fixed leg of a CDS with maturity T in terms of the intensity and rate, using an expression that will bring us to the formulae used in the numerical part.

Theorem 3.2.1. *The protection and premium legs of a CDS with settle date 0 and maturity T have the following expressions:*

$$ProtectionLeg(0,T) = L_{GD} \cdot \mathbb{E}\left[\int_0^T \lambda(s) \exp\left(-\int_0^s r(u) + \lambda(u) du\right) ds\right] \tag{3.2.1}$$

$$PremiumLeg(0,T) = R \cdot \mathbb{E}\left[\int_0^T \exp\left(-\int_0^s r(u) + \lambda(u) du\right) ds\right] \tag{3.2.2}$$

$$AccrualTerm(0,T) = \mathbb{E}\left[\int_0^T \lambda(s) \cdot s \cdot \exp\left(-\int_0^s r(u) + \lambda(u) du\right) ds\right]. \tag{3.2.3}$$

Proof. From (3.1.1) with $T_a = 0, T_b = T$ we have:

$$\begin{aligned}
PremiumLeg: \quad & \mathbb{E}\left[\textstyle\sum_{i=0}^T D\left(0, T_i\right) \alpha_i R \mathbb{1}_{\{\tau > T_i\}} \mid \mathcal{G}_t\right] \\
& = \textstyle\sum_{i=0}^T \alpha_i R \cdot \mathbb{E}\left[D\left(0, T_i\right) \mathbb{1}_{\{\tau > T_i\}} \mid \mathcal{G}_t\right] \\
(\textit{from}\ (2.3.1)) \quad & = \textstyle\sum_{i=0}^T \alpha_i R \cdot \bar{B}(0, T_i). \\
(|t_{i+1} - t_i| \to 0) \quad & = R \cdot \mathbb{E}\left[\int_0^T \exp\left(-\int_0^s r(u) + \lambda(u) du\right) ds\right].
\end{aligned}$$

We now consider infinitesimal increments; in this way the time becomes continuous and therefore we get from the summation to the integral notation. From (3.1.2) with $T_a = 0, T_b = T$ we have:

$$\begin{aligned}
\textit{ProtectionLeg}: \quad & \mathbb{E}\left[\mathbb{1}_{\{0<\tau\leq T\}} D(t,\tau) L_{GD} \mid \mathcal{G}_t\right]. \\
& = L_{GD}\mathbb{E}\left[\mathbb{1}_{\{0<\tau\leq T\}} \exp\left(-\int_0^\tau r_s ds\right) \mid \mathcal{G}_t\right] \\
\textit{(From [8])} \quad & = 1_{\{\tau>0\}}\mathbb{E}\left(\int_0^T \mathbb{1}_{\{0<s\leq T\}} \lambda_s \exp\left(-\int_0^s (r_u+\lambda_u)\,du\right) ds \mid \mathcal{F}_t\right) \\
& = 1_{\{\tau>0\}}\mathbb{E}\left(\int_0^T \lambda_s \exp\left(-\int_0^s (r_u+\lambda_u)\,du\right) ds \mid \mathcal{F}_t\right).
\end{aligned}$$

From (3.1.3) with $T_a = 0, T_b = T$ we have:

$$\begin{aligned}
\textit{AccrualTerm}: \quad & \mathbb{E}\left[D(t,\tau)\left(\tau - T_{\beta(\tau)-1}\right) R\mathbb{1}_{\{0<\tau<T\}}\right]. \\
\textit{(From [8])} \quad & = 1_{\{\tau>0\}} R \cdot \mathbb{E}\left(\int_0^T \mathbb{1}_{\{0<s\leq T\}} \lambda_s \cdot s \cdot \exp\left(-\int_0^s (r_u+\lambda_u)\,du\right) ds \mid \mathcal{F}_t\right) \\
& = 1_{\{\tau>0\}} R \cdot \mathbb{E}\left(\int_0^T \lambda_s \cdot s \cdot \exp\left(-\int_0^s (r_u+\lambda_u)\,du\right) ds \mid \mathcal{F}_t\right).
\end{aligned}$$

□

The premium leg is, when approximated, a sum of risky bonds with different maturities. We know from Theorem (2.6.1) that in credit Cheyette the risky bond price is available analytically as a function of state factors at time t. The protection leg has instead a complicated term that is worth studying in more detail. In order to do so, we discretise the formula (3.3.2) and obtain the following theorem. Numerically speaking, the time step can be discretised and the integrals approximated by Riemann sums.

In our experiments, we will consider, for now, Postponed Payments Running CDS, therefore, from now onward we will consider the Protection and Premium legs only in modeling the CDS.

Theorem 3.2.2. *The protection and premium discretized legs of a CDS that starts at time 0 with maturity T, assuming $R = \alpha_i = L_{GD} = 1$, with pay dates $T_1, ..., T_n = T$ have the following expressions:*

$$\begin{aligned}
\textit{ProtectionLeg}(0,T_n) \quad & \approx \mathbb{E}\left[\sum_0^{n-1} \exp\left(-\int_0^{T_{i+1}} r(u)du\right)\cdot\right. \\
& \left.\left[\exp\left(-\int_0^{T_i}\lambda(u)du\right) - \exp\left(-\int_0^{T_{i+1}}\lambda(u)du\right)\right]\right] \\
& = \sum_0^{n-1}\left[\mathbb{E}\left[\exp\left(-\int_0^{T_{i+1}} r(u)du\right)\cdot\exp\left(-\int_0^{T_i}\lambda(u)du\right)\right] - \bar{B}(0,T_{i+1})\right]. \\
\\
\textit{PremiumLeg}(0,T_n) \quad & \approx \sum_0^n \bar{B}(0,T_i).
\end{aligned} \tag{3.2.4}$$

The first term inside the sum of the floating leg is not available analytically. The idea is now to approximate this term, assuming independence of rates and intensity. This approach doesn't take into account an eventual convexity that could be present. Therefore, we will then come back to this approximation in Chapter 4. Before getting into the details of the approximation, we first precise the following notation:

$$S(t,T)^{T_1} := \mathbb{E}^{T_1}\left[\exp\left(-\int_t^T \lambda(u)du\right)\right] \tag{3.2.5}$$

where $T_1 > T$ and the exponent denotes the T_1-forward measure we are referring to.

The approximation is the following:

$$
\begin{aligned}
&\mathbb{E}\left[\exp\left(-\int_0^{T_{i+1}} r(u)du\right)\cdot\exp\left(-\int_0^{T_i}\lambda(u)du\right)\right]\\
&=\mathbb{E}^{T_{i+1}}\left[\exp\left(-\int_0^{T_{i+1}} r(u)du\right)\cdot\exp\left(-\int_0^{T_i}\lambda(u)du\right)\right]\\
&=B(0,T_{i+1})\cdot\mathbb{E}^{T_{i+1}}\left[\exp\left(-\int_0^{T_i}\lambda(u)du\right)\right]\\
&=B(0,T_{i+1})\cdot S(0,T_i)^{T_{i+1}}\\
&\approx B(0,T_{i+1})\cdot S(0,T_i)^{T_i}\\
&=\bar{B}(0,T_i)\cdot B(0,T_i,T_{i+1}).
\end{aligned}
\tag{3.2.6}
$$

The approximation is therefore:

$$
S(0,T_i)^{T_{i+1}}\approx S(0,T_i)^{T_i} \tag{3.2.7}
$$

Remark 3.2.3. Note that the approximation (3.2.7) is equivalent to the assumption of independence between rates and credit.

Finally, the formulae for the Protection and Premium leg in Theorem 3.2.2 become:

$$
\begin{aligned}
\text{ProtectionLeg}(0,T_n)\quad&\approx\sum_0^n\left[\mathbb{E}\left[\exp\left(-\int_0^{T_{i+1}} r(u)du\right)\cdot\exp\left(-\int_0^{T_i}\lambda(u)du\right)\right]-\bar{B}(0,T_{i+1})\right]\\
&=\sum_0^{n-1}\left[B(0,T_{i+1})\cdot S(0,T_i)^{T_i}-\bar{B}(0,T_{i+1})\right]\\
&=\sum_0^{n-1}\left[B(0,T_{i+1})\cdot S(0,T_i)^{T_i}-B(0,T_{i+1})S(0,T_{i+1})^{T_{i+1}}\right].\\
\text{PremiumLeg}(0,T_n)\quad&\approx\sum_0^n\bar{B}(0,T_i)=\sum_0^n B(0,T_i)S(0,T_i)^{T_i}.
\end{aligned}
\tag{3.2.8}
$$

Proposition 3.2.4. *The following formula links the default factor we want to calibrate to the risky bonds that appear in the CDS legs formulae.*

$$
\bar{B}(0,T)=\mathbb{E}\left[\exp\left(-\int_0^{t_f} r_s+\lambda_s ds\right)S(t_f,T)B(t_f,T)\right] \tag{3.2.9}
$$

where $B(t_f,\ T)$ is given by (1.3.5) *and is calibrated to the risk-free bonds that are on the yield curve, in a similar way to what we are now doing with the CDSs on the default curve.*

3.2.1 The Calibration Algorithm

We now want to calibrate the CDSs that lie on the default curve. We call t_f the final settlement date for the CMDS, i.e. the last date in which the swap buyer pays the floating rate to the swap seller. We want to develop a calibration method that avoids us to simulate the intensity (and the rate) up to maturity. The idea is to simulate the diffusion processes up to t_f only and then use the reconstitution formulae (1.3.5) and (2.5.2) to calibrate both the bonds and the CDSs on the default curve. In particular the default factor between 0 and T will be calibrated for $t > t_f$, whilst there are several way to calibrate the CDSs for $t \leq t_f$, for instance the drift of the simulated hazard rate can be centered directly.

The first thing worth noticing is that the default curve has only a finite set of building instruments with maturities $T_1, ..., T_n$ that we want to calibrate our model to. These CDS are therefore not enough to center the default factor at intermediate time points. The idea here is to assume that the default factor is piece-wise constant, i.e. for each maturity $T_i \in \{T_{t_f}, ..., T_n\}$, S_t is constant along the interval $[T_{i-1}, T_i]$ and t_f is, again, the final settle date for the CMDS rate we want to price. This approach will allow us to interpolate the default factor between t_f and T, in the following way:

$$
\frac{S(0,T)}{S(0,t_f)}=\exp\left(-\sum_{t_f}^{T_n}S_i(t_{i+1}-\max(t_f,t_i))-S_{T_n}(T-T_n)\right). \tag{3.2.10}
$$

Recall that the first step we are trying to complete is the calibration one. We want to calibrate the model using CDSs on the default curve. Therefore, we will need the values for $B(0,T), \bar{B}(0,T)$ with $T > t_f$.

Remark 3.2.5. Notice that, using Proposition 3.2.9, we need the credit and rate to be simulated only up to t_f.

Algorithm 1 Calibration algorithm for $t \geq t_f$

---

t_f is the final settlement date of the CMDS we want to price.
Simulate λ, r up to t_f.
Calibrate the drift of λ, r up to t_f, using the risk-free bond on the yield curve and the CDSs on the default curve that have maturities $T_i \leq t_f$.
$T_1 < ... < T_n$ are the CDS maturities on the default curve with $T_i > t_f$.
for i in [1, ..., n] **do**
    Make a guess for S_i.
    $T_{i1} < ... < T_{in}$ are the i^{th}-CDS pay dates.
    for j in [1, ..., n] **do**
        Compute $\dfrac{S(0,T_{ij})}{S(0,t_f)}$ *using* (3.2.10).
        Compute $S(t_f,T_{ij})$ *using* (2.5.2).
        Compute $\bar{B}(0,T_{ij})$ *using* (3.2.9).
        Compute $B(0,T_{ij})$ *interpolating the bonds on the curve.*
    end for
    Now compute the protection and premium leg for the CDS starting at 0 *using* (3.2.8).
    We compare the obtained price with the one on the corresponding CDS on the default curve.
    if the 2 prices are close enough **then** $T_i \to T_{i+1}$.
    else make another guess for S_i and repeat.
    end if
end for

The algorithm that refers to the calibration of the model to the CDS on the default curve, is shown in Algorithm 1. Now we can compute the CMDS rate and therefore price the Swap contract.

3.3 CMDS rate

At this point, we have calibrated the default factor to the CDSs on the default curve and we can use the calibrated default factor to determine the CMDS rate that appears in the Multi Swap product, using similar formulae to the ones we stated in the previous section for CDSs.

Theorem 3.3.1. *The protection and premium leg of a CDS that settles at t and has maturity T have the following expressions:*

$$\begin{aligned} ProtectionLeg(t,T) &= L_{GD} \cdot \mathbb{E}\left[\int_t^T \lambda(s) \exp\left(-\int_t^s r(u) + \lambda(u) du\right) ds \mid \mathcal{F}_t\right] \\ PremiumLeg(t,T) &= \mathbb{E}\left[\int_t^T R \exp\left(-\int_t^s r(u) + \lambda(u) du\right) ds \mid \mathcal{F}_t\right]. \end{aligned} \tag{3.3.1}$$

Proof. The proof is analogous to Theorem 3.2.1. □

Theorem 3.3.2. *The Floating and fixed discretised legs of a CMDS with maturity T have the following expressions, assuming $L_{GD} = 1$:*

$$\begin{aligned} ProtectionLeg(t,T) &\approx \mathbb{E}\left[\sum_t^{n-1} \exp\left(-\int_t^{T_{i+1}} r(u)du\right)\right. \\ &\quad \cdot \left.\left[\exp\left(-\int_t^{T_i} \lambda(u)du\right) - \exp\left(-\int_t^{T_{i+1}} \lambda(u)du\right)\right] \mid \mathcal{F}_t\right] \\ &= \sum_t^{n-1}\left[\mathbb{E}\left[\exp\left(-\int_t^{T_{i+1}} r(u)du\right) \cdot \exp\left(-\int_t^{T_i} \lambda(u)du\right) \mid \mathcal{F}_t\right] - \bar{B}(t,T_{i+1})\right] \\ PremiumLeg(t,T) &\approx \sum_t^n R\bar{B}(t,T_i). \end{aligned} \tag{3.3.2}$$

Here we have the following constant maturity default swap rate i.e. the fair default swap spread for a default swap starting at t and ending at T:

$$\begin{aligned}
R_{\text{CMDS}}(t,T) &= \frac{L_{GD}\cdot \mathbb{E}\left[\int_t^T \lambda(s)\exp\left(-\int_t^s r(u)+\lambda(u)du\right)ds \mid \mathcal{F}_t\right]}{\mathbb{E}\left[\int_t^T R\exp\left(-\int_t^s r(u)+\lambda(u)du\right)ds \mid \mathcal{F}_t\right]} \\
\text{(discretisation)} &\approx \frac{\sum_t^{n-1}\left[\mathbb{E}\left[\exp\left(-\int_t^{T_{i+1}} r(u)du\right)\cdot\exp\left(-\int_t^{T_i}\lambda(u)du\right)\mid \mathcal{F}_t\right]-\bar{B}(t,T_{i+1})\right]}{\sum_t^n \bar{B}(t,T_i)} \\
\text{(no approximation)} &= \frac{\sum_t^{n-1}\left[B(t,T_{i+1})\cdot S(t,T_i)^{T_{i+1}}-\bar{B}(t,T_{i+1})\right]}{\sum_t^n \bar{B}(t,T_i)} \\
\text{(approximation)} &\approx \frac{\sum_t^{n-1}\left[B(t,T_{i+1})\cdot S(t,T_i)^{T_i}-\bar{B}(t,T_{i+1})\right]}{\sum_t^n \bar{B}(t,T_i)}.
\end{aligned} \tag{3.3.3}$$

After having priced the CMDS rate, we will use it to price a security that is on the Deutsche Bank library, called Multi Swap.

Definition 3.3.3 (Swap)**.** Agreement between two parties to exchange cash flows of two different investment positions at specified future times according to certain agreed-upon terms of rules.

Definition 3.3.4 (Cancelable Swap)**.** A type of swap where one or both parties to the swap has, or have, the right but not the obligation to terminate the swap on a specific date before maturity. It can be either a callable swap or a puttable swap.

Definition 3.3.5 (Callable Swap)**.** A type of cancelable swap where the fixed-rate payer can terminate the swap on a predetermined date prior to maturity.

Definition 3.3.6 (Multi Callable Swap or Bermudan Callable Swaps)**.** A type of callable swap where the fixed-rate payer can terminate the swap on more than two predetermined dates in the future prior to maturity. Effectively, the swap issuer sells a number of options each of which gives the buyer the right to exercise on a predetermined date if favorable.

Definition 3.3.7 (Multi Swap)**.** A Multi Swap is a Bermudan callable swap where the underlying swap has 2 legs, each of which can have a coupon being a linear combination of arbitrary many rates plus one spread.

For now, we neglect the convexity correction. However, the model may be improved introducing some sort of convexity adjustment to take into account the correlation between rates and credit, that hasn't been considered by assuming (3.2.7). The approach we carry on here is quite different from the one in [9]. The idea here is to assume rate and credit intensity to follow a Hull-White extended Vasicek model with correlation ρ and compute (3.2.5) directly. The Hull-White extended Vasicek model allows us to obtain an analytical formula for (3.2.5) with no approximation. We will then need some function that transforms the stochastic volatility of the Cheyette model in a deterministic one. Before deriving the analytical formula for the default factor under a different forward measure, we recall basic definitions and results on the Hull-White model.

Chapter 4

Convexity adjustment

4.1 The Hull-White extended Vasicek model

We now recall the Hull-White extended Vasicek model that we will use extensively in the following section to present a convexity adjustment for our calibration and pricing model.

The Hull-White extended Vasicek model is a short rate model, described by the following SDE:

$$dr(t) = [\theta(t) - \alpha(t)r(t)]dt + \sigma(t)dW(t). \tag{4.1.1}$$

Remark 4.1.1. In the simplified case where $\theta(t) = \theta$ and $\alpha(t) = \alpha$, (4.1.1) can be solved using Ito's lemma, obtaining:

$$r(t) = e^{-\alpha t}r(0) + \frac{\theta}{\alpha}\left(1 - e^{-\alpha t}\right) + \sigma e^{-\alpha t}\int_0^t e^{\alpha u}dW(u) \tag{4.1.2}$$

and therefore:

$$r(t) \sim \mathcal{N}\left(e^{-\alpha t}r(0) + \frac{\theta}{\alpha}\left(1 - e^{-\alpha t}\right), \frac{\sigma^2}{2\alpha}\left(1 - e^{-2\alpha t}\right)\right). \tag{4.1.3}$$

Remark 4.1.2. In the case in which we assume α only to be constant, we have a similar expression for the short rate r and an analogous distribution for it:

$$r(t) = e^{-\alpha t}r(0) + \int_0^t e^{\alpha(s-t)}\theta(s)ds + \sigma e^{-\alpha t}\int_0^t e^{\alpha u}dW(u) \tag{4.1.4}$$

$$r(t) \sim \mathcal{N}\left(e^{-\alpha t}r(0) + \int_0^t e^{\alpha(s-t)}\theta(s)ds, \frac{\sigma^2}{2\alpha}\left(1 - e^{-2\alpha t}\right)\right). \tag{4.1.5}$$

As (4.1.5) shows, the process of the short rate under the Hull-White model behaves asymptotically as a stationary Normal distribution with mean and variance obtained by taking $t \to \infty$. This brings us to the following remark.

Remark 4.1.3 (Hull-White parameters' interpretation)**.** It is fundamental to give an idea of the heuristics of the model before getting into the next steps.

- $\frac{\theta(t)}{\alpha}$ is the *long term mean* level: the short rate r will tend to revert to this level after a long time. This can be easily seen by taking the limit $t \to \infty$ in the mean term in (4.1.5). The largest $\frac{\theta(t)}{\alpha}$, the highest the rates will be in the future on average.
- α is the *speed of mean reversion.* It says how fast the short rate r reverts to $\frac{\theta(t)}{\alpha}$. The bigger α is, the fastest the process converges to the stationary state. In other terms, α explains how fast the trajectories of the short rate r align with the long term mean $\frac{\theta(t)}{\alpha}$.
- $\sigma(t)$ is the *instantaneous volatility.* The bigger this value is, the more noise there is in the simulation.
- $\frac{\sigma(t)^2}{2\alpha}$ is the *long term variance.* It is worth noticing that $\sigma(t)$ and α fight each others in this expression, therefore it is not sufficient to have a small $\sigma(t)$ in order to have a small long term variance.

We first prove that the Hull-White model can be obtained from the HJM model when the volatility satisfies the separability condition.

Proposition 4.1.4. *The Hull-White extended Vasicek model can be equivalently formulated in term of the forward rate.*

Proof. Assume

$$\sigma_f(t,T) := \psi(t)\phi(T). \tag{4.1.6}$$

Notice that Cheyette volatility (1.3.2) satisfies (4.1.6) with $\psi(t) = \sigma_r(t)\mathrm{e}^{tk_f}$ and $\phi(T) = \mathrm{e}^{-Tk_f}$. Therefore:

$$G(t) := f(0,t) + \phi(t)\int_0^t \phi^2(u)\int_u^t \psi(s)dsdu$$

If we differenciate the above equation:

$$\begin{aligned} dr(t) &= r'(t)dt + \psi'(t)\textstyle\int_0^t \phi(s)dW(s) + \phi(t)\psi(t)dW(t) \\ &= \left(G'(t) + \psi'(t)\tfrac{r(t)-G(t)}{\psi(t)}\right)dt + \psi(t)\phi(t)dW(t) \end{aligned}$$

We now define

$$\theta(t) := G'(t) - \frac{\psi'(t)}{\psi(t)}G(t), \quad \alpha(t) := -\frac{\psi'(t)}{\psi(t)}, \quad \sigma(t) := \psi(t)\phi(t),$$

Therefore we obtain the Hull-White extended Vasicek dynamic for the short rate r:

$$dr(t) = [\theta(t) - \alpha(t)r(t)]dt + \sigma(t)dW(t).$$

□

In [10] we can find the following theorem:

Theorem 4.1.5 (Bond price under Hull-White extended Vasicek)**.** *The price of a bond where the short rate r follows* (4.1.1)*, is given by the following formula:*

$$B(t,T) = \exp\left(-A(t,T) - C(t,T)r_t\right) \tag{4.1.7}$$

where

$$\begin{aligned} A(t,T) &= \textstyle\int_t^T \left[\alpha(s)C(s,T) - \frac{1}{2}\sigma^2(s)C^2(s,T)\right]ds \\ C(t,T) &= \textstyle\int_t^T \exp\left(-\int_t^s \theta(u)du\right)ds. \end{aligned} \tag{4.1.8}$$

Remark 4.1.6. It is worth noticing that the introduction of time dependency in the Vasicek SDE's parameters, the bond price formula remains of an affine form. The fact that log bond prices are linear functions of the spot rate (and eventually other state variables) is quite convenient from a computational point of view since affine models lead to tractable pricing formulae as the ones we saw in Sections 2.5 and 2.6.

4.2 The convexity adjustment

The goal of the current section is to find the analytical formula for (3.2.5) to be used in (3.2.6) and therefore relax the independence between rates and credit assumption. There is empirical evidence of dependence between the interest rates and credit spreads (see [11] and [12]): for example, when the market recognise a decay of solvency within a company (or within a set of companies), risk-free bonds are bought bringing interest rates down. The first effect would broaden the credit spread and the second would decrease the interest rate. Therefore, it becomes of fundamental importance to be able to account for dependence between rates and credit to have a general, reliable model.

The idea here is to assume λ, r to follow two correlated Hull-Factor extended Vasicek model and compute $S(t,T)^{T_1}$ in (3.2.5) directly as a function of $S(t,T)^T$. The Gaussian model allows us to find an analytical formula for $S(t,T)^{T_1}$, which is highly convenient for the purpose of our paper. We derived it in the following Proposition.

Proposition 4.2.1. *Suppose the rate r and the intensity λ follow the following dynamics, under the risk-neutral measure:*

$$\begin{cases} dr_t = (\theta_r(t) - k_r(t)r_t)\,dt + \sigma_r(t)dW_t \\ d\lambda_t = (\theta_\lambda(t) - k_\lambda(t)\lambda_t)\,dt + \sigma_\lambda(t)dW_t \\ d\lambda dr = \rho dt \end{cases} \tag{4.2.1}$$

Then we have the following expression for the default factor between t and T, under the T_1-forward measure, where $T_1 > T$:

$$S(t,T)^{T_1} = S(t,T)\exp\left(-\int_t^T\int_t^u \exp\left(-\int_s^u k_\lambda(w)dw\right)\epsilon(s,T_1)dsdu\right) \tag{4.2.2}$$

where

$$\epsilon(s,T_1) = -\sigma_r(s)\sigma_\lambda(s)\rho\int_s^{T_1}\exp\left(-\int_s^k k_r(w)dw\right)dk. \tag{4.2.3}$$

Proof. Details can be seen in A.2. □

Remark 4.2.2. It is important to underline that the volatility $\sigma_\lambda(t)$ that appears in (4.2.1) is not the same that appears in (2.4.4). Whilst the first one is deterministic, the second one is stochastic. This is what makes the Cheyette model a quasi-Gaussian, but not Gaussian, one. In order to use the convexity adjustment we are proposing, we need, as already said, a function that transforms our stochastic Cheyette volatility in a deterministic one. A natural idea would be to look for the $\sigma_\lambda(t)$ in Hull-White extended Vasicek that matches the Cheyette variance.

Chapter 5

Numerical results

5.1 Data description

In this section we analyse the data we used for our experiments. We recall that the idea is to price a Constant Maturity Default Swap Rate and the related Multi Swap contract, using the formulae of Chapter 2 and Chapter 3 and calibrate the model to the CDSs that are on the default curve. This first step is done without using any convexity adjustment, therefore it will be fundamental to analyse the impact of the correlation on the price to understand whether it will be necessary to implement the convexity adjustment as well in further works.

In Table 5.1[1], we present the CDSs data that we will be using to initially assess the impact of correlation on CDSs pricing. There are 21 CDSs on the default curve, but we will actually analyse only 3 of them (the ones with maturities, 2021-05-18, 2025-05-18, 2030-05-18, respectively). These CDSs have the same issue date 2000-05-18 and differ for the maturity date, that ranges from a 1y to a 21y CDS. Moreover, they share the same spread and recovery, respectively, 0.0602 and 0.4.

In Table 5.2[2], we present the CDSs data that we will be using for the calibration. There are 25 CDSs on the default curve and they all have the same issue date 2000-04-19, that corresponds to the evaluation date as well. They differ for the maturity date, that ranges from a 1y to a 25y CDS. The frequency is set as SEMI, which means that the cash flows related to the protection leg happen every 6 months for each CDS from the issue date to the maturity date. Finally, they share the same spread and recovery, respectively, 0.0255 and 0.15.

As it has been remarked in 3.1.10, it is possible to extract default probabilities from CDSs prices. This brings us to the last column in Table 5.1 and Table 5.2, that is the result of bootstrapping the default probabilities related to the CDSs on the default curve.

In the Multi Swap that we are trying to reprice (see Table 5.3) with our new model, there are several input values that is worth explaining. First, the Initial and Final Notional are, respectively, the notional used for the first and last exchange. On the receive leg a positive amount is received, on the pay leg a positive amount is paid. The effective date corresponds to the start date of the leg. The roll date is the first roll date; subsequent roll dates follow this date with frequency determined by the Frequency input. We then have the Rate Index Type. In our case, the payer leg's Rate Index Type is CMDS, which is the Constant Maturity Credit Default Swap rate. This is the rate at which a par default swap, beginning at *set date + rate starting time*, and ending at *set date + rate starting time + rate term*, prices to zero. In this case, it's the rate at which a default swap with notional 100, beginning on the 2006-01-03 and ending at 2026-01-03, prices to zero. The Exercise Start date is the first date that exercise can occur. Analogously, the Exercise End date is the last date that exercise can occur.

5.2 Assumption of independence between rates and credit

As previously discussed in Section 4.2, it is not always correct to assume independence between credit and rates. In particular, systemic entities CDSs might have a non-neglectable dependency with risk-free rates, and the same applies to long maturity CDSs. In this second section of the

[1] Data source: Deutsche Bank internal database.
[2] Data source: Deutsche Bank internal database.

Issue Date	Roll Date	Maturity Date	Spread	Recovery	Frequency	Default Probabilities
2000-05-18	2000-05-18	2000-11-18	0.0602	0.4	SEMI	0.049967109
2000-05-18	2000-05-18	2021-05-18	0.0602	0.4	SEMI	0.096685879
2000-05-18	2000-05-18	2021-11-18	0.0602	0.4	SEMI	0.141826995
2000-05-18	2000-05-18	2022-05-18	0.0602	0.4	SEMI	0.184029092
2000-05-18	2000-05-18	2022-11-18	0.0602	0.4	SEMI	0.224800572
2000-05-18	2000-05-18	2023-05-18	0.0602	0.4	SEMI	0.262913884
2000-05-18	2000-05-18	2023-11-18	0.0602	0.4	SEMI	0.300119739
2000-05-18	2000-05-18	2024-05-18	0.0602	0.4	SEMI	0.334699987
2000-05-18	2000-05-18	2024-11-18	0.0602	0.4	SEMI	0.367563952
2000-05-18	2000-05-18	2025-05-18	0.0602	0.4	SEMI	0.398797249
2000-05-18	2000-05-18	2025-11-18	0.0602	0.4	SEMI	0.428640714
2000-05-18	2000-05-18	2026-05-18	0.0602	0.4	SEMI	0.456695931
2000-05-18	2000-05-18	2026-11-18	0.0602	0.4	SEMI	0.483800102
2000-05-18	2000-05-18	2027-05-18	0.0602	0.4	SEMI	0.509140353
2000-05-18	2000-05-18	2027-11-18	0.0602	0.4	SEMI	0.533622787
2000-05-18	2000-05-18	2028-05-18	0.0602	0.4	SEMI	0.556635886
2000-05-18	2000-05-18	2028-11-18	0.0602	0.4	SEMI	0.578978223
2000-05-18	2000-05-18	2029-05-18	0.0602	0.4	SEMI	0.599417144
2000-05-18	2000-05-18	2029-11-18	0.0602	0.4	SEMI	0.619495953
2000-05-18	2000-05-18	2030-05-18	0.0602	0.4	SEMI	0.638266855
2000-05-18	2000-05-18	2030-11-18	0.0602	0.4	SEMI	0.656112171

Table 5.1: CDS on the Default_Curve_1

Issue Date	Roll Date	Maturity Date	Spread	Recovery	Frequency	Default Probabilities
2000-04-19	2000-04-19	2001-04-19	0.0255	0.15	SEMI	0.02897419
2000-04-19	2000-04-19	2002-04-19	0.0255	0.15	SEMI	0.05701834
2000-04-19	2000-04-19	2003-04-19	0.0255	0.15	SEMI	0.11062939
2000-04-19	2000-04-19	2004-04-19	0.0255	0.15	SEMI	0.13627006
2000-04-19	2000-04-19	2005-04-19	0.0255	0.15	SEMI	0.15398909
2000-04-19	2000-04-19	2006-04-19	0.0255	0.15	SEMI	0.16116978
2000-04-19	2000-04-19	2007-04-19	0.0255	0.15	SEMI	0.18534235
2000-04-19	2000-04-19	2008-04-19	0.0255	0.15	SEMI	0.20907153
2000-04-19	2000-04-19	2009-04-19	0.0255	0.15	SEMI	0.23172573
2000-04-19	2000-04-19	2010-04-19	0.0255	0.15	SEMI	0.25373747
2000-04-19	2000-04-19	2011-04-19	0.0255	0.15	SEMI	0.27523258
2000-04-19	2000-04-19	2012-04-19	0.0255	0.15	SEMI	0.29610897
2000-04-19	2000-04-19	2013-04-19	0.0255	0.15	SEMI	0.31638362
2000-04-19	2000-04-19	2014-04-19	0.0255	0.15	SEMI	0.33629710
2000-04-19	2000-04-19	2015-04-19	0.0255	0.15	SEMI	0.35528668
2000-04-19	2000-04-19	2016-04-19	0.0255	0.15	SEMI	0.37375542
2000-04-19	2000-04-19	2017-04-19	0.0255	0.15	SEMI	0.39179357
2000-04-19	2000-04-19	2018-04-19	0.0255	0.15	SEMI	0.40931215
2000-04-19	2000-04-19	2019-04-19	0.0255	0.15	SEMI	0.42632613
2000-04-19	2000-04-19	2020-04-19	0.0255	0.15	SEMI	0.44293644
2000-04-19	2000-04-19	2021-04-19	0.0255	0.15	SEMI	0.45889388
2000-04-19	2000-04-19	2022-04-19	0.0255	0.15	SEMI	0.47447972
2000-04-19	2000-04-19	2023-04-19	0.0255	0.15	SEMI	0.48961664
2000-04-19	2000-04-19	2024-04-19	0.0255	0.15	SEMI	0.50431786
2000-04-19	2000-04-19	2025-04-19	0.0255	0.15	SEMI	0.51874446
2000-04-19	2000-04-19	2026-01-03	0.0255	0.15	SEMI	0.52860483

Table 5.2: CDS on the Default_Curve_2

numerical part, we want to strengthen our thesis with a correlation analysis. Note that the analyses of the current section use a Monte Carlo pricer to price the CDSs on the default curve, using the formulae of Theorem 3.2.1.

	Receiver Leg	Payer Leg
Currency	USD	USD
Initial Notional	0	0
Final Notional	0	0
Notional	180	100
Effective Date	2004-07-01	2004-07-01
Roll Date	2004-07-01	2004-07-01
End Date	2010-01-01	2006-01-01
Frequency	SEMI	MONTHLY
Rate Starting Time	0y	0y
Rate Term	6m	20y
Rate Index Type	LIBOR	CMDS
Spread	0	-0.025
Exercise Start	2004-07-01	
Exercise End	2004-07-01	
Set Date		2006-01-03

Table 5.3: MULTISWAP to be priced.

The idea here is to reprice the 1y, 5y, 10y CDS on the Default_curve_1 (see Table 5.1) using different correlation values for the interest rate and the credit intensity. We recall that the CDSs on the default curve are priced at par so their value is 0. As usual, it is not the CDS price that is traded (which will always be 0, by definition), but the spread that makes the contract fair. In Table 5.4 we reprice the 1y CDS and we can see that the price difference ranges from approximately -0.04% to 0.04%. The price difference increases in Table 5.5 where we reprice the 5y CDS and the price difference oscillates between -1.9% to 1.9%. Finally, in Table 5.6 the price difference increases even more, reaching -14% and 14.6 % in case of high correlation. Note that the upfront is the price of the CDS, so it is 0, as expected, in the case of 0 correlation and it increases for higher correlation values.

The results just stated confirm the hypothesis that the correlation between interest rate and credit intensity could be negligible only for short maturities CDSs. Indeed, in our experiments, the price difference with max correlation doesn't exceed 0.04%, in module.

Correlation	Upfront
-1	-0.00038416
-0.8	-0.00030753
-0.6	-0.00023091
-0.4	-0.00015354
-0.2	-0.00007639
0	0
0.2	0.00007639
0.4	0.00015277
0.6	0.00022916
0.9	0.00030555
1	0.00038193

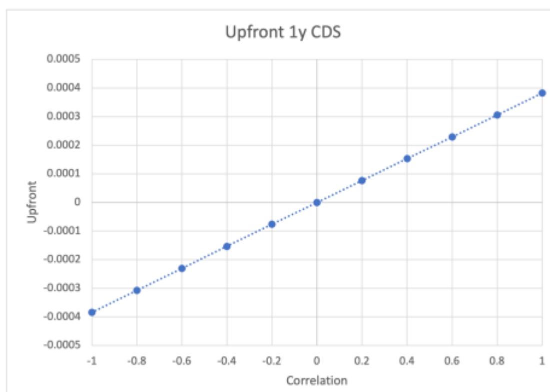

Table 5.4: Reprice of the 1y-maturity CDS on Default_Curve_1 with different correlations between rates and credit.

We now aim to show that CDSs referring to systemic entities have higher dependency with risk-free rates. Being a systemically important entity can be broadly defined as an entity whose failure would cause financial instability that would threaten the economy. For example, if a systemically important entity collapsed, interest rates would immediately increase sharply because of liquidity issues and the central banks intervention would be required. A bank, such as BNP Paribas, that has been analysed here, is clearly part of this class. We therefore now consider entities that are

Correlation	Upfront
-1	-0.018664698
-0.8	-0.014933365
-0.6	-0.011202039
-0.4	-0.007468721
-0.2	-0.003734552
0	0
0.2	0.003735788
0.4	0.007472899
0.6	0.011211222
0.9	0.014951306
1	0.018693497

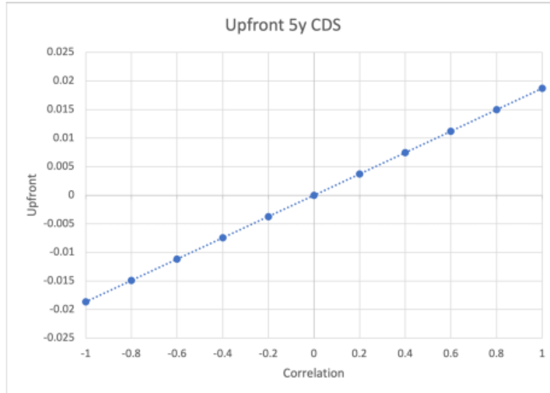

Table 5.5: Reprice of the 5y-maturity CDS on Default_Curve_1 with different correlations between rates and credit.

Correlation	Upfront
-1	-0.144302883
-0.8	-0.115549735
-0.6	-0.086760445
-0.4	-0.057906854
-0.2	-0.028986956
0	0
0.2	0.029055379
0.4	0.058179248
0.6	0.087371074
0.8	0.116630855
1	0.145963285

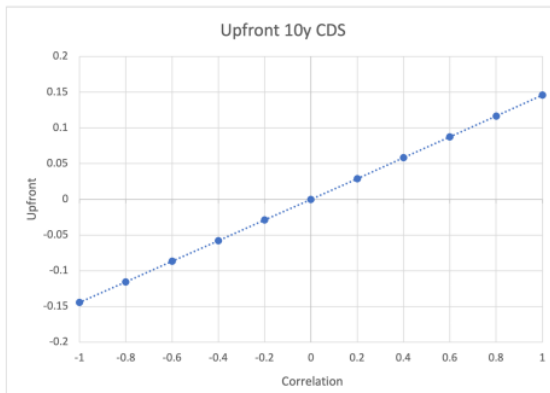

Table 5.6: Reprice of the 10y-maturity CDS on Default_Curve_1 with different correlations between rates and credit.

part of the Itraxx CDS index [3] and belong to different industries, from finance to pharmaceutical and communication companies, and compute the correlation with the respective risk-free rate in the period between January 2017 and January 2021. Note that for European companies, the Euro OverNight Index Average (EONIA) rate has been chosen as a proxy for the risk-free rate, whilst for UK companies, the Sterling Overnight Interbank Average Rate (SONIA) has been used as a proxy of the risk-free rate for sterling markets. The results are shown in Table 5.7.

Company	CDS maturity	Risk-free rate	Correlation
BNP Paribas	5Y	EONIA	76.39 %
Astrazeneca	5Y	SONIA	71.55 %
Danone	5Y	EONIA	58.24 %
Vodafone	5Y	SONIA	50.92 %
Bayer	5Y	EONIA	12.75 %

Table 5.7: Correlation study against risk-free rates.

As expected, BNP Paribas is highly correlated with EONIA. Astrazeneca is also highly correlated and this can be linked to the Covid crisis. Finally, the correlation decreases for Vodafone,

[3] Data source: Bloomberg.

Danone and Bayer, whose financial stability is clearly less linked to the market stability.

The results just stated confirm the hypothesis that the correlation between interest rate and credit intensity is not negligible for systemic companies.

5.3 Results

In this section we outline the results that we have with the new model (which we will refer to as "CMDS_Affine") presented in Chapter 3 and compare them with standard Monte Carlo simulation (which we will refer to as "CMDS_MonteCarlo"), where the credit intensity and interest rate have been simulated up to the maturity of the CMDS rate (2026-01-03) and the formulae in (3.3.3) have been used.

In this part, no convexity adjustment is considered, therefore it is interesting to analyse the difference of both the CMDS rate and Multi Swap price between the first method and the new method, that has been described in the previous sections, where the credit intensity and rate have been simulated only up to the settle date of the CMDS rate and affine formulae have been used from that point onward.

Note that in this section the CDSs to be calibrated are taken from Default_Curve_2 (see Table 5.2).

Fixed Credit Local Volatility, Interest Rate Local Volatility = 0.1%

The difference in CMDS rates, for correlation set to zero, is 0.0031 (CMDS_Affine_rate = 0.0214 USD, CMDS_MonteCarlo_rate = 0.0246). In terms of the MultiSwap price, we observed a difference equal to 0.2102 (CMDS_Affine_price = 1.4683 USD, CMDS_MonteCarlo_price = 1.6821 USD). Table 5.8 shows how the CMDS rate difference changes when varying the correlation between credit intensity and rates (to highlight the behaviour as a function of the correlation, the CMDS_Affine_rate price at 0 correlation has been shifted to match the CMDS_MonteCarlo_rate). As expected, the difference between the two prices is minimum when the correlation is 0, and increases when it shifts towards +1 and -1.

We repeat the same analysis for the MultiSwap prices: Table 5.9 shows the price differences between the two models. The behaviour is the same as for the rate, which confirms that the CMDS_Affine implementation is working as expected.

Fixed Credit Local Volatility, Interest Rate Local Volatility = 1%

We expect the CMDS rate and Multi Swap price differences to decrease accordingly when the credit-rates correlation and the fixed volatility decrease. We therefore investigate the relationship between price difference and correlation for a fixed vol scaled up to 1%.
Table 5.10 shows the CMDS rate difference as a function of the correlation while Table 5.11 shows the behaviour of the Multi Swap price difference: they both outline the same trend as for vol=0.1% (which confirms the consistency of our implementation), but the magnitude of the differences is higher ($\sim$ 10 times bigger, so proportional to the increase in the volatility).
We can therefore conclude that the difference between the two pricing model decreases when correlation decreases. Additionally, the impact of correlation is also influenced by the volatility: to a higher volatility corresponds a bigger impact of correlation (in fact, the change is actually proportional to the increase in volatility). The same effect is reflected on the Multi Swap derivative price (see Table 5.11).

Correlation	CMDS rate difference
-1	-0.00000455
-0.8	-0.00000286
-0.4	-0.00000143
0	0.00000000
0.4	0.00000143
0.8	0.00000285
1	0.00000357

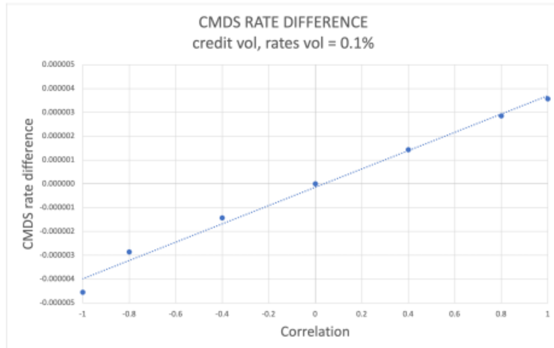

Table 5.8: Price of the CMDS rate difference with credit volatility = rates volatility = 0.1% with respect to different correlation values, using Default_Curve_2.

Correlation	Swap price difference
-1	-0.0003
-0.8	-0.0002
-0.4	-0.0001
0	0.0000
0.4	0.0001
0.8	0.0002
1	0.0002

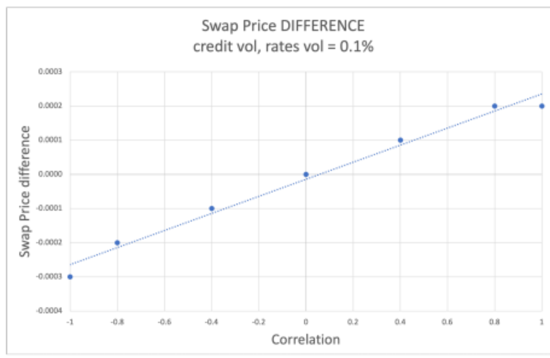

Table 5.9: Price of the Multi Swap with credit volatility = rates volatility = 0.1% with respect to different correlation values, using Default_Curve_2.

Correlation	CMDS rate difference
-1	-0.00005647
-0.8	-0.00004517
-0.4	-0.00002257
0	0.00000000
0.4	0.00002677
0.8	0.00004506
1	0.00005630

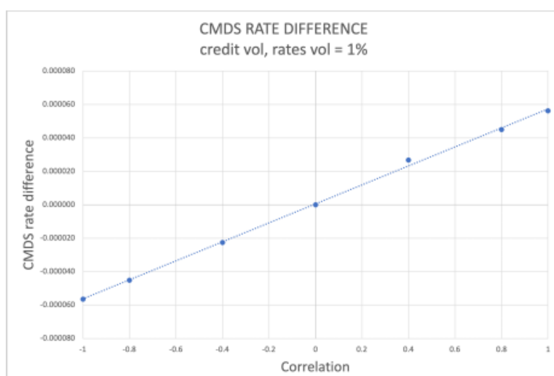

Table 5.10: Price of the CMDS rate with credit volatility = rates volatility = 1% with respect to different correlation values, using Default_Curve_2.

Correlation	Swap price difference
-1	-0.0039
-0.8	-0.0031
-0.4	-0.0015
0	0.0000
0.4	0.0018
0.8	0.0031
1	0.0039

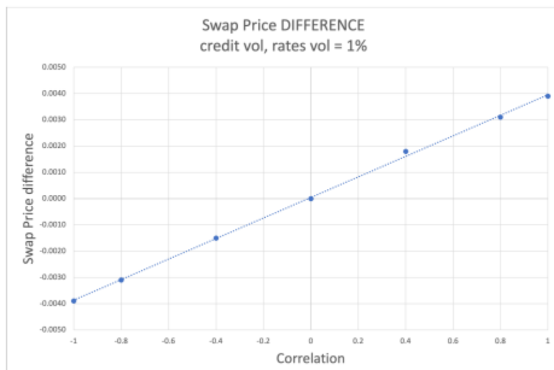

Table 5.11: Price of the Multi Swap with credit volatility = rates volatility = 1% with respect to different correlation values, using Default_Curve_2.

Conclusion

In this paper we focused on modelling the stochastic term structure of the credit spread using a credit skew term structure model: we choose to model instantaneous credit spread by using the Cheyette representation of the Heath-Jarrow-Morton (HJM) framework. This choice ensures that we are both in a arbitrage free and finite dimension Markovian framework, which is economically and computationally advantageous. On one hand, we want a model to be arbitrage free so that no trader can make a risk-free profit by buying a security and selling the replicating portfolio. If any arbitrage opportunities do arise, they quickly disappear as traders taking advantage of the arbitrage push the derivative's price until it equals the value of replicating portfolios. On the other hand, the finite-dimension Markovian framework allows us to derive analytical, in fact affine, formulae for the risky bonds and default factors, which are the building blocks for the protection and premium leg of a CDS and CMDS rate.

Since CDSs are liquid instruments, we didn't use the developed model to price them, but to calibrate the default factor to the CDSs on the default curve (see Table 5.2) and then use the calibrated factor to value the CMDS rate that appears in the Multi Swap security (see Table 5.3). Note that the calibration is different for CDSs on the curve that have maturity smaller or equal than the final settle date of the CMDS rate and for the CDSs on the curve that have maturity higher than the final settle date of the CMDS rate. This is due to the fact that, in our new model, the credit intensity is simulated only up to the final settle date of the CMDS rate. In the first case, we could calibrate, for instance, the drift of the hazard rate, whilst in the second case the default factor directly is calibrated, assuming it has an exponential form (see (3.2.10)).

So far, we assumed independence between credit spread and rates, that can be translated mathematically with the assumption in (3.2.7). We then discussed a potential improvement to relax the assumption of independence between rates and credit and therefore to take into account for the correlation, that we saw to be not negligible in the numerical part for long maturity and systemic companies related CDSs. The idea here is to assume λ, r to follow two-dimension correlated Hull-Factor extended Vasicek model. The Gaussian model allows us to find an analytical formula for the default factor in the general case (see Proposition 4.2.1).

In the numerical part, we implemented the new method without convexity and integrated it in the C++ library of Deutsche Bank. Moreover, we analysed the effect of correlation on the CMDS rate and Multi Swap price, taking as benchmark the Monte Carlo pricer, where the credit spread and rates were simulated up to the maturity of the CMDS rate and all the CDSs on the default curve have been calibrated adjusting the drift of the credit hazard rate. The difference between the two models has the expected behavior: it decreases when correlation decreases. Additionally, the impact of correlation is also influenced by the credit local volatility: to a smaller volatility corresponds a smaller impact of correlation (in fact, the change is actually proportional to the reduction in volatility). The same effect is reflected on the Multi Swap derivative price.

Further research could focus on three main points: first, we will implement the convexity adjustment described above and assess the impact of it in calibrating long-maturity CDSs and pricing the CMDS rate and related contracts. Secondly, we will implement a discretisation scheme to prevent negative hazard rates. Indeed, as previously outlined in the introductory part, a significant drawback for the usage of Cheyette model in credit is that this could lead to negative survival probabilities. Finally, different forms for the calibrated part of the default factor (see 3.2.10) will be explored and compared to the current one.

Appendix A

Technical Proofs

A.1 Risky Bond Price proof

We call $\mathcal{H}_t := \sigma(\{\tau < s\}, s \leq t)$.
We first want to prove that:

$$E\left(1_{\{\tau \geq T\}} \mid \mathcal{F}_T \vee \mathcal{H}_t\right) = \mathbb{1}_{\{\tau > t\}} \exp\left(-\int_t^T \lambda_s ds\right).$$

In order to do so, we observe that $\{\tau > t\} \in \mathcal{H}_t$ therefore:

$$\begin{aligned}
\mathbb{E}\left(\mathbb{1}_{\{\tau \geq T\}} \mid \mathcal{F}_T \vee \mathcal{H}_t\right) &= \mathbb{1}_{\{\tau > t\}} \mathbb{E}\left(\mathbb{1}_{\{\tau \geq T\}} \mid \mathcal{F}_T \vee \mathcal{H}_t\right) \\
&= \mathbb{1}_{\{\tau > t\}} \frac{\mathbb{Q}\left(\{\tau \geq T\} \cap \{\tau > t\} \mid \mathcal{F}_T\right)}{\mathbb{Q}\left(\tau > t \mid \mathcal{F}_T\right)} \\
&= \mathbb{1}_{\{\tau > t\}} \frac{\mathbb{Q}\left(\tau \geq T \mid \mathcal{F}_T\right)}{\mathbb{Q}\left(\tau > t \mid \mathcal{F}_T\right)} \\
&= \mathbb{1}_{\{\tau > t\}} \frac{\exp\left(-\int_0^T \lambda_s ds\right)}{\exp\left(-\int_0^t \lambda_s ds\right)} \\
&= \mathbb{1}_{\{\tau > t\}} \exp\left(-\int_t^T \lambda_s ds\right).
\end{aligned} \tag{A.1.1}$$

We now prove the risky bond price formula:

$$\begin{aligned}
&\mathbb{E}\left(\exp\left(-\textstyle\int_t^T r_s ds\right) \mathbb{1}_{\{\tau > T\}} \mid \mathcal{G}_t\right) \\
&\quad = \mathbb{E}\left(\mathbb{E}\left(\exp\left(-\textstyle\int_t^T r_s ds\right) \mathbb{1}_{\{\tau > T\}} \mid \mathcal{F}_T \vee \mathcal{H}_t\right) \mid \mathcal{G}_t\right) \\
&\quad = \mathbb{E}\left(\exp\left(-\textstyle\int_t^T r_s ds\right) \mathbb{E}\left(\mathbb{1}_{\{\tau > T\}} \mid \mathcal{F}_T \vee \mathcal{H}_t\right) \mid \mathcal{G}_t\right) \\
&\quad = \mathbb{1}_{\{\tau > t\}} \mathbb{E}\left(\exp\left(-\textstyle\int_t^T r_s + \lambda_s ds\right) \mid \mathcal{G}_t\right)
\end{aligned}$$

Recall that U in (2.1.2) is a random variable independent of $\mathcal{F}_T$. We now use [13] to deduce the following:

$$\begin{aligned}
&\mathbb{E}\left(\exp\left(-\textstyle\int_t^T r_s + \lambda_s ds\right) \mid \mathcal{F}_t \vee \sigma(U)\right) \\
&\quad = \mathbb{E}\left(\exp\left(-\textstyle\int_t^T r_s + \lambda_s ds\right) \mid \mathcal{F}_t\right).
\end{aligned}$$

A.2 Default factor under the T_1-forward measure and Hull-White

Recall that λ follows the following SDE under the risk-neutral measure:

$$d\lambda_t = \left(\theta_\lambda(t) - k_\lambda(t)\lambda_t\right) dt + \sigma_\lambda(t) dW_t \tag{A.2.1}$$

We now apply the change of measure to switch from the risk-neutral measure to the T_1-forward measure:

$$\frac{dP}{dQ}(t) = \frac{B(t,T_1)}{B(0,T_1)B(t)} = \frac{Z_t}{B(0,T_1)}$$

where, from [14]:

$$\begin{aligned}
&\tfrac{dZ}{Z}(t) = \Sigma(t,T_1)dW(t)\\
&\Sigma(t,T_1) := -\textstyle\int_0^T \sigma_f(t,u)du\\
&\sigma_f(t,T_1) = \sigma_r(t)\exp\left(-\textstyle\int_t^T k_r(u)du\right).
\end{aligned}$$

From Girsanov theorem we know that the drift of the process doesn't change under a different measure, whilst the Brownian motion becomes the following:

$$d\widetilde{W}(t) = dW(t) - \Sigma(t,T_1)dt.$$

Therefore, the dynamics of λ under the T_1 forward measure is:

$$d\lambda_t = \left(\widetilde{\theta_\lambda}(t) - k_\lambda(t)\lambda_t\right)dt + \sigma_\lambda(t)d\widetilde{W}_t \tag{A.2.2}$$

where

$$\begin{aligned}
\widetilde{\theta_\lambda} &:= \theta_\lambda - \tfrac{\sigma_\lambda(t)\cdot\Sigma(t,T_1)}{k_\lambda(t)}\\
&= \theta_\lambda - \tfrac{\rho\sigma_\lambda(t)\int_t^{T_1}\sigma_r(u)\exp\left(-\int_u^{T_1}k_r(s)ds\right)du}{k_\lambda(t)}.
\end{aligned}$$

Using the dynamics (A.2.2) in Theorem 4.1.5, we obtain the following formula for the discount factor:

$$S(t,T)^{T_1} = \exp\left(-A(t,T) - C(t,T)\lambda_t\right) \tag{A.2.3}$$

where

$$\begin{aligned}
A(t,T) &= \textstyle\int_t^T\left[k_\lambda(s)C(s,T) - \frac{1}{2}\sigma_\lambda^2(s)C^2(s,T)\right]ds\\
C(t,T) &= \textstyle\int_t^T \exp\left(-\int_t^s \widetilde{\theta}(u)du\right)ds.\\
&= \textstyle\int_t^T \exp\left(-\int_t^s \theta(u)du\right)\cdot\\
&\exp\left(-\textstyle\int_t^s \rho\sigma_\lambda(u)\int_u^{T_1}\left(\sigma_r(w)\exp\left(-\int_w^{T_1}k_r(j)dj\right)dw\right)du\right)ds.
\end{aligned} \tag{A.2.4}$$

and λ is the solution of the SDE (A.2.2)

We can now proceed in two different ways: we can either solve the SDE (A.2.2) and rearrange the terms in (A.2.3) and (A.2.4) to obtain the desired formula, or we can pursue the proof in the following way.

We first solve the SDE (A.2.2) by using $e^{\int_t^T k_\lambda(u)du}$ as the multiplying factor, therefore we obtain:

$$\lambda_T = \lambda_t e^{-\int_t^T k_\lambda(s)ds} + \int_t^T \sigma_\lambda(u)e^{-\int_u^T k_\lambda(s)ds}dW_u + \int_t^T \theta_\lambda(u)e^{-\int_u^T k_\lambda(s)ds}du \tag{A.2.5}$$

where

$$\theta_\lambda = \frac{\rho\sigma_\lambda(t)\int_t^{T_1}\sigma_r(u)\exp\left(-\int_u^{T_1}k_r(s)ds\right)du}{k_\lambda(t)} + \widetilde{\theta_\lambda}.$$

We now split the dynamics of λ in its stochastic and deterministic parts:

$$\begin{aligned}
d\widetilde{\lambda}_t &= -k_\lambda(t)\widetilde{\lambda}_t dt + \sigma_\lambda(t)dW_t\\
d\bar{\lambda}_t &= (\theta_\lambda(t) - k_\lambda(t)\bar{\lambda}_t)dt.
\end{aligned} \tag{A.2.6}$$

The solutions of the SDEs in (A.2.6) are:

$$\begin{aligned}
\widetilde{\lambda_T} &= \widetilde{\lambda}_t e^{-\int_t^T k_\lambda(s)ds} + \textstyle\int_t^T e^{-\int_u^T k_\lambda(s)ds}\sigma_\lambda(u)dW_u\\
\bar{\lambda}_t &= \bar{\lambda}_t e^{-\int_t^T k_\lambda(s)ds} + \textstyle\int_t^T e^{-\int_u^T k_\lambda(s)ds}\theta_\lambda(u)du.
\end{aligned} \tag{A.2.7}$$

Were the first one has been found by using $e^{\int_t^T k_\lambda(u)du}$ as the multiplying factor, whilst the second is simply an ODE.

In order to make computation easier, from here we assume k_λ to be constant to make computation easier. However, the final result remains true even with a non constant k_λ.

We now proceed defining the following variables.

$$\begin{aligned}
\widetilde{\Lambda}(T) &:= \int_0^T \widetilde{\lambda}_u du \\
&= \widetilde{\Lambda}(t) + \widetilde{\lambda}_t \int_t^T e^{-k_\lambda(u-t)} du + \int_t^T \int_t^u \sigma_\lambda(s) e^{-k_\lambda(u-s)} dW_s du \\
&= \widetilde{\Lambda}(t) + \frac{\widetilde{\lambda}_t(1-e^{-k_\lambda(T-t)})}{k_\lambda} + \int_t^T \frac{\sigma_\lambda(s)(1-e^{-k_\lambda(T-s)})}{k_\lambda} dW_s. \qquad \text{(A.2.8)}
\end{aligned}$$

$$\begin{aligned}
\bar{\Lambda}(T) &:= \int_0^T \bar{\lambda}_u du \\
&= \bar{\Lambda}(t) + \frac{\bar{\lambda}_t(1-e^{-k_\lambda(T-t)})}{k_\lambda} + \int_t^T \int_t^u \theta_\lambda(s) e^{-k_\lambda(u-s)} ds du.
\end{aligned}$$

We now compute the default factor:

$$\begin{aligned}
S(t,T) &= \mathbb{E}\left(e^{-\int_t^T \lambda_u du} \mid \mathcal{F}_t\right) \\
&= e^{-\int_t^T \bar{\lambda}_u du} \cdot \mathbb{E}\left(e^{-\int_t^T \widetilde{\lambda}_u du} \mid \mathcal{F}_t\right) \qquad \text{(A.2.9)} \\
&= e^{-(\bar{\Lambda}(T)-\bar{\Lambda}(t))} \cdot \mathbb{E}\left(e^{-(\widetilde{\Lambda}(T)-\widetilde{\Lambda}(t))} \mid \mathcal{F}_t\right)
\end{aligned}$$

where $\widetilde{\Lambda}(T) - \widetilde{\Lambda}(t)$ is a Normal random variable with the following mean and variance:

$$\begin{aligned}
\mathbb{E}\left(\widetilde{\Lambda}(T) - \widetilde{\Lambda}(t) \mid \mathcal{F}_t\right) &= \frac{\widetilde{\lambda}_t(1-e^{-k_\lambda(T-t)})}{k_\lambda}. \\
\text{Var}\left(\widetilde{\Lambda}(T) - \widetilde{\Lambda}(t) \mid \mathcal{F}_t\right) &= \int_t^T \frac{\sigma_\lambda(s)^2(1-e^{-k_\lambda(T-s)})^2}{k_\lambda^2} ds \qquad \text{(A.2.10)} \\
&= \int_t^T \sigma_\lambda(s)^2 B(s,T)^2 ds \\
&= 2C(t,T)
\end{aligned}$$

where

$$\begin{aligned}
B(s,T) &:= \frac{(1-e^{-k_\lambda(T-s)})}{k_\lambda} \\
C(t,T) &:= \frac{\int_t^T \sigma_\lambda(s)^2 B(s,T)^2}{2}.
\end{aligned} \qquad \text{(A.2.11)}$$

Therefore, the default factor becomes the following, since the second part is nothing more that the expectation of the exponential of a normal random variable:

$$S(t,T) = e^{-(\bar{\Lambda}(T)-\bar{\Lambda}(t))} \cdot e^{-\widetilde{\lambda}_t B(t,T) + C(t,T)}. \qquad \text{(A.2.12)}$$

It is worth noticing that the first factor only depends on the T_1-forward measure. Let's therefore try to split it into its T_1-dependent and T_1-independent parts:

$$\begin{aligned}
\bar{\Lambda}(T) - \bar{\Lambda}(t) &= \frac{\bar{\lambda}_t(1-e^{-k_\lambda(T-t)})}{k_\lambda} + \int_t^T \int_t^u \theta_\lambda(s) e^{-k_\lambda(u-s)} ds du \\
&+ \int_t^T \int_t^u \Sigma(s,T_1)\sigma_\lambda(s)\rho(s) e^{-k_\lambda(u-s)} ds du. \\
&= \frac{\bar{\lambda}_t(1-e^{-k_\lambda(T-t)})}{k_\lambda} + \int_t^T \int_t^u \theta_\lambda(s) e^{-k_\lambda(u-s)} ds du \qquad \text{(A.2.13)} \\
&+ \int_t^T \int_t^u \sigma_r(s) \left(\int_s^{T_1} e^{-k_r(k-s)} dk\right) \sigma_\lambda(s)\rho(s) e^{-k_\lambda(u-s)} ds du.
\end{aligned}$$

If we allowed k_r, k_λ to be time dependent, we would have the following expression instead.

$$\begin{aligned}\bar{\Lambda}(T) - \bar{\Lambda}(t) &= \frac{\bar{\lambda}_t(1 - e^{-k_\lambda(T-t)})}{k_\lambda} + \int_t^T \int_t^u \theta_\lambda(s) e^{-k_\lambda(u-s)} ds du \\ &+ \int_t^T \int_t^u \Sigma(s, T_1)\sigma_\lambda(s)\rho(s) e^{-k_\lambda(u-s)} ds du. \\ &= \frac{\bar{\lambda}_t(1 - e^{-k_\lambda(T-t)})}{k_\lambda} + \int_t^T \int_t^u \theta_\lambda(s) e^{-k_\lambda(u-s)} ds du \\ &+ \int_t^T \int_t^u \sigma_r(s) \left(\int_s^{T_1} e^{-\int_s^k k_r(w)dw} dk\right) \sigma_\lambda(s)\rho(s) e^{-\int_u^s k_\lambda(w)dw} ds du.\end{aligned} \tag{A.2.14}$$

The T_1-dependent part appears in the last term only, therefore the convexity adjustment term that we are looking for is the following:

$$\exp\left(\int_t^T \int_t^u e^{-\int_s^u k_\lambda(w)dw} \sigma_r(s)\sigma_\lambda(s)\rho(s) \int_s^{T_1} \left(e^{-\int_s^k k_r(w)dw} dk\right) ds du\right). \tag{A.2.15}$$

The default factor under the T_1-forward measure has consequently the following formula:

$$S(t,T)^{T_1} = S(t,T) \exp\left(-\int_t^T \int_t^u \exp\left(-\int_s^u k_\lambda(w)dw\right) \epsilon(s, T_1) ds du\right) \tag{A.2.16}$$

where

$$\epsilon(s, T_1) = -\sigma_r(s)\sigma_\lambda(s)\rho \int_s^{T_1} \exp\left(-\int_s^k k_r(w)dw\right) dk. \tag{A.2.17}$$

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